Asymptotic Yamabe-Problem Obstruction-Densities

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Abstract

The Yamabe problem is that of finding a metric with constant scalar curvature conformal to a given metric. We study this problem in the context of conformal hypersurface embeddings with a *defining density* σ. This leads to the study of solutions given as an asymptotic expansion in σ, but the existence of such solutions is obstructed in a manner proportional to a conformal invariant we refer to as the *obstruction density*. Our main goal is the calculation of this obstruction explicitly in the cases of embedded surfaces and volumes. This provides two nontrivial conformal hypersurface invariants, which prove interesting to mathematical physics, independent of their origin in this problem.

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Chapter 1

Introduction

On a smooth manifold $\mathcal{M}$ of dimension d, we call any submanifold of dimension $n = d - 1$ a *hypersurface* of $\mathcal{M}$. Hypersurfaces are found in many applications of differential geometry, from solving boundary value problems to examining properties of the boundary of the manifold itself. In this thesis we consider hypersurfaces specifically in the context of conformal geometry, and study one particular kind of boundary value problem in this framework.

In particular, let $(\mathcal{M}, g)$ be a smooth manifold with metric g, which we will assume throughout this thesis to be Riemannian (*i.e.*, g is positive-definite). A conformal structure on $\mathcal{M}$ is an equivalence class of metrics $[g]$, under the equivalence relation $g \sim \hat{g}$, where

$$\hat{g} = \Omega^2 g \ ,$$

for some positive function $\Omega : \mathcal{M} \to \mathbb{R}^+$. We call the transformation of the metric according to this relation a *conformal transformation*, and say that g is *conformal* to $\hat{g}$.

One important class of objects we can use to study these conformal transformations are *densities*, which come with a conformal weight w. Under a conformal transformation, a density f of weight w obeys the transformation law:

$$\hat{f} = \Omega^w f \ .$$

Densities provide a particularly clean way of classifying conformal invariants, and play a fundamental role in conformal geometry as a result.

Given only a conformal class of metrics $[g]$, there is no preferred representative metric g to choose for calculations of objects in $\mathcal{M}$. However, given a single density τ of weight $w = 1$, we can introduce a double equivalence class instead, defined by

$$(g, \tau) \sim (\Omega^2 g, \Omega^{-1} \tau) \ .$$

We call τ a *conformal scale*. In this double equivalence class, there is a canonical metric defined by the representative $(g, 1)$, provided τ is nowhere vanishing.

It is also interesting to consider a more general class of densities σ of weight 1, which vanish on a nonempty set Σ. We consider specifically the case where this zero locus defines a hypersurface in $\mathcal{M}$, and call σ a *defining density*. Away from Σ, the defining density can be treated as a conformal scale, but in doing so we introduce asymptotic behavior near Σ in the canonical metric.

We consider the problem of finding a particular defining density σ for which the canonical representative metric g has constant scalar curvature. In the case of a true conformal scale τ, this is the Yamabe problem, which has a solution for a compact Riemannian manifold, but not in the general case (see, *e.g.*, [7]). The introduction of the zero locus Σ turns our problem into an asymptotic version of the Yamabe problem, relevant for hypersurface embeddings and manifolds with boundary. In particular, we will show that the condition on σ can be written as a PDE:

$$(\nabla \sigma)^2 - 2\sigma \frac{(\Delta + J^g)\sigma}{d} = 1 \ , \tag{1.1}$$

where J^g is proportional to the scalar curvature under the metric g, and $(g, \sigma) \sim (\hat{g}, 1)$, where $\hat{g}$ has constant scalar curvature. This equation also plays the role of a conformally invariant version of the requirement that the normal vector to a surface (given here by $\nabla \sigma$) has unit length. In fact, along Σ, this is precisely what Equation (1.1) requires.

The above PDE can be solved order by order in σ until a critical order, where we have a nonzero coefficient that depends on the dimension of Σ (see [5]). We call this coefficient the *obstruction density*, and our main goal is to find an expression for this conformally invariant object in terms of hypersurface quantities.

In [5] it is also shown that the solution can be improved by adding a term proportional to $\log \sigma$, at the cost of smoothness near the boundary. The coefficient of the log term is related to the same obstruction density we seek. This approach in particular becomes relevant for mathematical physics (*e.g.*, [1]). Specifically, applications have included attempts to calculate the entropy in an entangled system, through a calculation of a renormalized surface area (see [8]). This surface area is found via a similar expansion as the coefficient of a logarithmic divergence. For example, the obstruction density for surfaces appears in these contexts as the variation of the Willmore energy (see [1]), and has the form

$$\Delta H + 2H(H^2 - K)\,. \tag{1.2}$$

(see Chapter 4). Our main result is given by Equation (4.13), which is the analogue to Equation (1.2) for embedded volumes.

The thesis is structured as follows: in Chapter 2, we provide a review of identities and definitions from Riemannian, conformal, and hypersurface geometry. We continue with a brief introduction to tractors, the fundamental objects of conformal geometry. Chapter 3 studies the case of a defining density in much more detail, and produces relevant identities we will need for the calculation of the obstructions. Chapter 4 contains the details of the calculation, and the obstruction densities for surfaces and volumes are provided by Equations (4.7) and (4.13), respectively.

Chapter 2

Background

We will first need some basic results and definitions from Riemannian and conformal geometry, as well as a brief discussion of *tractors*, which provide a fundamental calculus for conformal geometry. Tensors in general do not respect the structure of a conformal class of metrics, and are more naturally replaced by tractors in this context.

2.1 Riemannian Geometry

Let $(\mathcal{M}, g)$ be a manifold of dimension d with Riemannian metric g. The curvature of this Riemannian manifold is described by a set of fundamental tensors, which depend only on the metric g: the *Riemann Curvature Tensor* is defined by its action on vector fields[1],

$$R^{\#}_{ab} v^c := R_{ab}{}^{c}{}_{d} v^d = [\nabla_a, \nabla_b] v^c \ ,$$

where ∇ is the Levi-Civita connection of g. It obeys the symmetries

$$R_{abcd} = -R_{bacd} = R_{cdab} \ ,$$

as well as the Bianchi identity:

$$\nabla_a R_{bcde} + \nabla_b R_{cade} + \nabla_c R_{abde} = 0 \ . \tag{2.1}$$

[1]We make use of Penrose abstract index notation throughout, labelling tensor fields with indices to denote their type.

We define the Ricci tensor as the Riemman tensor traced on its first and third indices,

$$R_{ab} = g^{cd} R_{cadb} = R_{ba} \, ,$$

and its trace $R^a_a =: R$, is the *scalar curvature*. This quantity is the subject of the Yamabe problem discussed above–we will attempt to find a metric for which R becomes constant.

2.2 Hypersurface Geometry

Consider a hypersurface Σ in $\mathcal{M}$, with the special property that Σ is the zero locus of some function[2] s, called the *defining function* of Σ. The existence of a defining functions provides a decomposition of tensors in $\mathcal{M}$ near Σ:

$$T = T_0 + sT_1 + s^2 T_2 + \dots \, , \tag{2.2}$$

where each T_k is non-vanishing along Σ, and this expansion is valid in a neighborhood of Σ. Alternatively, this decomposition can be viewed as a method of extending the tensor T_0, defined along Σ, to a tensor T in the bulk space $\mathcal{M}$. We stress that viewed in this manner, the extension T is not unique.

We will frequently choose a particular extension of a tensor T_0 along Σ in order to compute certain expressions explicitly. To justify any given choice of extension, it is important to recognize which expressions are in fact extension independent. In particular, we consider the equivalence relation $T \sim T'$, where

$$T' = T + sU \, , \tag{2.3}$$

for some smooth tensor U in $\mathcal{M}$. Tensors along Σ can then be described by an equivalence class $[T]$ of tensors in $\mathcal{M}$. Extension independent expressions are precisely those that respect this equivalence relation.

Another important consequence of the defining function is that the vector $n_a := \nabla_a s$ is necessarily perpendicular to the hypersurface. For this reason, n_a is called the *conormal vector* to Σ. We will make the further assumption on s that n_a has nonzero length at every point of Σ. Then, we

[2]We will also assume smoothness of the defining function for the calculations that follow.

can normalize n_a by dividing by its length along the hypersurface, to create a unit normal vector,

$$\hat{n}_a := \frac{n_a}{|n|}\,.$$

The unit normal vector along Σ simply gives the direction in $\mathcal{M}$ away from the hypersurface. Changes in tensors along this direction are calculated via the *normal derivative* $\nabla_n := n^a\nabla_a$, and corresponding unit normal derivative $\nabla_{\hat{n}}$. The unit normal vector also provides another decomposition of tensors in $\mathcal{M}$, this time into their tangential and normal pieces: for example,

$$v_a \stackrel{\Sigma}{=} v_a^\top + \hat{n}_a \hat{n}.v\,,$$

where $\hat{n}.v := \hat{n}^b v_b$, the symbol $\stackrel{\Sigma}{=}$ denotes equality only along the hypersurface Σ, and $v_a^\top$ denotes a tensor with no normal component ($n.v^\top \stackrel{\Sigma}{=} 0$). This decomposition suggests the definition of a projection operator $\gamma_b^a := g_b^a - \hat{n}^a\hat{n}_b$, so that $v_a^\top \stackrel{\Sigma}{=} \gamma_a^b v_b$.

The projection γ maps all tensors defined along Σ to tensors *tangential* to Σ. These tangential tensors define a part of the tangent bundle $T\mathcal{M}$ which we denote $(TM)^\top$. This turns out to be isomorphic to the tangent bundle of Σ:

$$(T\mathcal{M})^\top \cong T\Sigma\,. \tag{2.4}$$

We note that $\gamma^2 = \gamma$, so that γ is also a tangential tensor along Σ. In fact, the corresponding tensor on the interior of Σ is the intrinsic metric $\bar{g}$, which we call the *first fundamental form*.

Using the isomorphism (2.4), and extensions of the form in Equation (2.2), we can extend tensors in $T\Sigma$ into tensors in $T\mathcal{M}$. Expressions extended in this way must be extension independent, respecting the equivalence relation defined by Equation (2.3). The most fundamental operator with this property is the *tangential derivative*, defined by:

$$\nabla_a^\top := \nabla_a - \hat{n}_a\nabla_{\hat{n}} = \gamma_a^b\nabla_b\,.$$

A quick calculation shows that the tangential derivative of s gives 0:

$$\nabla_a^\top s = \nabla_a s - \hat{n}_a\hat{n}^c\nabla_c s = n_a - \hat{n}_a|n| = 0\,,$$

so that for any tensors T and T' in the same equivalence class

$$\nabla_a^\top T \stackrel{\Sigma}{=} \nabla_a^\top T'\,.$$

One natural question at this stage is to consider how the tangential derivative behaves under the isomorphism (2.4). In particular, one might expect that the tangential derivative behaves similarly to the intrinsic derivative $\bar{\nabla}$ corresponding to the Levi Civita connection of $\bar{g}$ defined above. In fact, these two operators are not quite the same, and we measure their difference with a tensor II we call the *second fundamental form*; if v is a tangential vector, then

$$\bar{\nabla}_a v^b \stackrel{\Sigma}{=} \nabla^\top_a v^b + \hat{n}^b \mathrm{II}_{ac} v^c \,. \tag{2.5}$$

Noting that the normal component of the right hand side of Equation (2.5) must vanish, we find

$$\mathrm{II}_{ab} \stackrel{\Sigma}{=} \nabla^\top_a \hat{n}_b \,.$$

Though not immediately obvious from its definition, a short calculation reveals that II_{ab} is a symmetric tensor: $\nabla^\top_a \hat{n}_b = \nabla^\top_b \hat{n}_a$. Its trace is related to the *mean curvature* H by the following equation:

$$H := \frac{1}{n} \mathrm{II}^a_a = \frac{1}{n} \nabla^\top . \hat{n} \,, \tag{2.6}$$

an average of the eigenvalues of II (hence the *mean* in *mean curvature*). This gives the decomposition of II into its trace and trace-free components: $\mathrm{II}_{ab} = \mathring{\mathrm{II}}_{ab} + H\gamma_{ab}$.

Underlying each of these definitions is a link between quantities *intrinsic* to a surface (those that can be calculated with no knowledge of the larger manifold $\mathcal{M}$), and those *extrinsic* to the surface (those depending on the embedding itself). In fact, Equation (2.5) shows that a completely intrinsic quantity, $\bar{\nabla}_a v^b$, can be calculated in terms of necessarily extrinsic tensors (including the normal vector $\hat{n}^b$). This sort of relationship motivated much of the classical work on hypersurface embeddings (see [3]). One of the most fundamental relationships between intrinsic and extrinsic quantities is described by Gauss' *Theorema Egregium*, which states that the determinant of II for a surface embedded in a Euclidean space can be given in terms of only intrinsic curvature terms.

We will require a generalization of Gauss' result, which we call the Gauss equation. This includes an additional term, the tangential piece of the extrinsic curvature (zero for the Euclidean space of Gauss):

$$\bar{R}_{abcd} \stackrel{\Sigma}{=} R^\top_{abcd} + \mathrm{II}_{ac}\mathrm{II}_{bd} - \mathrm{II}_{ad}\mathrm{II}_{bc} \,. \tag{2.7}$$

Written this way, Equation (2.7) provides the link between the curvature of the embedded space itself and the curvature of the space it lies in. Considering a two dimensional surface embedded in Euclidean space gives the *Theorema Egregium* from Equation (2.7).

Another useful relation along these lines is the Codazzi-Mainardi Equation, which provides the curl of II:

$$\bar{\nabla}_a \mathrm{II}_{bc} - \bar{\nabla}_b \mathrm{II}_{ac} = (R_{abcd}\hat{n}^d)^\top . \tag{2.8}$$

Tracing Equation (2.8) yields a useful relation for the gradient of H ($d \geq 3$):

$$\bar{\nabla}_a H = \frac{1}{d-2}\bar{\nabla}.\mathring{\mathrm{II}}_a - P(a,n)^\top , \tag{2.9}$$

where $P(a,n)^\top$ denotes $(P_{ab}n^b)^\top$, and P_{ab} is the Schouten tensor to be introduced in the next section.

2.3 Conformal Geometry

A conformal structure on a Riemannian manifold is an equivalence class of metrics $[g]$, where $g \sim \hat{g}$ if

$$\hat{g} = \Omega^2 g ,$$

and Ω is some smooth, nonvanishing function on $\mathcal{M}$. A naïve approach to studying conformal geometry is to simply calculate how the tensors introduced earlier transform under this relation. While this is not the main approach we will take in this thesis, it is useful to list some of these transformation laws.

Our first example is the Riemann tensor. To give this transformation law, however, it is easier to first decompose $R_{ab}{}^c{}_d$ into its trace-free and pure trace components:

$$R_{ab}{}^c{}_d = W_{ab}{}^c{}_d + g^c_a P_{bd} - g_{ad}P^c_b - g^c_b P_{ad} + g_{bd}P^c_a , \tag{2.10}$$

where $W_{ab}{}^c{}_d$ and P_{ab} are the *Weyl tensor* and *Schouten tensor*, respectively. The former is the completely trace-free part of $R_{ab}{}^c{}_d$, which vanishes identically in $d < 4$. In fact, it turns out that $W_{ab}{}^c{}_d$ is conformally invariant, which immediately tells us an interesting fact: if g is a conformally flat metric (i.e., there is some $\hat{g}$ conformal to g such that $\hat{g}$ has vanishing Riemann tensor), then the Weyl tensor corresponding to g vanishes, for $d \geq 4$. In this sense, the Weyl tensor is the obstruction to g being conformally flat in dimensions greater than 3.

The Schouten tensor P_{ab} appearing in Equation (2.10) is a symmetric, trace-adjusted Ricci tensor, defined in $d \geq 3$ by

$$P_{ab} := \frac{1}{d-2}\left(R_{ab} - \frac{R}{2(d-1)}g_{ab}\right) .$$

Denoting the trace of P by J, we find

$$J = \frac{R}{2(d-1)} ,$$

so that the definition above can be rephrased as:

$$R_{ab} = (d-2)P_{ab} + Jg_{ab} .$$

The Schouten tensor does transform as simply as the Weyl tensor, but from the above we see that its trace is proportional to the scalar curvature R, the subject of the Yamabe problem. As a result, we will need its transformation:

$$P_{ab} \mapsto P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{2}\Upsilon_c \Upsilon^c g_{ab} .$$

where $\Upsilon_a = \Omega^{-1}\nabla_a \Omega$. Tracing this result gives us the transformation law for J:

$$J \mapsto \Omega^{-2}(J - \nabla . \Upsilon - \frac{(d-2)}{2}\Upsilon^2) . \tag{2.11}$$

With these transformations written down, we turn now to a more natural approach to conformal geometry, which requires the introduction of *tractors*.

2.4 Tractor Calculus

In Riemannian geometry, the fundamental objects we study are the vectors, residing in the tangent bundle to a manifold $\mathcal{M}$. These vectors are defined in a way that respects coordinate transformations, making them convenient for descriptions of quantities defined on the manifold itself. However, vectors and tensors do not in general respect a *conformal* structure on $\mathcal{M}$, which makes them less than ideal for conformal geometry. Instead, we build our calculus around *standard tractors*, which we construct in a way that will respect the conformal structure of $\mathcal{M}$.

One approach to this construction is through a direct example: in [2] the authors build a tractor calculus in the context of finding conformally Einstein metrics, for example. Here we will take a more general approach, defining tractors through their behavior under conformal transformations. This is analogous to considering the transformations of vectors under coordinate transformations as a fundamental feature–ultimately the approach will provide a clear path to conformal invariants.

For a Riemannian manifold $\mathcal{M}$ with dimension d, the group corresponding to conformal isometries of $\mathcal{M}$ is $SO(d+1,1)$. Quantities which respect this symmetry are most naturally built from $d+2$ dimensional objects as a result. In particular, standard tractors are members of a rank $d+2$ vector bundle over $\mathcal{M}$, represented by triplets (v^+, v_b, v^-). We require that tractors transform according to a particular $SO(d+1,1)$ matrix,

$$\begin{pmatrix} v^+ \\ v_a \\ v^- \end{pmatrix} \mapsto \begin{pmatrix} \Omega & 0 & 0 \\ -\Upsilon_a & \delta_a^b & 0 \\ -\frac{1}{2\Omega}\Upsilon^2 & \frac{1}{\Omega}\Upsilon^b & \frac{1}{\Omega} \end{pmatrix} \begin{pmatrix} v^+ \\ v_b \\ v^- \end{pmatrix}, \tag{2.12}$$

under a conformal transformation. We denote these tractors[3] V^A, so that Equation (2.12) can be written:

$$V^A \mapsto U^A{}_B V^B\,, \tag{2.13}$$

where $U^A{}_B$ is the matrix in Equation (2.12).

$$h_{AB} := \begin{pmatrix} 0 & 0 & 1 \\ 0 & g^{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{2.14}$$

in the sense that

$$U^T h\, U = h\,.$$

This is directly analogous to the case of tensors in flat space, where the matrix corresponding to a change of coordinates preserves the metric. In

[3] We will use capital indices for tractors, and the corresponding lower case indices for their vector components. Note that when V^A has an index up, v_a will have its index down in the middle slot.

fact, the matrix h appearing in this formula gives a metric for tractors. In particular, given any tractors V^A and W^A, we can form the scalar

$$h_{AB}V^AW^B\,.$$

The structure of the two tractors above ensures, crucially, that this quantity is conformally invariant. This fundamental feature of tractors is what makes them so useful for conformal geometry: we can immediately deduce conformal invariance from the index structure of a tractor expression. We will use the tractor metric to define the lower index tractors: $V_B = h_{AB}V^A$.

A natural question that arises with this structure is the following: is there a covariant connection $\nabla^{\mathcal{T}}$ that respects this conformal structure, in the sense that

$$\nabla^{\mathcal{T}}V \mapsto U\,\nabla^{\mathcal{T}}V\,,$$

when $V \mapsto UV$ as in Equation (2.13)? Indeed, such a connection exists, which we will call the *tractor connection*: it is given by

$$\nabla^{\mathcal{T}}_a \begin{pmatrix} v^+ \\ v_b \\ v^- \end{pmatrix} := \begin{pmatrix} \nabla_a v^+ - v_a \\ \nabla_a v_b + P_{ab}v^+ + g_{ab}v^- \\ \nabla_a v^- - P_{ab}v^b \end{pmatrix}\,. \tag{2.15}$$

In fact, this definition also ensures that the action of $\nabla^{\mathcal{T}}$ commutes with that of the tractor metric. That is, h_{AB} is parallel with respect to this connection:

$$\nabla^{\mathcal{T}}h = 0\,.$$

This shows that the connection defined by Equation (2.15) is really the analogue to the Levi-Civita connection on $\mathcal{M}$ for a metric g.

As in Riemannian Geometry, we can construct a curvature tensor from the commutator of two tractor connections acting on a standard tractor. We denote this tensor by Ω, and call it the *tractor curvature*:

$$[\nabla^{\mathcal{T}}_a, \nabla^{\mathcal{T}}_b]I^C = \Omega_{ab}{}^C{}_D\, I^D\,,$$

and Ω is represented as a matrix in its tractor indices by

$$\Omega_{ab}{}^C{}_D = \begin{pmatrix} 0 & 0 & 0 \\ 2\nabla_{[a}P_{b]}{}^c & W_{ab}{}^c{}_d & 0 \\ 0 & -2\nabla_{[a}P_{b]d} & 0 \end{pmatrix}. \tag{2.16}$$

The curl of the Schouten tensor appearing in Equation (2.16) is called the *Cotton tensor*, C_{abc}:

$$C_{abc} = \nabla_{[a} P_{b]c}\,.$$

This is an interesting tensor in its own right: since the Weyl tensor vanishes for $d = 3$, the tractor transformation rules imply that C_{abc} is a conformal invariant. Thus, in $d = 3$ it replaces W as the conformally flat obstruction: Weyl is identically zero, so a conformally flat metric is given by one with zero Cotton tensor instead.

The similarity between the behavior of the tractor connection and the Levi-Civita connection in Riemannian geometry lead us at this stage to generally drop the superscript $\mathcal{T}$ from the tractor connections. Instead, the type of connection required will be inferred from the object being acted on. This convention extends to the commutator of two connections, which we define as an operator:

$$[\nabla_a, \nabla_b] =: R^{\#}_{ab}\,.$$

In this sense, $R^{\#}_{ab}$ will be called the *total curvature*: it acts on both tensor and tractor indices, according to the appropriate curvature operator, and obeys the usual rules from differential geometry for objects with multiple indices.

So far we have only described operations on tractors and described their structure. We now provide a few crucial examples of tractors. The first is a very simple tractor called the *canonical tractor* X^A:

$$X^A = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \qquad (2.17)$$

From its definition it is clear that X^A should be given by the same expression, even after a conformal transformation. From the transformation law (2.12),

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ \Omega^{-1} \end{pmatrix}.$$

So to ensure that Equation (2.17) remains valid after the transformation, we must have

$$X^A \mapsto \Omega\, U^A{}_B X^B\,.$$

This tells us that the canonical tractor X has weight $w = 1$.

Another method to produce tractors is to begin with a density f of weight w, and to operate with the following differential operator (called the *Thomas D-operator* [2]):

$$D^A f = \begin{pmatrix} (d+2w-2)w \\ (d+2w-2)\nabla_a \\ -(\Delta + wJ) \end{pmatrix} f\,. \tag{2.18}$$

In fact, this definition is valid for tractors as well, by replacing the Levi-Civita connection with the tractor connection $\nabla^{\mathcal{T}}$. In each case, the Thomas D-operator maps objects with weight w to tractors (with one extra tractor index) of weight $w-1$. Using the transformations $f \mapsto \Omega^w f$ and Equation (2.11) for J, a short calculation shows that $D^A f$ indeed satisfies Equation (2.12), with an extra factor of Ω^{w-1}:

$$D^A f \mapsto \Omega^{w-1} U^A{}_B D^B f\,.$$

A useful extension of the Thomas D-operator is to simply remove the factor of $(d+2w-2)$ in cases where it is not zero. In that case, we have the modified Thomas D-operator $\widehat{D}$:

$$\widehat{D}^A f := D^A \frac{1}{d+2w-2} f = \begin{pmatrix} w \\ \nabla_a \\ \frac{-(\Delta+wJ)}{d+2w-2} \end{pmatrix} f\,. \tag{2.19}$$

In fact, this modified Thomas D-operator becomes quite useful when acting on products: since weights add for products, the weight operator is a Leibniz operator, but its square is not. By removing the extra factor of w from the definition, the first two slots of $\widehat{D}$ contain Leibnizian operators, and the third slot obeys a fairly simple rule as well:

$$\widehat{D}^A(fg) = (\widehat{D}^A f)g + f(\widehat{D}^A g) - \frac{2}{d+2w_f+2w_g-2} X^A (\widehat{D}^B f)(\widehat{D}_B g)\,, \tag{2.20}$$

where X^A is given by Equation (2.17), and w_f, w_g are the weights of f and g, respectively. We see that the failure of the modified Thomas D-operator to obey a Leibnizian rule is proportional only to the bottom slot of the tractor. When each of these terms is defined (*i.e.*, for the proper weights), Equation (2.20) can simplify calculations considerably.

The Thomas D-operator has some additional useful properties that we will state without proof for later use; the square of the Thomas D-operator is identically 0:

$$D_A D^A \equiv 0 \ ,$$

and two Thomas D-operators commute on scalar functions:

$$D_A D_B f = D_B D_A f \ .$$

Finally, the action of the modified Thomas D-operator on the canonical tractor X gives the tractor metric:

$$\widehat{D}_A X_B = h_{AB} \ .$$

Note that we can define tractors for both the bulk space $\mathcal{M}$ and the hypersurface Σ. We denote the standard tractor bundles by $\mathcal{TM}$ and $\mathcal{T}\Sigma$ respectively. We will see a relationship between sections of these bundles in the next chapter, analogous to Equation (2.4), once we have developed an analog to the normal vector.

Chapter 3

Conformal Hypersurfaces

We now consider embeddings defined by a defining density σ. Since σ is also a defining function as described in Chapter 2, we can define the normal vector to Σ by $n_a = \nabla_a \sigma$. Since we have a conformal scale defining our embedding, however, we have more than just the classical hypersurface results: we are now in a position to study conformal hypersurface invariants. To accomplish this, we will proceed in a similar manner to the approach to hypersurface geometry in Chapter 2.

Given the importance in Riemannian geometry of the normal vector, it is natural to attempt to build this vector into a tractor. One option is to define a new tractor, I^A, from the Thomas D-operator, analogous to a derivative of σ. We call this the *scale tractor*:

$$I^A := \widehat{D}^A \sigma = \begin{pmatrix} \sigma \\ \nabla_a \sigma \\ -\frac{\Delta\sigma + J\sigma}{d} \end{pmatrix} =: \begin{pmatrix} \sigma \\ n_a \\ \rho \end{pmatrix} ,$$

where we have used the fact that σ is a weight $w = 1$ density. This definition is valid everywhere in $\mathcal{M}$, but it is especially useful along Σ, since the top slot of I vanishes there. In particular, this shows that along Σ, n is a conformally invariant vector.

The conformal invariance of n indicates that there is another natural approach to building a tractor with the normal vector as its middle component: we can build a tractor directly along Σ without use of σ (which we will call the *normal tractor*) by:

$$N^A = \begin{pmatrix} 0 \\ n_a \\ -H \end{pmatrix} .$$

It is not too difficult to show that N^A obeys the tractor transformation law, Equation (2.13), along Σ. We shall see that there is a connection between I and N later, but for the moment we can define tangential tractors using just the latter, in the same way as tangential tensors: we say a tractor V^A is *tangential* if $N_A V^A \stackrel{\Sigma}{=} 0$, and similarly for objects with more than one tractor index.

Once again, we can construct a projector which acts on any tractor and produces the tangential piece. This is the tractor version of the first fundamental form, Σ^{AB}, defined by $\Sigma^{AB} \stackrel{\Sigma}{=} h^{AB} - N^A N^B$, which allows us to construct the tangential part of $\mathcal{TM}$, which we denote $(\mathcal{TM})^\top$. As in Equation (2.4), we have

$$(\mathcal{TM})^\top \cong \mathcal{T}\Sigma\,.$$

When a canonical extension of N is needed, we can try to use the scale tractor I. In fact, the condition for this extension to be valid is that $I^2 \stackrel{\Sigma}{=} 1$, which is analogous to asking that the normal vector be a unit vector along the hypersurface in Riemannian geometry. In particular, with this extension we can construct an analog to the second fundamental form:

$$P^{AB} := \widehat{D}^A I^B\,.$$

Note that P^{AB} is trace-free and symmetric, since two D operators commute on scalars, and $D^2 = 0$.

With the normal vector and its canonical extension, we can also form a conformally invariant analog to a normal derivative:

$$\begin{aligned} I.\widehat{D} &= \sigma\left(-\frac{1}{d+2w-2}(\Delta + wJ)\right) + n^a\nabla_a\sigma + \rho w \\ &= \nabla_n + \rho w - \frac{\sigma}{d+2w-2}(\Delta + wJ)\,. \end{aligned}$$

The leading behavior near Σ is on its own a conformally invariant version of the normal derivative called the *Robin* operator δ_R. The operator is given in our calculations by:

$$\delta_R = \nabla_n - wH\,,$$

which follows from $\rho \stackrel{\Sigma}{=} -H$, which we will show in the next chapter (Equation (4.1)). Near Σ, the Robin operator δ_R dominates, since the Laplacian term suppressed by the power of σ. However, away from Σ, we see that the leading derivate term becomes a Laplacian operator. For this reason, the operator $I.\widehat{D}$ is known as a *Laplace-Robin operator*. We stress once again that this operator can act on tractors as well as on densities, using the appropriate connection in each case.

Now we are ready to tackle the Yamabe problem. As described in Chapter 1,we want to find a scale σ such that $\hat{g} = \sigma^{-2}g$ has constant scalar curvature. Since they are proportional, this is equivalent to the condition that J is a constant. Using Equation (2.11) and noting that $\Omega = \sigma^{-1}$, we find

$$J \mapsto \sigma^2 J + \sigma\Delta\sigma - \frac{d}{2}n^2 \,. \tag{3.1}$$

The Yamabe problem requires the combination on the right hand side to be a constant. Consider, however, the square of the scale tractor,

$$I^2 = h_{AB}I^A I^B = -2\frac{\sigma\Delta\sigma + J\sigma^2}{d} + n^2 \,.$$

This is precisely Equation (3.1), up to a constant. That is,solution to the Yamabe problem is equivalent to the statement that I^2 is constant! Now, if we require that $n^2 = 1$ along the hypersurface, then $I^2 \stackrel{\Sigma}{=} 1$. In that case, the problem is reduced to solving $I^2 = 1$. In practice, we will try to solve $I^2 = 1$ order by order in σ, starting with $I^2 \stackrel{\Sigma}{=} 1$. Of course, there is no guarantee that any σ exists that satisfies this condition through all orders, but in fact the failure of a solution to exist is interesting in its own right.

Chapter 4

Obstruction Densities

To solve the equation $I^2 = 1$, we are really looking for a new scale $\bar{\sigma}$ in terms of a given defining density σ, for which $I^2(\bar{\sigma}) = 1$. As a first step, we will note that given a defining density σ with nowhere vanishing normal vector n_a, it is always possible to solve for a new density $\hat{\sigma}$ which satisfies $(\nabla\hat{\sigma})^2 \stackrel{\Sigma}{=} 1$ (see [5]). This is an important result, as it automatically gives a scale for which $I^2 \stackrel{\Sigma}{=} 1$. We take this as our initial defining density.

The most we can assume about this scale is that $I^2 = 1 + \sigma A_1$, for some function A_1, since we know nothing about how it extends off Σ. The next step is to build a new density which has $I^2 = 1 + \sigma^2 A_2$, if possible, and to repeat this process order by order until all A_k have vanished. For a scale σ with $I^2 = 1 + \sigma^k A_k$, we will try to accomplish this by writing the new scale $\bar{\sigma}$ in terms of σ as:

$$\bar{\sigma} = \sigma + \sigma^{k+1} f_k \, .$$

In fact, with this definition, it is possible to solve the equation

$$\bar{I}^2 = 1 + \sigma^{k+1} A_{k+1} \, ,$$

for f_k, given $k \neq d$ ([5]), where d is the dimension of $\mathcal{M}$. This result gives a recursive formula that provides a scale which satisfies $I^2 = 1$ up to order σ^d. However, moving past this dimension is not so easy. In fact, once we reach this critical order, no smooth change in scale,

$$\bar{\sigma} = \sigma \left(1 + \sigma f_1 + \sigma^2 f_2 + \ldots\right) \, ,$$

will affect this $\mathcal{O}(\sigma^d)$ term (see [5]). There is an interesting twist on this result, however: no matter the scale we started with, performing this recursive process will always lead to the *same* scale $\hat{\sigma}$ satisfying

$$I_{\hat{\sigma}}^2 = 1 + \sigma^d \mathcal{B}\,,$$

where $\mathcal{B}$ is a density which obstructs a solution to the Yamabe problem (which we thus call the *obstruction density*). The scale $\hat{\sigma}$ we find in this manner is called the *conformal unit scale*. The above illustrates that this scale is actually uniquely determined, and in the remainder of this work, we will work with this scale, denoting it simply σ.

This choice of scale leads to a number of simplified formulæ, in the same way that a unit scale in classical hypersurface geometry simplifies many results there. First of all, we can consider the bottom slot of the scale tractor I:

$$\begin{aligned} -\frac{1}{d}(\Delta\sigma + J\sigma) &\stackrel{\Sigma}{=} -\frac{1}{d}\nabla^a n_a \stackrel{\Sigma}{=} -\frac{1}{d}(\nabla^\top.\hat{n} + n^a\nabla_n n_a) \\ &\stackrel{\Sigma}{=} -\frac{1}{d}(nH + \frac{1}{2}\nabla_n(1-2\rho\sigma)) \stackrel{\Sigma}{=} -\frac{1}{d}(nH - \rho)\,, \end{aligned}$$

where we have used $n^2 = 1 - 2\rho\sigma + \sigma^d\mathcal{B}$ and the fact that $\nabla^\top\sigma \stackrel{\Sigma}{=} 0$. Solving this equation for ρ along Σ, we find:

$$\rho \stackrel{\Sigma}{=} -H\,. \tag{4.1}$$

This result tells us that $I^A \stackrel{\Sigma}{=} N^A$, so the scale tractor indeed acts as a natural extension of the normal tractor in this scale.

With this scale in mind, we can use another method to actually calculate $\mathcal{B}$: by applying the $I.D$ operator n times to the scale tractor I, and using Equation (2.20), it was shown in [5] that:

$$(-1)^n\frac{n!(n+1)!}{2}\mathcal{B} = \bar{D}_A[\Sigma^A_B(I.D^n I^B - I.D^{n-1}[X^B K])]\big|_\Sigma\,. \tag{4.2}$$

where $K := P_{AB}P^{AB}$. This formula provides a method of calculating $\mathcal{B}$: we compute primarily normal derivatives of two extrinsic tractors, yet the use of $\bar{D}$ and Σ show that the result is intrinsic to the surface. In fact, the operator $I.D^n$ can be replaced by a tangential operator P_n when acting on tractors of the appropriate weight. Using these ideas, we can now use this formula to explicitly compute the obstruction density for surfaces ($n = 2$) and volumes ($n = 3$).

4.1 Surfaces

In the case $n = 2$, Equation (4.2) reduces to:

$$\mathcal{B} = \frac{1}{6}\bar{D}_A[\Sigma^A_B(P_2 N^B - [I.D(X^B K)])]\big|_\Sigma\,, \tag{4.3}$$

where $P_2 \overset{\Sigma}{=} I.D^2$ is a conformally invariant Yamabe operator defined on Σ. Explicitly, the formula for P_2 is quite simple when acting on tractors of the appropriate weight (see [5]):

$$P_2 N^C \overset{\Sigma}{=} -\Delta^\top I^C\,,$$

where $\Delta^\top := g^{ab}\nabla^\top_a \nabla^\top_b$. We stress that since the derivatives appearing above are tangential derivatives, we are allowed to extend N in any way we wish, as described in Chapter 2. Here, we have chosen the scale tractor as our extension, which is justified by the result in Equation (4.1).

The calculation reduces to two important terms: a tangential differential operator acting on the normal tractor, and a normal derivative on $X^B K$. We begin with the calculation of the first term, using the result above for P_2. In particular, from Equation (2.15):

$$\begin{aligned}\nabla^\top_a I^B \overset{\Sigma}{=} \gamma^c_a \nabla_c I^B &\overset{\Sigma}{=} \gamma^c_a \begin{pmatrix} 0 \\ \nabla_c n_b + \sigma P_{ab} + g_{cb}\rho \\ \nabla_c \rho - P_{cb} n^b \end{pmatrix} \\ &\overset{\Sigma}{=} \begin{pmatrix} 0 \\ \mathrm{II}_{ab} - H\gamma_{ab} \\ \nabla^\top_a \rho - P(a,n)^\top \end{pmatrix}.\end{aligned}$$

The middle slot of this tractor is simply $\mathring{\mathrm{II}}$, and the bottom slot can be calculated: since $\rho \overset{\Sigma}{=} -H$ and ρ is a scalar,

$$\nabla^\top_a \rho \overset{\Sigma}{=} -\bar{\nabla}_a H \overset{\Sigma}{=} -\frac{1}{n-1}\bar{\nabla}.\mathring{\mathrm{II}}_a + P(a,n)^\top\,,$$

where we have used Equation (2.9) in the last step. Plugging this in, we find:

$$\nabla^\top_a I^B \overset{\Sigma}{=} \begin{pmatrix} 0 \\ \mathring{\mathrm{II}}_{ab} \\ -\frac{\bar{\nabla}.\mathring{\mathrm{II}}_a}{n-1} \end{pmatrix}. \tag{4.4}$$

Taking one more tangential derivative of this tractor, and using

$$\nabla^{\top a}\mathring{\mathrm{I\!I}}_{ab} \stackrel{\Sigma}{=} \bar{\nabla}.\mathring{\mathrm{I\!I}}_b - n_b \mathrm{I\!I}^{ac}\mathring{\mathrm{I\!I}}_{ac} \stackrel{\Sigma}{=} \bar{\nabla}.\mathring{\mathrm{I\!I}}_b - n_b \operatorname{tr}\mathring{\mathrm{I\!I}}^2\,,$$

we find:

$$\Delta^\top I^B \stackrel{\Sigma}{=} \begin{pmatrix} 0 \\ \bar{\nabla}.\mathring{\mathrm{I\!I}}_b - n_b \operatorname{tr}\mathring{\mathrm{I\!I}}^2 - \gamma_b^a \frac{1}{n-1}\bar{\nabla}.\mathring{\mathrm{I\!I}}_a \\ -\frac{1}{n-1}\bar{\nabla}.\bar{\nabla}.\mathring{\mathrm{I\!I}} - P^\top_{ab}\mathring{\mathrm{I\!I}}^{ab} \end{pmatrix}$$

$$\stackrel{\Sigma}{=} \begin{pmatrix} 0 \\ -n_b \operatorname{tr}\mathring{\mathrm{I\!I}}^2 \\ -\bar{\nabla}.\bar{\nabla}.\mathring{\mathrm{I\!I}} - P^\top_{ab}\mathring{\mathrm{I\!I}}^{ab} \end{pmatrix}\,.$$

The other half of the calculation requires that we compute $I.D(X^B K)$. We first write down a more explicit form of P^{AB} using the results above. Since we know it is symmetric, and that I has weight 0, our calculation becomes much simpler:

$$P^{AB} = \begin{pmatrix} wI_B \\ \nabla^{\mathcal{T}}_a I_B \\ -\frac{1}{d-2}(-\Delta^{\mathcal{T}} I_B - wJI_B) \end{pmatrix}$$

$$\stackrel{\Sigma}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathring{\mathrm{I\!I}}_{ab} & -\frac{1}{n-1}\bar{\nabla}.\mathring{\mathrm{I\!I}}_b \\ 0 & -\frac{1}{n-1}\bar{\nabla}.\mathring{\mathrm{I\!I}}_a & \star \end{pmatrix}\,,$$

where we have refrained from calculating the bottom-right component of P, since it will not be needed. Tracing this expression with the tractor metric h, we find:

$$K \stackrel{\Sigma}{=} \operatorname{tr}\mathring{\mathrm{I\!I}}^2\,. \tag{4.5}$$

We note here that K has an interpretation as the Lagrangian density for Polyakov's rigid string ([9]). The variation of this quantity gives the Willmore invariant in this dimension. Returning to our calculation, it was shown in [5] that for $d=3$, $I.DK \stackrel{\Sigma}{=} 0$. So, our second term can be written:

$$\begin{aligned} I.D(X^B K) &\stackrel{\Sigma}{=} (d-4) I.\widehat{D}(X^B K) \stackrel{\Sigma}{=} (d-4)(I.\widehat{D}X^B)K \\ &\stackrel{\Sigma}{=} (d-4)K(I^B) \\ &\stackrel{\Sigma}{=} \begin{pmatrix} 0 \\ -Kn_b \\ HK \end{pmatrix} . \end{aligned}$$

Adding this result to the first term, Equation (4.1) becomes:

$$\mathcal{B} \stackrel{\Sigma}{=} \frac{1}{6} \bar{D}_A \left(X^A[-\bar{\nabla}.\bar{\nabla}.\mathring{\mathrm{I\!I}} - P^{\top}_{ab}\mathring{\mathrm{I\!I}}^{ab} - HK] \right) .$$

Now, using Equation (2.20) and the fact that the Thomas D-operator acting on X produces the metric, a short calculation shows that for functions f of weight w,

$$\bar{D}_A \left(X^A f \right) \stackrel{\Sigma}{=} (n+w)(n+2w+2)f . \tag{4.6}$$

Here, f is of weight -3, so we have:

$$\mathcal{B} \stackrel{\Sigma}{=} -\frac{1}{3} \left(\bar{\nabla}.\bar{\nabla}.\mathring{\mathrm{I\!I}} + P^{\top}_{ab}\mathring{\mathrm{I\!I}}^{ab} + HK \right) . \tag{4.7}$$

In order to put this in a more familiar form, we can use Equation (2.9) to replace the $\bar{\nabla}.\bar{\nabla}.\mathring{\mathrm{I\!I}}$ term. After a short calculation, we find:

$$\mathcal{B} = -\frac{1}{3} \left(\bar{\Delta} H + 2H(H^2 - K) \right) ,$$

which is proportional to the Willmore invariant quoted in Equation (1.2).

4.2 Volumes

The calculation for volumes is significantly more involved than the surface calculation done above, but in principle it follows the same procedure. We start with Equation (4.2), which now reduces to:

$$\mathcal{B} \stackrel{\Sigma}{=} -\frac{1}{72} \bar{D}_A [\Sigma^A_B (I.D^3 I^B - I.D^2[X^B K])] . \tag{4.8}$$

As in the $n=2$ case, we can replace the $I.D^3$ term with a tangential operator P_3 (in fact, this is a general result when acting on a weight-0 object like I). As in the surface calculation, we have two main components: the tangential operator and the normal derivatives of KX^B. We begin by calculating the P_3 part: from [5], the form of P_3 is

$$\begin{aligned} P_3 N^B \overset{\Sigma}{=} &\left(-8P(n.a) + 4\nabla^\top_c \mathring{\mathrm{II}}^c_a + 8\nabla^\top_a H\right) \nabla^{\top a} I^B \\ &+ 8\mathring{\mathrm{II}}^{ac}\nabla^\top_a \nabla^\top_c I^B - 8n^c R^{\#}_{ca}\nabla^\top_a I^B\,, \end{aligned} \tag{4.9}$$

where the $R^{\#}_{ca}$ indicates the total curvature operator described in Chapter 2. We have already computed the tractor in the first and third terms in Equation (4.4). Taking an additional tangential derivative gives

$$\nabla^\top_a \nabla^\top_c I^B \overset{\Sigma}{=} \begin{pmatrix} -\mathring{\mathrm{II}}_{ac} \\ \nabla^\top_a \mathring{\mathrm{II}}^b_c - \frac{1}{d-2}\gamma^b_a \bar{\nabla}.\mathring{\mathrm{II}}_c \\ -\mathring{\mathrm{II}}^b_c P^\top_{ab} - \frac{1}{d-2}\nabla^\top_a \bar{\nabla}.\mathring{\mathrm{II}}_c \end{pmatrix}.$$

The third term of Equation (4.9) can be calculated using the tractor curvature Ω in Equation (2.16). The result is:

$$-8n^c R^{\#}_{ac}\nabla^\top_a I^B \overset{\Sigma}{=} \begin{pmatrix} 0 \\ -8n^a W_a{}^{cb}{}_d \mathring{\mathrm{II}}^d_c + 8n^a R_a{}^d \mathring{\mathrm{II}}^b_d \\ -8\mathring{\mathrm{II}}^d_c \nabla_n P^c{}_d + 8\mathring{\mathrm{II}}^{cd} n^a \nabla^\top_c P_{ad} - 4n^a R_a{}^d \bar{\nabla}.\mathring{\mathrm{II}}_d \end{pmatrix}.$$

To simplify these expressions, several identities are needed. We make use of Equations (2.5) through (2.9) primarily, as well as decompositions of these quantities into tangential and normal pieces. After some calculation, we arrive at the result,

$$(I.D)^3 I^B \overset{\Sigma}{=} \begin{pmatrix} -8K \\ 4\nabla_b K \\ -8P^\top.\mathring{\mathrm{II}}^2 - 4\mathring{\mathrm{II}}^{ac}\nabla^\top_a \bar{\nabla}.\mathring{\mathrm{II}}_c - 4(\bar{\nabla}.\mathring{\mathrm{II}})^2 + 8\mathring{\mathrm{II}}^{ab} n^c C_{bca} \end{pmatrix}. \tag{4.10}$$

To handle the second term in Equation (4.8), we first note that $X^B K$ has weight -1, so that

$$\begin{aligned} I.D(X^B K) &= \big[(d-4)\,(\nabla_n + H) - \sigma(\Delta - J)\big](X^B K) \\ &= -\sigma(\Delta - J)(X^B K)\,. \end{aligned}$$

Acting with $I.D$ lowers the weight by 1, however, so the full expression along Σ becomes:

$$\begin{aligned} I.D^2(X^B K) &\stackrel{\Sigma}{=} -2(\nabla_n + 2H)(-\sigma(\Delta - J))(X^B K) \\ &\stackrel{\Sigma}{=} 2(\Delta - J)(X^B K) \\ &\stackrel{\Sigma}{=} 2(\Delta^{\mathcal{T}} X^B)K + 2X^B(\Delta K) + 4(\nabla^{\mathcal{T}}_a X^B)(\nabla^a K) - 2JKX^B \, . \end{aligned}$$

Using Equation (2.15), we can calculate:

$$\nabla^{\mathcal{T}}_a X^B \stackrel{\Sigma}{=} \begin{pmatrix} 0 \\ g_{ab} \\ 0 \end{pmatrix} \implies \Delta^{\mathcal{T}} X^B \stackrel{\Sigma}{=} \begin{pmatrix} -d \\ 0 \\ -J \end{pmatrix} ,$$

and so:

$$I.D^2(X^B K) \stackrel{\Sigma}{=} \begin{pmatrix} -8K \\ 4\nabla_b K \\ 2\Delta K - 4JK \end{pmatrix} .$$

Remembering that the weight of K is -2, this can be written more simply as:

$$I.D^2(X^B K) \stackrel{\Sigma}{=} -2D^B K \, . \tag{4.11}$$

To calculate this term, we need to compute normal derivatives of K. In particular, since ∇_n is not extension independent, we will need to be very careful about the extension we choose. In fact, in the derivation of our formula in Equation (4.2), we have implicitly used the middle slot of P^{AB} as our extension for $\mathring{\mathrm{I\!I}}$. From Equation (2.15), the middle slot is given by:

$$\mathring{\mathrm{I\!I}}_{ab} := \nabla_a n_b + \rho g_{ab} + \sigma P_{ab} \, . \tag{4.12}$$

To compute normal derivatives of this tensor, we will first need several identities. This calculation becomes fairly involved; we prove some of these identities here to illustrate the procedure. Our first example is the normal derivative of n:

$$\begin{aligned} \nabla_n n_b = n^c \nabla_c n_b &= n^c \nabla_c \nabla_b \sigma \\ &= n^c \nabla_b n_c = \frac{1}{2}\nabla_b(n^2) \\ &= \frac{1}{2}\nabla_b(1 - 2\rho\sigma + \sigma^d \mathcal{B}) \\ &= -\rho n_b - \sigma(\nabla_b \rho) + \mathcal{O}(\sigma^{d-1}) \, . \end{aligned}$$

Note that we are free to ignore terms of order σ^{d-2} after one normal derivative, since we can at most only remove two factors of σ total in this calculation. As a result, we will generally drop the extra terms proportional to $\mathcal{B}$ in the calculations that follow. Another important normal derivative is

$$\begin{aligned}\nabla_n \nabla_a n_b &= [\nabla_n, \nabla_a] n_b + \nabla_a(\nabla_n n_b) \\ &= n^c[\nabla_c, \nabla_a] n_b + [n^c, \nabla_a]\nabla_c n_b - \nabla_a(-\rho n_b - \sigma \nabla_b \rho) \\ &= n^c R_{cab}{}^d n_d - (\nabla_a n^c)(\nabla_c n_b) - \rho(\nabla_a n_b) \\ &\quad - 2\nabla_{(a}\rho n_{b)} - \sigma \nabla_a \nabla_b \rho \ .\end{aligned}$$

These two examples illustrate the technique used to compute a normal derivative away from Σ. After this calculation is done, we can reduce our expression to quantities that exist only along Σ, especially to conformal invariants of the hypersurface, for a better understanding of what we have found. To accomplish this, we have some additional identities, such as

$$\nabla_n \rho \stackrel{\Sigma}{=} P(n,n) + \frac{1}{d-2} K,$$

$$\begin{aligned}(\nabla_a n_b)(\nabla^a n^b) &\stackrel{\Sigma}{=} (\mathring{\mathrm{I\!I}}_{ab} - \rho g_{ab})(\mathring{\mathrm{I\!I}}^{ab} - \rho g^{ab}) \\ &\stackrel{\Sigma}{=} K + d\rho^2 \stackrel{\Sigma}{=} K + dH^2 \ ,\end{aligned}$$

and

$$\nabla_a \rho = \nabla_a^\top \rho + n_a \nabla_n \rho \stackrel{\Sigma}{=} -\bar{\nabla} H + n_a \left(P(n,n) + \frac{1}{d-2} K \right) \ .$$

One identity in particular requires additional work to compute, so we prove it explicitly:

$$
\begin{aligned}
\nabla_n R(n,a,b,n) &= n^c \nabla_c (n^d R_{dabn}) \\
&= (\nabla_n n^d) R_{dabn} + n^c n^d \nabla_c R_{dabn} \\
&\stackrel{\Sigma}{=} (Hn^d) R_{dabn} + (g^{cd} - \gamma^{cd}) \nabla_c R_{dabn} \\
&\stackrel{\Sigma}{=} HR(n,a,b,n) + \nabla^d R_{dabn} - \nabla^{\top d} R_{dabn} \\
&\stackrel{\Sigma}{=} HR(n,a,b,n) + (\nabla^d n^c) R_{dabc} - n^c (\nabla_b R_{dac}{}^d + \nabla_c R_{da}{}^d{}_b) \\
&\quad - \nabla^{\top d} R_{dabn} \\
&\stackrel{\Sigma}{=} HR(n,a,b,n) + (\mathring{\mathrm{I\!I}}^{dc} + Hg^{dc}) R_{dabc} - n^c (-\nabla_b R_{ac} + \nabla_c R_{ab}) \\
&\quad - \nabla^{\top d} R_{dabn} \\
&\stackrel{\Sigma}{=} HR(n,a,b,n) + \mathring{\mathrm{I\!I}}^{dc} R_{dabc} - HR_{ab} - \mathring{\mathrm{I\!I}}_b^c R_{ac} - HR_{ab} \\
&\quad + \nabla_b R(a,n) - \nabla_n R_{ab} - \nabla^{\top d} R_{dabn} \ .
\end{aligned}
$$

Rearranging this, we see that we can write the combination of normal derivatives on the Riemann and Ricci tensors in terms of surface quantities. Crucially, these two terms appear in exactly the correct ratios for the volume obstruction in particular, so we eliminate all the terms containing normal derivatives of ambient curvature.

Orchestrating these identities and the work in Chapter 2, we arrive at the following expression for the bottom slot of Equation (4.11):

$$
\begin{aligned}
2\Delta K - 4JK \stackrel{\Sigma}{=}\ & 4\mathring{\mathrm{I\!I}}^{ab} \bar{\Delta} \mathring{\mathrm{I\!I}}_{ab} + 4(\bar{\nabla}_k \mathring{\mathrm{I\!I}}_{ab})^2 + 4R(n,a,b,n)^2 + 8R(n,P,n) \\
& + 6R(n, \mathring{\mathrm{I\!I}}^2, n) + 4P^\top_{ab} P^{\top ab} + 8HP.\mathring{\mathrm{I\!I}} - 8P^\top.\mathring{\mathrm{I\!I}}^2 \\
& + 12H \operatorname{tr} \mathring{\mathrm{I\!I}}^3 + 6K^2 - 4\mathring{\mathrm{I\!I}}^{ab} \nabla^\top_d R(d,a,b,n) \\
& + 2\mathring{\mathrm{I\!I}}^{ab} \nabla^\top_a \bar{\nabla}.\mathring{\mathrm{I\!I}}_b + 4\mathring{\mathrm{I\!I}}^{ab} \nabla^\top_a P(b,n)^\top + 2(\bar{\nabla}.\mathring{\mathrm{I\!I}})^2 \ .
\end{aligned}
$$

Combining this result with Equation (4.10), and using Equation (4.6) once again to extract the bottom slot (note that the top two slots cancel once again), we obtain the final result for the obstruction density:

$$\begin{aligned}\mathcal{B} \stackrel{\Sigma}{=} \frac{1}{24}\Big[&8\mathring{\mathrm{I\!I}}^{ab}\bar{\Delta}\mathring{\mathrm{I\!I}}_{ab} + 4\big(\bar{\nabla}_c\mathring{\mathrm{I\!I}}_{ab}\big)^2 + 6(\bar{\nabla}.\mathring{\mathrm{I\!I}})^2 \\ &- 8\bar{J}K + 4W(n,a,b,n)^2 - 24W(n,\mathring{\mathrm{I\!I}}^2,n) \\ &+ 4\mathring{\mathrm{I\!I}}^{ab}\mathring{\mathrm{I\!I}}^{cd}W_{cabd} + 4K^2 - 8HW(n,\mathring{\mathrm{I\!I}},n) \\ &- 8\mathring{\mathrm{I\!I}}^{ab}\nabla^{\top}_c(W(c,a,b,n)^{\top}) + 8\mathring{\mathrm{I\!I}}^{ab}n^cC_{cba}\Big]\,.\end{aligned} \tag{4.13}$$

This result can be understood in several different ways: the forthcoming work [4] examines some alternative derivations of the leading terms of Equation (4.13). In particular, the first four terms are proportional to the manifestly invariant $(\bar{D}_A L_{BC})(\bar{D}^A L^{BC})$, where L_{AB} is the intrinsic tractor corresponding to P_{AB} along Σ. The remaining terms thus also form a conformal hypersurface invariant, which in particular implies the final three terms,

$$HW(n,\mathring{\mathrm{I\!I}},n) + \mathring{\mathrm{I\!I}}^{ab}\nabla^{\top}_c(W(c,a,b,n)^{\top}) - \mathring{\mathrm{I\!I}}^{ab}n^cC_{cba}\,,$$

also must be invariant.

Another approach to produce the leading terms of $\mathcal{B}$ is to calculate the variation of the integral

$$\int_{\Sigma} dA\,\mathrm{tr}\,\mathring{\mathrm{I\!I}}^3\,.$$

(see [4]). This provides in part a generalization of the surface obstruction result, where $\mathcal{B}$ can be found by the variation of $\int_{\Sigma} dA\,\mathrm{tr}\,\mathring{\mathrm{I\!I}}^2$, the rigid string action (see [9],[1]).

Chapter 5

Conclusion

We have calculated explicitly for the first time the volume obstruction density, given in Equation (4.13), in full generality. As we have seen in the two dimensional case, this obstruction density provides a nontrivial conformal invariant independent of its origin. In principle, the same methods can be used to produce the obstruction density for higher dimensions.

There are several possible approaches to calculate the obstruction density in three and higher surface dimensions. If one only wishes to obtain the leading behavior, a flat bulk space $\mathcal{M}$ provides a more tractable calculation, removing most of the sub-leading terms (as Equation (4.13) shows). In fact, this approach also provides a way of finding yet another conformal hypersurface invariant, which is interesting in its own right.

Additionally, there are methods aside from Equation (4.2), which may prove more effective in higher dimensions. In [5] and [4] the variational approach discussed above indeed proves much more tractable for the volume obstruction, but it is only currently known that the leading structure will be produced in this manner. Still, this again leads necessarily to another conformal hypersurface invariant. This approach is interesting for physics as well: in two dimensions, the variation we compute is that of the rigid string action, for example, and in higher dimensions these variations may prove useful in computations of entanglement entropy (e.g., by [1]), where conformally invariant generalizations of the Willmore Energy are needed.

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