TOPOLOGICAL QUANTUM FIELD THEORY FROM THE VIEWPOINT OF CELLULAR GRAPHS

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Abstract. The edge-contraction axioms are introduced for topological recursion of multilinear maps that arise from cellular graphs, and it is proven that these axioms are equivalent to topological quantum field theories over a Frobenius algebra, assuming the consistency of the axioms. As an example, the concrete formulas of topological quantum field theories on the finite-dimensional abelian group algebra is computed.

Contents

1. Introduction

The purpose of this paper is to relate the moduli space $\mathcal{M}_{g,n}$ (or the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$) to topological quantum field theories over a Frobenius algebra, via the theory of cellular graphs. As a result, we obtain a graphical description of topological quantum field theory, along with a convenient tool of computing it.

The *moduli space* $\mathcal{M}_{g,n}$ is defined as the set of biholomorphic isomorphism classes of complex structures on a compact, oriented surface of genus g, with n points specified[1]. For example, it is well-known that $\mathcal{M}_{0,3}$ is a single point since the Riemann sphere $\mathbb{P}^1$ with any three marked points is biholomorphic to $\mathbb{P}^1$ with $\{0, 1, \infty\}$ specified. It is also known that $\mathcal{M}_{1,1}$ consists of the elliptic curves. Every elliptic curve is a quotient $\mathbb{C}/\Lambda$ for some lattice $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$ (hence is topologically a torus), and is uniquely determined by $\tau \in \mathbb{H} = \{z \in \mathbb{C} | \operatorname{Im} z > 0\}$, up to a fractional linear transformation $\tau \mapsto \frac{a\tau+b}{c\tau+d}$, where $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{PGL}_2(\mathbb{Z})$. That is,

$$\mathcal{M}_{1,1} = \mathbb{H}/\,\mathrm{PGL}_2(\mathbb{Z}).$$

[1] Equivalently, it is also the space of smooth algebraic curves of genus g with n marked points.

In general, the moduli space $\mathcal{M}_{g,n}$ is very mysterious; however, one may understand it with the aid of ribbon graphs. A *ribbon graph* is a graph together with a cyclic ordering on the set of half-edges incident to each vertex. We may attach oriented disks to a ribbon graph Γ in a unique way compatible with the ordering of the half-edges to obtain a compact oriented topological surface $C(\Gamma)$. A ribbon graph Γ is said to have topological type (g, n) if $C(\Gamma)$ has genus g and is obtained by attaching n disks. Moreover, if we assign a positive real number ("length") to each edge of a ribbon graph, then we obtain a *metric* ribbon graph. It is a theorem, due to Harer [8], Mumford [13], Strebel [17], Thurston and others [16], that there is an orbifold isomorphism

$$\mathcal{M}_{g,n} \times \mathbb{R}_+^n \cong RG_{g,n},$$

where $RG_{g,n}$ denotes the space of metric ribbon graphs of type (g, n). For more details, see [11]; and for topological recursion on ribbon graphs, see [3].

Because attaching 2-cells to a ribbon graph gives a surface, a ribbon graph is essentially the 1-skeleton of a cell-decomposition of a compact oriented topological surface of genus g with n 2-cells. The dual graph of a ribbon graph of type (g, n), i.e., the 1-skeleton of a closed oriented surface with n (labeled) 0-cells, is called a *cellular graph* of type (g, n).

Why do cellular graphs matter? One reason is that cellular graphs serve as an important tool of topological recursion. We consider n-linear maps on a Frobenius algebra that arise from the set $\Gamma_{g,n}$ of cellular graphs of type (g, n). On cellular graphs, one may contract edges to obtain graphs of simpler types. We impose a set of axioms, called *edge-contraction axioms* (ECA), to relate maps that arise from graphs before and after contraction. One of our goals of the paper is to explore the following conjecture due to Mulase.

Conjecture 1.1 (Graph-independence, Conjecture 3.2)**.** *The assignment of multilinear maps obeying the ECA depends only on the topological type* (g, n)*, not any particular choice of graphs.*

Our first main result is a series of evidence for graph-independence:

Theorem 1.2 (Propositions 3.3, 3.4, Theorem 4.1)**.** *For* $g = 0, 1$*, graph-independence holds for all* n *and all Frobenius algebra* A*; for the Frobenius algebra* $\mathbb{C}[G]$*, where* G *is a finite abelian group, graph-independence holds for all* (g, n)*.*

On the other hand, it is observed that two-dimensional topological quantum field theories are equivalent to commutative Frobenius algebras [9, 3.3.2]. A *topological quantum field theory* (TQFT) is by definition a quantum field theory that is *a priori* topologically invariant. Atiyah [2] gives a set of axioms, now called the *Atiyah–Segal axioms*, to describe an n-dimensional TQFT as an assignment of vector spaces $V(\Sigma)$ to $(n-1)$-manifold Σ and of linear maps to cobordism $M : V(\Sigma) \to V(\Sigma')$ [9, 1.2.23]. In categorical language, an n-dimensional TQFT is a *symmetric monoidal functor*

$$(\mathcal{B}ord_n^{or}, \amalg) \to (Vect, \otimes),$$

where $\mathcal{B}ord_n^{or}$ is the category whose objects are closed oriented $(n-1)$-manifolds, and morphisms are oriented n-dimensional bordisms [18]. However, for our purpose, we define a (two-dimensional) TQFT as a collection of multilinear maps on a Frobenius algebra, indexed by (g, n), where $g \geq 0$ and $n > 0$.

By the equivalence between the TQFT and Frobenius algebras, in principle it is possible to reconstruct the TQFT from any given Frobenius algebras. Our second main result, also due to Mulase, is that topological recursion on cellular graphs provides a method of reconstruction. More precisely,

Theorem 1.3 (Theorems 3.1 and 3.5). *Assuming graph-independence, a collection $\{\Omega_{g,n}\}$ of totally symmetric multilinear maps is a TQFT if and only if it satisfies the ECA.*

In summary, the main goal of this paper is the completion the following chain of correspondences:

$$\mathcal{M}_{g,n} \xleftrightarrow{\times\mathbb{R}_+^n,\ \sim} RG_{g,n} \xleftrightarrow{\text{dual graph}} \Gamma_{g,n} \xleftrightarrow{\text{Frob. Alg.}} \Omega_{g,n}$$

1.1. **Organization of the paper.** The paper is organized as follows. We begin in Section 2 with definitions of and introductions to the three main objects of our study—Frobenius algebras, TQFT, and cellular graphs. In Section 3, we introduce the edge-contraction axioms and discuss their consequences. In particular, we will discuss the graph-independence conjecture and prove the main theorem. In Section 4, we use the ECA to compute the TQFT defined on finite-dimensional abelian group algebras. This serves both as an example of computation of TQFT using ECA, as well as an evidence of *graph-independence* in this particular case.

1.2. **A note on research development and latest progress.** In fact, the conjecture only stayed open for a very short time; around the completion of this paper, Mulase gives a proof in their paper [6] to appear. Unfortunately, we have not been able to record the proof here, and we would not grant graph-independence a theorem status in this paper.

Along the development of the theory, immediately obtained were the relation between TQFT and ECA assuming graph-independence, as well as graph-independence in the cases $(0, n)$ and $(1, 1)$. Indeed, graph-independence were once believed false, and in hope of finding a counterexample for graph-independence in $(1, 2)$ case, the author computed concrete values of assigned maps using ECA reduction for the non-abelian group algebras $Z(\mathbb{C}[G])$, where G takes S_3, D_4, and $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$, but surprisingly found the conjecture to be hold in all three cases. This was the turning point of the development of the full theory, and for this reason, we include the concrete computations in Appendix B.

The author humbly remarks that, in addition to the three examples above, the $(1, n)$ graph-independence (Proposition 3.4) and computation of the TQFT on abelian group algebras (Proposition 4.1) were also completed independently by the author.

Acknowledgement. I take this opportunity to express my deepest gratitude to my research advisor, Professor Motohico Mulase, for his mathematical mentoring and academic advice. I also thank him for teaching me this subject, and for his guidance and patience in my preparation of this thesis; without him, this thesis would be impossible. I am also grateful to Motohico Mulase and Olivia Dumitrescu for the access to the working draft of their paper, which has been a great source of reference. Lastly, I would also like to thank my mathematics teachers: Professor Greg Kuperberg, Professor Dmitry Fuchs, and Professor Anne Schilling, among others.

2. Background

In this section, we give precise definitions of the objects of study.

2.1. **Frobenius algebra.**

Definition 2.1. A *Frobenius algebra* is a finite-dimensional associative unital algebra A over a field K, equipped with a non-degenerate symmetric bilinear form $\eta : A \times A \to K$, called the *Frobenius form*, which satisfies

$$\eta(v_1, v_2v_3) = \eta(v_1v_2, v_3)$$

for all $v_1, v_2, v_3 \in A$.

Remark 2.2.

- We define the *co-unit* (or *trace*) $\epsilon : A \to K$ by
$$\epsilon(v) = \eta(1, v) = \eta(v, 1).$$
Clearly ϵ is a linear functional. Note that in general, $\eta(v_1, v_2) = \epsilon(v_1 v_2)$ for any $v_1, v_2 \in A$, so specifying the Frobenius form is equivalent to specifying the co-unit.[2]
- The non-degeneracy of η implies the canonical isomorphism
$$\begin{array}{cccc} \lambda & : & A & \to & A^* \\ & & v & \mapsto & \eta(v, -) \end{array} \tag{2.1}$$
- We define the co-multiplication $\delta : A \to A \otimes A$ by requiring the diagram
$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[d, "\lambda"', "\wr"] & A \otimes A \arrow[d, "\wr"', "\lambda \otimes \lambda"] \\ A^* \arrow[r, "\mu^*"] & A^* \otimes A^* \end{tikzcd}$$
to commute. Here $\mu : A \otimes A \to A$ denotes the multiplication map in A.
- We introduce a basis $\langle e_1, e_2, ..., e_r\rangle$ of A with $e_1 = 1$. For this basis, let $\eta_{ij} = \eta(e_i, e_j)$, and $[\eta^{ij}] = [\eta_{ij}]^{-1}$. Then
$$\epsilon(e_a) = \eta(1, e_a) = \eta_{1a},$$
and
$$\delta(e_a) = \sum_{b,c,i,j} \eta^{bi}\eta^{cj}\eta(e_a, e_b e_c) e_i \otimes e_j. \tag{2.2}$$
The last identity follows from the formulas
$$\mu^* \circ \lambda(v_1, v_2) = \eta(v_1 v_2)$$
for any $v_1, v_2 \in A$, and
$$\lambda^{-1}(\phi) = \sum_{b,i} \eta^{bi}\phi(e_b)e_i$$
for any $\phi \in A^*$.

Example 2.1. Any matrix algebra $A = \mathrm{Mat}_n(K)$ over a field K is a Frobenius algebra with co-unit the usual trace of matrices (this justifies the alternative name "trace" for ϵ).

Example 2.2. $A = K[G]$, where K is any field and G is any finite group, is a Frobenius algebra, with co-unit
$$\epsilon\left(\sum_g c_g g\right) = c_1$$
[10, Ex. (3.15 E)]. $K[G]$ is commutative if and only if G is abelian. In general, the center $Z(K[G])$ of the group algebra is always a commutative Frobenius algebra.

[2] The term "Frobenius form" is sometimes referred to the linear functional ϵ rather than the bilinear form η.

Example 2.3. Let X be a closed oriented manifold of dimensional n, and let $H^*(X) = \bigoplus_{q=0}^{n} H^q(X)$ denote the de Rham cohomology on X, which together with the wedge product forms an algebra over $\mathbb{R}$. It is a Frobenius algebra with Frobenius form

$$\int : \begin{array}{ccc} H^*(X) \times H^*(X) & \to & \mathbb{R} \\ (\alpha, \beta) & \mapsto & \displaystyle\int_X \alpha \wedge \beta \end{array}$$

The Poincaré duality implies the Frobenius form is non-degenerate, for it states the duality between $H^i(X)$ and $H^{n-i}(X)$.

Technically speaking, this is not exactly a Frobenius algebra according to our definition. For the Frobenius form is not symmetric, but rather *graded*-symmetric, since the algebra is itself graded-commutative, that is, if $\alpha \in H^p(X)$ and $\beta \in H^q(X)$, then

$$\beta \wedge \alpha = (-1)^{pq} \alpha \wedge \beta.$$

Nonetheless, it is worth mentioning because of its relation to *cohomological field theory* (CohFT).

As we are only interested in commutative case, in the sequel the term "Frobenius algebra" shall always mean a *commutative* Frobenius algebra. The main examples of Frobenius algebra considered in this paper are abelian group algebras over $\mathbb{C}$.

2.2. **Topological quantum field theory.** For the purpose of this paper, we define TQFT as follows:

Definition 2.3. Let A be a Frobenius algebra over a field K. A TQFT is a collection of totally symmetric linear maps $\Omega_{g,n} : A^{\otimes n} \to K$ indexed by (g, n) (where $g \geq 0$ and $n > 0$) that satisfies

(1) $\Omega_{0,3}(v_1, v_2, v_3) = \eta(v_1, v_2 v_3) = \eta(v_1 v_2, v_3)$;
(2) $\Omega_{g,n+1}(v_1, ..., v_n, 1) = \Omega_{g,n}(v_1, ..., v_n)$;
(3) $\Omega_{g,n}(v_1, ..., v_n) = \sum_{a,b} \eta^{ab} \Omega_{g-1,n+2}(e_a, e_b, v_1, ..., v_n)$;
(4) $\Omega_{g,n}(v_1, ..., v_n) = \sum_{a,b} \eta^{ab} \Omega_{g_1,|I|+1}(e_a, v_I) \Omega_{g_2,|J|+1}(e_b, v_J)$ whenever $I \sqcup J = \{1, ..., n\}$; (Here, as well as in the sequel, $v_I = \bigotimes_{i \in I} v_i$.)

for all $v_i \in A$.

We remark that parts (2)–(4) in the definition above correspond to the *natural morphisms* between moduli spaces $\overline{\mathcal{M}}_{g,n}$ of *stable* algebraic curves of genus g with n nonsingular marked points [19]. Specifically, (2) corresponds to the *forgetful morphism*

$$\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$$

given by forgetting the last marked point; (3) and (4) correspond to the *gluing morphisms*

$$\overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n},$$
$$\overline{\mathcal{M}}_{g_1,n_1} \times \overline{\mathcal{M}}_{g_1,n_2} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2},$$

where the first map glues the first and the last marked points of the curve, and the second maps glues the two curves at their first marked points. For more discussion about the moduli space $\overline{\mathcal{M}}_{g,n}$ and Gromov-Witten theory, we refer to [19]. For the equivalence between two-dimensional TQFT and Frobenius algebras, we refer to [1, 4, 5, 14, 15]. For more discussion of other relations between the TQFT and Frobenius algebras, we refer to [9] and [18].

2.3. Cellular graphs.

Definition 2.4. A *cellular graph* is a 1-skeleton of a cell-decomposition of an oriented, connected surface of genus g with labeled *vertices*, or 0-cells. At each vertex, an outgoing arrow is attached to an incident half-edge. A cellular graph with genus g and n vertices is said to have type (g, n), and the set of all cellular graphs of type (g, n) is denoted by $\Gamma_{g,n}$.

For fixed (g, n), cellular graphs are characterized by the degrees of the vertices, namely

$$\Gamma_{g,n} = \bigsqcup_{(\mu_1,...,\mu_n)\in\mathbb{Z}_+^n} \Gamma_{g,n}(\mu_1, ..., \mu_n),$$

where $\Gamma_{g,n}(\mu_1, ..., \mu_n)$ denotes the subset of $\Gamma_{g,n}$ that consists of graphs whose Vertex i has degree μ_i $(0 \leq i \leq n)$. The number of cellular graphs also contains rich combinatorial information; for instance, it is showed in [7] that

$$|\Gamma_{0,1}(2m)| = \frac{1}{m+1}\binom{2m}{m} = C_m$$

is the n-th Catalan number. For more details of cellular graphs and generalization of Catalan numbers, we refer to [7, 12, 20].

3. Edge-contraction axioms

In this section, we introduce the edge-contraction axioms, and explore its relations with the TQFT.

For a Frobenius algebra A over K, we consider assignment of multilinear maps to each $\gamma \in \Gamma_{g,n}$, on which we impose the following axioms, called the *edge-contraction axioms* (ECA). By abuse of notation, we also denote by γ the map assigned to the graph γ. (For simplicity, we assume that the assigned maps are *totally symmetric*, although this is in fact a consequence of the ECA.)

ECA 0: If $\gamma \in \Gamma_{0,1}$ consists of a single vertex and no edges, then

$$\gamma(v) = \epsilon(v) \tag{3.1}$$

for any $v \in A$.

ECA 1: If the arrowed edge at Vertex 1 of γ connects to Vertex j, where $j \neq 1$, then

$$\gamma(v_1, ..., v_n) = \gamma^j(v_1 v_j, v_2, ..., \widehat{v}_j, ..., v_n) \tag{3.2}$$

for any $v_i \in A$, where $\gamma^j \in \Gamma_{g,n-1}$ is obtained from γ by contracting the arrowed edge at Vertex 1, as Figure 1 shows.

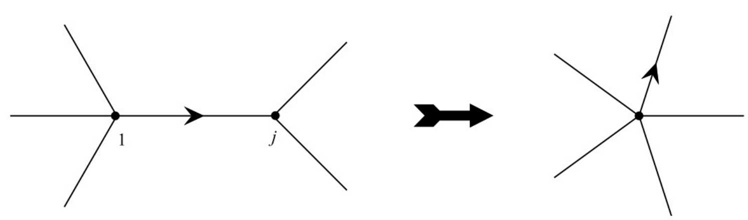

Figure 1. contracting edge

ECA 2: If the arrowed edge at Vertex 1 is a loop, then

$$\gamma(v_1, ..., v_n) = \gamma'(\delta(v_1), v_2, ..., v_n),$$

for any $v_i \in A$, where γ' is obtained from γ by contracting the loop at Vertex 1, as Figure 2 shows.

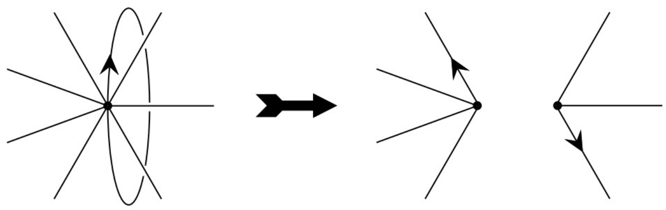

FIGURE 2. contracting loop

Note that here the coproduct $\delta(v_1)$ has two components, hence it occupies the first and the second slots of γ'.

Depending on the type of the arrowed loop in γ, in the lase case above the resulting graph γ' may be connected or disconnected. If γ' is connected, we call the loop a *loop of a handle*, and if γ' is disconnected, we call the loop a *separating loop*. More precisely, using (2.2) we may rewrite these two cases of ECA 2 as follows:

ECA 2-1: (loop of a handle) If $\gamma' \in \Gamma_{g-1,n+1}$, then

$$\gamma(v_1, ..., v_n) = \sum_{b,c,i,j} \eta^{bi}\eta^{cj}\eta(v_1, e_b e_c)\gamma'(e_i, e_j, v_2, ..., v_n); \tag{3.3}$$

ECA 2-2: (separating loop) If $\gamma' = \gamma^I \sqcup \gamma^J$, where $\gamma^I \in \Gamma_{g_1,|I|+1}$ and $\gamma^J \in \Gamma_{g_2,|J|+1}$, with $I \sqcup J = \{2, ..., n\}$, then

$$\gamma(v_1, ..., v_n) = \sum_{b,c,i,j} \eta^{bi}\eta^{cj}\eta(v_1, e_b e_c)\gamma^I(e_i, v_I)\gamma^J(e_j, v_J). \tag{3.4}$$

If $\gamma \in \Gamma_{g,n}$, we define $m(\gamma) = 2g - 2 + n$. Then contraction of edges reduces $m(\gamma)$ in most of the cases: for ECA 1, we have $m(\gamma^j) = 2g - 2 + (n - 1) = m(\gamma) - 1$; for ECA 2-1, we have $m(\gamma') = 2(g - 1) - 2 + (n + 1) = m(\gamma) - 1$; for ECA 2-2[3], unless $g_1 = 0$ and $I = \emptyset$, we have $m(\gamma^I) < m(\gamma)$, and it is similar for $m(\gamma^J)$. Moreover, if $g_1 = 0$ and $I = \emptyset$, although $m(\gamma^I) = m(\gamma)$, the degree of Vertex 1 is γ^I less than that in γ; by induction on that degree, it follows that the recursion may proceed in all cases above. Finally, if γ is connected, then the minimum of $m(\gamma)$ is -1, admitted by $\gamma \in \Gamma_{0,1}$; in this case $\gamma = \epsilon \in A^*$ as we shall see later. Hence, the ECA completely determine the assignment of multilinear maps to connected cellular graphs.

Less obviously, the ECA also determine the assignment of maps to disconnected graphs. For one may "paste" two connected components of a disconnected cellular graph γ by attaching a separating loop to form another graph γ', then ECA 2-2 determines γ in terms of γ'. By induction on number of connected components, it follows that indeed the assignment to all cellular graphs are completely determined by the ECA.

[3]Indeed, even in this case we have

$$m(\gamma^I) + m(\gamma^J) = 2g_1 - 2 + (|I| + 1) + 2g_2 - 2 + (|J| + 1) = 2(g_1 + g_2) - 2 + (|I| + |J|) = m(\gamma) - 1.$$

However, since $m(\gamma^I)$ or $m(\gamma^J)$ could take value -1, there is no guarantee that the m value decreases for all components.

The reason why the ECA are important is that establishes connections between TQFT and cellular graphs, hence gives an alternative description of TQFT. Mulase proves the following result:

Theorem 3.1. *Given a TQFT $\{\Omega_{g,n} : A \to K\}$, we associate to each $\gamma \in \Gamma_{g,n}$ a totally symmetric multilinear map $A^{\otimes n} \to K$, also denoted by γ, in the following way:*

(i) If $\gamma \in \Gamma_{0,1}$, then set $\gamma(v) = \epsilon(v) = \Omega_{0,1}$;
(ii) If $\gamma \in \Gamma_{0,2}$, then set $\gamma(v_1, v_2) = \eta(v_1, v_2) = \Omega_{0,2}$;
(iii) If $\gamma \in \Gamma_{g,n}$, where $m(\gamma) = 2g-2+n > 0$, then set $\gamma = \Omega_{g,n}$.

Then γ satisfies the ECA.

Proof. The proof is direct verification of the ECA. We write

$$e_i e_j = \sum_k m_{ij}^k e_k. \tag{3.5}$$

ECA 0: This is satisfied by definition (1).

ECA 1: From Definition 2.3 (4), taking $g_1 = 0$, $g_2 = g$, $I = \{1,2\}$, and $J = \{3, ..., n\}$, we have

$$\begin{aligned}\Omega_{g,n}(v_1, ..., v_n) &= \sum_{a,b} \Omega_{0,3}(v_1, v_2, e_a)\eta^{ab}\Omega_{g,n-1}(e_b, v_3, ..., v_n) \\ &= \sum_{a,b} \eta(v_1 v_2, e_a)\eta^{ab}\Omega_{g,n-1}(e_b, v_3, ..., v_n) \qquad \text{by Def. 2.3 (1)}\end{aligned} \tag{3.6}$$

by Definition 2.3 (1). We want to show

$$\Omega_{g,n}(v_1, v_2, v_3, ..., v_n) = \Omega_{g,n-1}(v_1 v_2, v_3, ..., v_n),$$

and it suffices to consider $v_1 = e_i$, $v_2 = e_j$. We obtain

$$\begin{aligned}\Omega_{g,n}(e_i, e_j, v_3, ..., v_n) &= \sum_{a,b} \eta(e_i e_j, e_a)\eta^{ab}\Omega_{g,n-1}(e_b, v_3, ..., v_n) \\ &= \sum_{a,b,k} m_{ij}^k \eta_{ka}\eta^{ab}\Omega_{g,n-1}(e_b, v_3, ..., v_n) \\ &= \sum_{b,k} m_{ij}^k \delta_k^b \Omega_{g,n-1}(e_b, v_3, ..., v_n) \\ &= \sum_{b} m_{ij}^b \Omega_{g,n-1}(e_b, v_3, ..., v_n) \\ &= \Omega_{g,n-1}(e_i e_j, v_3, ..., v_n),\end{aligned}$$

as desired.

ECA 2-1: We want to show

$$\Omega_{g,n}(v_1, v_2, ..., v_n) = \Omega_{g,n}(\delta(v_1), v_2, ..., v_n),$$

and it suffices to consider $v_1 = e_a$. We obtain

$$\begin{aligned}\Omega_{g,n}(e_a, v_2, ..., v_n) &= \sum_{b,c} \eta^{bc}\Omega_{g-1,n+2}(e_a, e_b, e_c, v_2, ..., v_n) \qquad \text{by Def. 2.3 (3)} \\ &= \sum_{b,c,i,j} \eta^{bc}\eta^{ij}\Omega_{0,3}(e_a, e_b, e_i)\Omega_{g-1,n+1}(e_j, e_c, v_2, ..., v_n) \qquad \text{by (3.6)}\end{aligned}$$

$$= \sum_{b,c,i,j} \eta^{bc}\eta^{ij}\eta(e_a, e_b e_i)\Omega_{g-1,n+1}(e_j, e_c, v_2, ..., v_n)$$
$$= \Omega_{g-1,n+1}(\delta(e_a), v_2, ..., v_n) \qquad \text{by (2.2)}$$

as desired.

ECA 2-2: We fix I and J with $I \sqcup J\{2, ..., n\}$, and we want to show

$$\Omega_{g,n}(v_1, v_I, v_J) = \sum_{b,c,i,j} \eta^{bi}\eta^{cj}\eta(e_a, e_b e_c)\Omega_{g_1,|I|+1}(e_i, v_I)\Omega_{g_2,|J|+1}(e_j, v_J),$$

and again it suffices to consider $v_1 = e_a$. We compute

$$\Omega_{g,n}(e_a, v_I, v_J) = \sum_{i,j} \Omega_{g_1,|I|+2}(e_a, e_i, v_I)\Omega_{g_2,|J|+1}(e_j, v_J)$$
$$\text{by Def. 2.3 (4), with } I' = I \cup \{1\}, J' = J$$
$$= \sum_{i,j,b,c} \eta^{ij}\eta^{bc}\Omega_{0,3}(e_a, e_i, e_b)\Omega_{g_1,|I|+1}(e_c, v_I)\Omega_{g_2,|J|+1}(e_j, v_J) \qquad \text{by (3.6)}$$
$$= \sum_{i,j,b,c} \eta^{ij}\eta^{bc}\eta(e_a, e_i e_b)\Omega_{g_1,|I|+1}(e_c, v_I)\Omega_{g_2,|J|+1}(e_j, v_J)$$

as desired. □

The main question, therefore, is the converse to the above theorem. Since the TQFT are topologically invariant, it is a necessary condition for the converse to hold that the assignments of maps determined by the ECA are also topologically invariant. This is indeed a conjecture by Mulase.

Conjecture 3.2 (Graph-independence)**.** *For each $\gamma \in \Gamma_{g,n}$, we associate an n-linear map $\gamma : A^{\otimes n} \to K$ in such a way that the ECA are satisfied. If $\gamma_1, \gamma_2 \in \Gamma_{g,n}$, then $\gamma_1 = \gamma_2$ as an element of $(A^*)^{\otimes n}$.*

For cellular graphs of small genus, there is evidence for graph-independence, as verified by brute-force computation. More precisely, Mulase proves the following.

Proposition 3.3.

- *If $\gamma \in \Gamma_{0,n}$, then*

$$\gamma(v_1, ..., v_n) = \epsilon(v_1 v_2 \cdots v_n). \tag{3.7}$$

- *If $\gamma \in \Gamma_{1,1}$, then*

$$\begin{aligned} \gamma(v) &= \sum_{a,b} \eta(v, e_a e_b)\eta^{ab} \\ &= \sum_{a,b} \gamma_{0,3}(v, e_a, e_b)\eta^{ab}, \end{aligned} \tag{3.8}$$

where $\gamma_{0,3} \in \Gamma_{0,3}$.

The case $g = 1$ in the above proposition may be generalized further to arbitrary n:

Proposition 3.4 (C)**.** *If $\gamma \in \Gamma_{1,n}$, then*

$$\begin{aligned} \gamma(v_1, ..., v_n) &= \sum_{a,b} \eta(v_1 \cdots v_n, e_a e_b)\eta^{ab} \\ &= \operatorname{Tr} \mu_{v_1 \cdots v_n}, \end{aligned} \tag{3.9}$$

where $\mu_v : A \to A$ is the linear map given by the (left) multiplication by $v \in A$.

In particular, graph-independence holds in all cases above. The proofs to these propositions are based on direct computations using structure constants of the Frobenius algebra. Since the computations are quite cumbersome, we leave them in Appendix A.

3.1. **Main results.** As a consequence of graph-independence, it is established that TQFT and ECA are equivalent, so the cellular graphs give another description of the TQFT.

Theorem 3.5 (Main theorem). *Assuming graph-independence, we denote by $\Omega_{g,n}(v_1, ..., v_n)$ the associated map $\gamma(v_1, ..., v_n)$ for any $\gamma \in \Gamma_{g,n}$. Then $\{\Omega_{g,n}\}$ is a TQFT.*

Proof. The proof is again by direct verification of axioms in Definition 2.3.

(1) $\Omega_{0,3}(v_1, v_2, v_3) = \eta(v_1, v_2 v_3)$ by (3.7).

(2) Take an arbitrary $\gamma \in \Gamma_{g,n}$, and construct an extension $\widetilde{\gamma} \in \Gamma_{g,n+1}$ by adding a vertex of degree 1. Then by ECA 1,

$$\begin{aligned}\Omega_{g,n+1}(1, v_1, ..., v_n) &= \widetilde{\gamma}(1, v_1, ..., v_n) = \gamma(1 \cdot v_1, ..., v_n)\\ &= \gamma(v_1, ..., v_n) = \Omega_{g,n}(v_1, ..., v_n).\end{aligned}$$

(3) Take a $\gamma \in \Gamma_{g,n+1}$ such that Vertex $(n+1)$ has an arrowed loop, which is a loop of a handle. Suppose the contraction of the loop leads to a graph $\gamma' \in \Gamma_{g-1,n+2}$. Then by ECA 2-1,

$$\begin{aligned}\Omega_{g,n}(v_1, ..., v_n) &= \Omega_{g,n+1}(v_1, ..., v_n, 1) = \gamma(v_1, ..., v_n, 1) = \gamma'(v_1, ..., v_n, \delta(1))\\ &= \sum_{a,b,i,j} \eta(1, e_a e_b)\eta^{ai}\eta^{bj}\gamma'(v_1, ..., v_n, e_i, e_j)\\ &= \sum_{a,b,i,j} \eta_{ab}\eta^{ai}\eta^{bj}\gamma'(v_1, ..., v_n, e_i, e_j)\\ &= \sum_{b,i,j} \delta_b^i \eta^{bj}\gamma'(v_1, ..., v_n, e_i, e_j)\\ &= \sum_{i,j} \eta^{ij}\gamma'(v_1, ..., v_n, e_i, e_j)\\ &= \sum_{i,j} \eta^{ij}\Omega_{g-1,n+2}(v_1, ..., v_n, e_i, e_j).\end{aligned}$$

(4) Let $I \sqcup J = \{1, 2, ..., n\}$ and $g = g_1 + g_2$. We choose a graph $\gamma \in \Gamma_{g,n+1}$ that has an arrowed loop at Vertex $(n+1)$, which is a separating loop. Suppose the contraction of this loop produces $\gamma \mapsto \gamma^I \sqcup \gamma^J$, where $\gamma^I \in \Gamma_{g_1,|I|+1}$ and $\gamma^J \in \Gamma_{g_2,|J|+1}$. Then by ECA 2-2,

$$\begin{aligned}\Omega_{g,n}(v_1, ..., v_n) &= \Omega_{g,n+1}(v_1, ..., v_n, 1)\\ &= \sum_{a,b,i,j} \eta(1, e_a e_b)\eta^{ai}\eta^{bj}\gamma^I(e_i, v_I)\gamma^J(e_j, v_J)\\ &= \sum_{a,b,i,j} \eta^{ij}\Omega_{g_1,|I|+1}(e_i, v_I)\Omega_{g_2,|J|+1}(e_j, v_J).\end{aligned}$$

This completes the proof. $\square$

4. Example: Abelian group algebra

In this section, we use the ECA to compute the explicit formulas of the TQFT on the Frobenius algebra $A = \mathbb{C}[G]$, where G is any finite abelian group. Here, we do not assume the graph-independence conjecture; rather, our computation provides evidence for graph-independence in this special case.

Theorem 4.1. *Let G be a finite abelian group, and $A = \mathbb{C}[G]$ the group algebra. Then any assignment of linear, totally symmetric map to $\gamma \in \Gamma_{g,n}$ obeying the ECA has formula*

$$\gamma(v_1, v_2, ..., v_n) = |G|^g \epsilon(v_1 v_2 \cdots v_n). \tag{4.1}$$

As a result, graph-independence is true for all (g, n) for the Frobenius algebra $\mathbb{C}[G]$.

Proof. Note that A has G as basis, and for any $\alpha, \beta \in G$, we have

$$\eta(\alpha, \beta) = \epsilon(\alpha\beta) = \delta_1^{\alpha\beta},$$

so the matrix $[\eta_{\alpha\beta}]$ is a permutation matrix (for any fixed ordering of the elements in G) corresponding to the permutation $\{\sigma : \alpha \mapsto \alpha^{-1}\}$ on G given by inversion. Furthermore, $[\eta_{\alpha\beta}]$ is symmetric, hence

$$[\eta^{\alpha\beta}] = [\eta_{\alpha\beta}]^{-1} = [\eta_{\alpha\beta}]^t = [\eta_{\alpha\beta}],$$

so $\eta^{\alpha\beta} = \delta_1^{\alpha\beta}$ also. Substitution into (2.2) gives

$$\begin{aligned}\delta(\alpha) &= \sum_{\beta,\sigma,\zeta,\xi} \eta(\alpha, \beta\sigma)\eta^{\beta\zeta}\eta^{\sigma\xi}\zeta \otimes \xi \\ &= \sum_{\beta,\sigma} \epsilon(\alpha\beta\sigma)\beta^{-1} \otimes \sigma^{-1} \\ &= \sum_{\beta} \beta^{-1} \otimes \alpha\beta.\end{aligned}$$

We denote by $\gamma(v_1, ..., v_n)$ the map $\mathbb{C}[G]^{\otimes n} \to \mathbb{C}$ associated to $\gamma \in \Gamma_{g,n}$ in a way so that the ECA are satisfied. We will prove the theorem by double inductions on $m(\gamma)$ and on the degree d of Vertex 1. The base case is $m(\gamma) = -1$, and $\gamma \in \Gamma_{0,1}$; in this case the formula holds by ECA 0.

For the inductive step, we assume that the theorem is true for all γ with $m(\gamma) < m$; consider $\gamma \in \Gamma_{g,n}$ with $m(\gamma) = m$. We argue by cases:

Case 1: The arrowed edge at Vertex 1 of γ connects to Vertex $j \neq 1$:

By ECA 1, contraction gives the graph $\gamma^j \in \Gamma_{g,n-1}$, which has $m(\gamma^j) = m(\gamma) - 1$, so

$$\begin{aligned}\gamma(v_1, v_2, ..., v_n) &= \gamma^j(v_1 v_j, v_2, \cdots, \widehat{v}_j, \cdots, v_n) \\ &= |G|^g \epsilon(v_1 v_2 \cdots v_n),\end{aligned}$$

where the last equality follows by induction.

Case 2-1: The arrowed edge at Vertex 1 is a loop of a handle, and the contracted graph γ' has type $(g - 1, n + 1)$:

By ECA 2-1, we have

$$\gamma(\alpha, v_2, ..., v_n) = \gamma'(\delta(\alpha), v_2, ..., v_n).$$

Since $m(\gamma') = m(\gamma) - 1$,

$$\gamma(\alpha, v_2, ..., v_n) = \sum_{\beta} \gamma'(\beta^{-1}, \alpha\beta, v_2, ..., v_n)$$

$$= \sum_{\beta} |G|^{g-1} \epsilon(\alpha v_2 \cdots v_n)$$
$$= |G|^g \epsilon(\alpha v_2 \cdots v_n)$$

by induction. Therefore, if $v_1 = \sum_\alpha c_\alpha \alpha$, then

$$\gamma(v_1, v_2, ..., v_n) = |G|^g \epsilon(v_1 v_2 \cdots v_n).$$

Case 2-2: The arrowed edge at Vertex 1 is a separating loop, and contraction results has two components $\gamma^I \in \Gamma_{g_1,|I|+1}$ and $\gamma^J \in \Gamma_{g_2,|J|+1}$, where $g_1 + g_2 = g$ and $I \sqcup J = \{2, ..., n\}$:

By ECA 2-2, we have

$$\gamma(\alpha, v_2, ..., v_n) = \gamma^I \gamma^J (\delta(\alpha), v_I, v_J).$$

For any $I \subset \{1, ..., n\}$, write $u_I = \prod_{i \in I} v_i$. Since suppose $g_1, g_2 \neq 0$, and $I, J \neq \emptyset$, then both $m(\gamma^I)$ and $m(\gamma^J)$ are less than $m(\gamma)$, by induction

$$\begin{aligned}\gamma(\alpha, v_2, ..., v_n) &= \sum_{\beta} \gamma^I(\beta^{-1}, v_I) \gamma^J(\alpha\beta, v_J) \\ &= \sum_{\beta} |G|^{g_1} \epsilon(\beta^{-1} u_I) |G|^{g_2} \epsilon(\alpha\beta u_J) \\ &= |G|^g \sum_{\beta} \epsilon(\beta^{-1} u_I) \epsilon(\alpha\beta u_J) \\ &= |G|^g \epsilon(\alpha v_2 \cdots v_n).\end{aligned}$$

To see the last equality above, we write

$$u_I = \sum_{\beta} \epsilon(\beta^{-1} u_I)\beta, \qquad u_J = \sum_{\sigma} \epsilon(\sigma^{-1} u_J)\sigma,$$

then

$$v_2 \cdots v_n = u_I u_J = \sum_{\beta,\sigma} \epsilon(\beta^{-1})\epsilon(\sigma^{-1})\beta\sigma,$$

so

$$\epsilon(\alpha v_2 \cdots v_n) = \sum_{\beta,\sigma} \epsilon(\beta^{-1} u_I)\epsilon(\sigma^{-1} u_J)\epsilon(\alpha\beta\sigma) = \sum_{\beta} \epsilon(\beta^{-1} v_I)\epsilon(\alpha\beta u_J).$$

It then follows again by linearity that

$$\gamma(v_1, v_2, ..., v_n) = |G|^g \epsilon(v_1 v_2 \cdots v_n).$$

Finally, without loss of generality, assume $g_1 = 0$ and $I = \emptyset$. Then we have $\gamma^I \in \Gamma_{0,1}$, and $\gamma^J \in \Gamma_{g,n}$

$$\begin{aligned}\gamma(\alpha, v_2, ..., v_n) &= \sum_{\beta} \gamma^I(\beta^{-1}) \gamma^J(\alpha\beta, v_2, ..., v_n) \\ &= \sum_{\beta} \epsilon(\beta^{-1}) \gamma^J(\alpha\beta, v_2, ..., v_n) \\ &= \gamma^J(\alpha, v_2, ..., v_n),\end{aligned}$$

so in general $\gamma = \gamma^J$ by linearity. But the degree of Vertex 1 in γ^J is less than that of γ, and the base cases where the degree is 0 or 1 are handled in ECA 0 or ECA 1, hence by induction we have

$$\gamma(v_1, v_2, ..., v_n) = |G|^g \epsilon(v_1 v_2 \cdots v_n).$$

Hence, no matter how we apply the ECA to contract the edges, we will always get the same result (4.1); the induction is now complete. ☐

Appendix A. Graph-independence for $g = 0, 1$

In this section we prove Propositions 3.3 and 3.4.

First we record Mulase's proof to Proposition 3.3. For the convenience of the reader, we restate the proposition below:

Proposition A.1.

- *If* $\gamma \in \Gamma_{0,n}$, *then*
$$\gamma(v_1, ..., v_n) = \epsilon(v_1 v_2 \cdots v_n).$$
- *If* $\gamma \in \Gamma_{1,1}$, *then*
$$\gamma(v) = \sum_{a,b} \eta(v, e_a e_b)\eta^{ab}.$$

We argue by cases, and divide the original statement into following chain of lemmas.

Lemma A.2. *If* $\gamma \in \Gamma_{0,1}$, *then* $\gamma(v) = \epsilon(v)$.

Proof. We proceed by induction on degree of γ. The base case is when γ has degree 0, i.e., it has no edges, then by ECA 0 we have $\gamma(v) = \epsilon(v)$.

Now suppose the lemma is true for all $\gamma \in \Gamma_{0,1}$ with degree less than $2m$. Let γ have degree $2m$, and denote by γ_1 and γ_2 the two components of the graph obtained by contracting the arrowed loop of γ. Then $\gamma_1, \gamma_2 \in \Gamma_{0,1}$, so ECA 2-2 gives

$$\begin{aligned}
\gamma(v) &= \sum_{a,b,i,j} \eta(v, e_a e_b)\eta^{ai}\eta^{bj}\gamma_1(e_i)\gamma_2(e_j) \\
&= \sum_{a,b,i,j} \eta(v, e_a e_b)\eta^{ai}\eta^{bj}\epsilon(e_i)\epsilon(e_j) \qquad \deg(\gamma_1), \deg(\gamma_2) < 2m \\
&= \sum_{a,b,i,j} \eta(v, e_a e_b)\eta^{ai}\eta^{bj}\eta_{1i}\eta_{1j} \\
&= \sum_{a,b} \eta(v, e_a e_b)\delta^a_1\delta^b_1 \\
&= \eta(v, e_1 e_1) = \eta(v, 1) = \epsilon(v).
\end{aligned}$$
☐

Lemma A.3. *If* $\gamma \in \Gamma_{0,2}$, *then* $\gamma(v_1, v_2) = \eta(v_1, v_2) = \epsilon(v_1 v_2)$.

Proof. We proceed by induction on the degree of Vertex 1. The base case is when the degree is 1, i.e., it connects vertices 1 and 2. Then contraction results in $\gamma^2 \in \Gamma_{0,1}$, so by ECA 1 and Lemma A.2 we have

$$\gamma(v_1, v_2) = \gamma^2(v_1 v_2) = \epsilon(v_1 v_2) = \eta(v_1, v_2).$$

Now suppose the lemma is true for all $\gamma \in \Gamma_{0,2}$ in which Vertex 1 has degree less than m. Let $\gamma \in \Gamma_{0,2}$ in which Vertex 1 has degree m.

Case 1: If the arrowed edge connects vertices 1 and 2, then ECA 1 implies the desired formula, just as in the base case.

Case 2: If the arrowed edge is a (separating) loop, again denote by γ_1 and γ_2 the components of the contracted graph. Note that $\gamma_1 \in \Gamma_{0,1}$, and since the Vertex 1 in γ_2 has degree less than m (in fact, the degree of γ_1 and the degree of Vertex 1 in γ_2 add to $m-1$), we have by inductive assumption that $\gamma_2 = \eta$ as an element of $(A^*)^{\otimes 2}$. Therefore

$$\begin{aligned}\gamma(v_1, e_c) &= \sum_{a,b,i,j} \eta^{ai}\eta^{bj}\eta(v_1, e_a e_b)\gamma_1(e_i)\gamma_2(e_j, e_c)\\ &= \sum_{a,b,i,j} \eta^{ai}\eta^{bj}\eta(v_1, e_a e_b)\epsilon(e_i)\eta(e_j, e_c)\\ &= \sum_{a,b,j} \delta_1^a\eta^{bj}\eta(v_1, e_a e_b)\eta_{jc}\\ &= \sum_{b} \delta_c^b\eta(v_1, e_b)\\ &= \eta(v_1, e_c),\end{aligned}$$

hence by linearity we have $\gamma(v_1, v_2) = \eta(v_1, v_2)$. This completes the induction. □

Lemma A.4. *If $\gamma \in \Gamma_{0,n}$, then $\gamma(v_1, ..., v_n) = \epsilon(v_1 v_2 \cdots v_n)$.*

Proof. We proceed by induction on n, and the base cases $n = 1, 2$ are taken care of in Lemmas A.2 and A.3. Now let $\gamma \in \Gamma_{0,n}$ with $n > 2$. If the arrowed edge at Vertex 1 connects to Vertex $j \neq 1$, then ECA 1 implies

$$\gamma(v_1, v_2, ..., v_n) = \gamma^j(v_1 v_j, v_2, ..., \widehat{v}_j, ..., v_n) = \epsilon((v_1 v_j)v_2 \cdots \widehat{v}_j \cdots v_n) = \epsilon(v_1 v_2 \cdots v_n)$$

as desired.

If the arrowed edge is a loop, then n does not decrease after contraction, so we use another induction on the degree of Vertex 1. Suppose Vertex 1 in γ has degree $m > 1$ (the case where degree is 1 is considered in the last paragraph), and let $\gamma = \gamma^I \sqcup \gamma^J$, where $\gamma^I \in \Gamma_{0,|I|+1}$ and $\gamma^I \in \Gamma_{0,|J|+1}$ with $I \sqcup J = \{2, ..., n\}$. Writing

$$u_I = \prod_{i\in I} v_i = \sum_p m_I^p e_p, \qquad u_J = \prod_{j\in J} v_j = \sum_q m_J^q e_q,$$

we have

$$\begin{aligned}\gamma(e_a, v_2, ..., v_n) &= \sum_{b,c,i,j} \eta^{bi}\eta^{cj}\eta(e_a, e_b e_c)\gamma^I(e_i, v_I)\gamma^J(e_j, v_J)\\ &= \sum_{b,c,i,j} \eta^{bi}\eta^{cj}\eta(e_a, e_b e_c)\eta(e_i, u_I)\eta(e_j, u_J)\\ &= \sum_{b,c,i,j,p,q} \eta^{bi}\eta^{cj}\eta(e_a, e_b e_c)m_I^p\eta_{ip}m_J^q\eta_{jq}\\ &= \sum_{b,c,p,q} \delta_p^b\delta_q^c\eta(e_a, e_b e_c)m_I^p m_J^q\\ &= \sum_{b,c} \eta(e_a, e_b e_c)m_I^b m_J^c\\ &= \eta(e_a, u_I u_J) = \epsilon(e_a v_2 \cdots v_n).\end{aligned}$$

By linearity, this gives

$$\gamma(v_1, ..., v_n) = \epsilon(v_1 \cdots v_n),$$

hence completes the double induction. □

Lemma A.5. *If $\gamma \in \Gamma_{1,1}$, then $\gamma(v) = \sum_{a,b} \eta(v, e_a e_b)\eta^{ab}$.*

Proof. We proceed by induction on degree of γ. The base case is when γ has degree 4, i.e., it consists of a meridian circle and a longitude circle on a torus, then contracting either of them results in $\gamma' \in \Gamma_{0,2}$. Hence ECA 2-1 gives

$$\begin{aligned}\gamma(v) &= \sum_{a,b,i,j} \eta(v, e_a e_b)\eta^{ai}\eta^{bj}\gamma'(e_i, e_j) \\ &= \sum_{a,b,i,j} \eta(v, e_a e_b)\eta^{ai}\eta^{bj}\eta_{ij} \\ &= \sum_{a,b,i} \eta(v, e_a e_b)\eta^{ai}\delta_i^b \\ &= \sum_{a,b} \eta(v, e_a e_b)\eta^{ab},\end{aligned}$$

as desired.

Now suppose the lemma is true for all $\gamma \in \Gamma_{1,1}$ with degree less than $2m$, and let γ has degree $2n$.

Case 1: If the arrowed loop is a loop of a handle, then same argument in the previous paragraph gives the desired formula.

Case 2: If the arrowed loop is a separating loop, then the contracted graph has two components, denoted by γ_1 and γ_2. Without loss of generality, let $\gamma_1 \in \Gamma_{0,1}$ and $\gamma_2 \in \Gamma_{1,1}$. Since γ_2 has degree less than $2m$, our desired formula holds for γ_2 by induction. We have

$$\begin{aligned}\gamma(e_a) &= \sum_{b,c,i,j} \eta(e_a, e_b e_c)\eta^{bi}\eta^{cj}\gamma_1(e_i)\gamma_2(e_j) \\ &= \sum_{b,c,i,j} \eta(e_a, e_b e_c)\eta^{bi}\eta^{cj}\eta_{i1}\gamma_2(e_j) \\ &= \sum_{b,c,j} \eta(e_a, e_b e_c)\delta_1^b\eta^{cj} \sum_{k,l} \eta(e_j, e_k e_l)\eta^{kl} \\ &= \sum_{c,j,k,l} \eta(e_a, e_c)\eta(e_j, e_k e_l)\eta^{cj}\eta^{kl} \\ &= \sum_{c,j,k,l} \eta_{ac}\eta(e_j, e_k e_l)\eta^{cj}\eta^{kl} \\ &= \sum_{k,l} \eta(e_a, e_k e_l)\eta^{kl},\end{aligned}$$

so by linearity

$$\gamma(v) = \sum_{k,l} \eta(v, e_k e_l)\eta^{kl}$$

as desired.

This completes the induction. □

Now we prove Proposition 3.4. We first note that

$$\begin{aligned}\sum_{b,c}\eta(e_a,e_be_c)\eta^{bc} &= \sum_{b,c}\eta(e_ae_b,e_c)\eta^{bc}\\ &= \sum_{b,c,d} m_{ab}^d\eta_{dc}\eta^{bc} = \sum_b m_{ab}^b\\ &= \mathrm{Tr}[m_{ab}^c]_{c,d} = \mathrm{Tr}\,\mu_{e_a},\end{aligned}$$

so by linearity we have $\sum_{b,c}\eta(v,e_be_c) = \mathrm{Tr}\,\mu_v$. Thus, it remains to prove the following

Proposition A.6. *If $\gamma\in\Gamma_{1,n}$, then*

$$\gamma(v_1,...,v_n) = \sum_{a,b}\eta(v_1\cdots v_n, e_ae_b)\eta^{ab},$$

where μ_v is the multiplication by v.

Proof. It suffices to consider $v_1 = e_a$, and we proceed by multiple inductions. We first apply induction on n. The base case $n = 1$ is the second half of Proposition 3.3. Now assume the proposition is true for $\gamma\in\Gamma_{1,m}$ where $m<n$, and consider $\gamma\in\Gamma_{1,n}$ whose Vertex 1 has degree 1. Then the edge connects to Vertex $j\neq 1$, and the contraction results in $\gamma^j\in\Gamma_{1,n-1}$, so by induction and ECA 1, we have

$$\begin{aligned}\gamma(v_1,v_2,...,v_n) &= \gamma^j(v_1v_j,v_2,...,\widehat{v}_j,...,v_n)\\ &= \sum_{b,c}\eta(v_1v_2\cdots v_n, e_be_c)\eta^{bc}.\end{aligned}$$

Now assume the statement is true for $\gamma\in\Gamma_{1,n}$ whose Vertex 1 has degree less than d, and consider $\gamma\in\Gamma_{1,n}$ whose Vertex 1 has degree d.

Case 1: If the arrowed edge connects Vertex 1 to Vertex $j\neq 1$, then the proof is the same as $d=1$ case.

Case 2-1: If the arrowed edge is a loop of a handle, suppose the contraction results in $\gamma'\in\Gamma_{0,n+1}$. Then ECA 2-1 implies

$$\begin{aligned}\gamma(e_a,v_2,...,v_n) &= \sum_{b,c,i,j}\eta^{bi}\eta^{cj}\eta(e_a,e_be_c)\gamma'(e_i,e_j,v_2,...,v_n)\\ &= \sum_{b,c,i,j}\eta^{bi}\eta^{cj}\eta(e_ae_b,e_c)\epsilon(e_ie_jv_2\cdots v_n) \qquad \text{by (3.7)}\\ &= \sum_{b,c,d,i,j}\eta^{bi}\eta^{cj}m_{ab}^d\eta_{dc}\epsilon(e_ie_jv_2\cdots v_n)\\ &= \sum_{b,d,i,j}\eta^{bi}\delta_d^j m_{ab}^d\epsilon(e_ie_jv_2\cdots v_n)\\ &= \sum_{b,i,j}\eta^{bi}m_{ab}^j\epsilon(e_ie_jv_2\cdots v_n)\\ &= \sum_{b,i}\eta^{bi}\epsilon(e_ie_ae_bv_2\cdots v_n)\\ &= \sum_{b,i}\eta^{bi}\eta(e_av_2\cdots v_n, e_ie_b).\end{aligned}$$

Case 2-1: If the arrowed edge is a separating loop, suppose the contraction results in $\gamma' = \gamma^I \sqcup \gamma^J$ where without loss of generality, $\gamma^I \in \Gamma_{0,|I|+1}$ and $\gamma^J \in \Gamma_{1,|J|+1}$, with $I \sqcup J = \{2, ..., n\}$. First assume $I \neq \emptyset$, then ECA 2-2 implies

$$\gamma(e_a, v_2, ..., v_n) = \sum_{b,c,i,j} \eta^{bi}\eta^{cj}\eta(e_a, e_b e_c)\gamma^I(e_i, v_I)\gamma^J(e_j, v_J)$$

Writing

$$u_I = \prod_{i\in I} v_i = \sum_p m_I^p e_p, \qquad u_J = \prod_{j\in J} v_j = \sum_q m_J^q e_q,$$

we have

$$\begin{aligned}
\gamma(e_a, v_2, ..., v_n) &= \sum_{b,c,i,j} \eta^{bi}\eta^{cj}\eta(e_a, e_b e_c)\gamma^I(e_i, v_I)\gamma^J(e_j, v_J) \\
&= \sum_{b,c,i,j,k,l} \eta^{bi}\eta^{cj}\eta(e_a e_b, e_c)\epsilon(e_i u_I)\eta(e_j u_J, e_k e_l)\eta^{kl} \qquad \text{by (3.7), (3.9)} \\
&= \sum_{b,c,d,i,j,k,l,p,q} \eta^{bi}\eta^{cj} m_{ab}^d \eta_{dc} m_I^p \eta_{ip} m_J^q \eta(e_j e_q, e_k e_l)\eta^{kl} \\
&= \sum_{b,d,j,k,l,p,q} \delta_p^b \delta_d^j m_{ab}^d m_I^p m_J^q \eta(e_j e_q, e_k e_l)\eta^{kl} \\
&= \sum_{b,j,k,l,q} m_{ab}^j m_I^b m_J^q \eta(e_j e_q, e_k e_l)\eta^{kl} \\
&= \sum_{b,k,l,q} m_I^b m_J^q \eta(e_a e_b e_q, e_k e_l)\eta^{kl} \\
&= \sum_{k,l} \eta(e_a u_I u_J, e_k e_l)\eta^{kl} \\
&= \sum_{k,l} \eta(e_a v_2 \cdots v_n, e_k e_l)\eta^{kl}
\end{aligned}$$

Finally, if $I = \emptyset$, then $\gamma^J \in \Gamma_{1,n}$ as well, then ECA 2-2 gives

$$\begin{aligned}
\gamma(e_a, v_2, ..., v_n) &= \sum_{b,c,i,j} \eta^{bi}\eta^{cj}\eta(e_a, e_b e_c)\gamma^I(e_i)\gamma^J(e_j, v_2, ..., v_n) \\
&= \sum_{b,c,d,i,j} \eta^{bi}\eta^{cj} m_{ab}^d \eta(e_d, e_c)\epsilon(e_i)\gamma^J(e_j, v_2, ..., v_n) \\
&= \sum_{b,d,j} \eta^{bi}\eta^{cj} m_{ab}^d \eta_{dc}\eta_{1i}\gamma^J(e_j, v_2, ..., v_n) \\
&= \sum_{b,d,j} \delta_1^b \delta_d^j m_{ab}^d \gamma^J(e_j, v_2, ..., v_n) \\
&= \sum_{j} m_{a1}^j \gamma^J(e_j, v_2, ..., v_n) \\
&= \gamma^J(e_a, v_2, ..., v_n).
\end{aligned}$$

But the degree of Vertex 1 in γ^J is less than d, so induction on d implies γ has the same formula as γ^J. $\square$

Appendix B. Examples: some non-abelian group algebras at (1,2) case

In this section, we compute the maps assigned to $\gamma_{1,2} \in \Gamma_{1,2}$ obeying ECA for the Frobenius algebras $Z(\mathbb{C}[G])$, where G takes the symmetric group S_3 of order 6, dihedral group D_4 of order 8, and $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$ of order 21; the first two are the smallest non-abelian groups[4], and the last one is the smallest non-abelian group of odd order.

In general, for any finite group G, the Frobenius algebra $Z(\mathbb{C}[G])$ has basis

$$\left\langle \sum_{\alpha \in [\![\alpha]\!]} \alpha \,\middle|\, [\![\alpha]\!] \text{ conjugacy class in } G \right\rangle;$$

for if $\sum_\alpha c_\alpha \alpha \in Z(\mathbb{C}[G])$, then commuting with $\beta \in G$ forces $c_\sigma = c_\beta$ for all $\sigma \in [\![\beta]\!]$.

By linearity of the assigned maps, it suffices to compute the values of these basis elements. Restricted to the $(g,n) = (1,2)$ case, ECA 1 reads

$$\gamma_{1,2}(e_a, e_b) = \gamma_{1,1}(e_a e_b), \tag{B.1}$$

and ECA 2-1 reads

$$\begin{aligned} \gamma_{1,2}(e_a, e_b) &= \sum_{c,d,i,j} \eta^{ci}\eta^{dj}\eta(e_a, e_c e_d)\gamma_{0,3}(e_i, e_j, e_b) \\ &= \sum_{c,d,i,j} \eta^{ci}\eta^{dj}\eta(e_a, e_c e_d)\epsilon(e_i e_j e_b) \qquad \text{by Prop. 3.3.} \end{aligned} \tag{B.2}$$

For ECA 2-2, we assume without loss of generality that $g_1 = 0$; then if $I = \{2\}$, ECA 2-2 reads

$$\begin{aligned} \gamma_{1,2}(e_a, e_b) &= \sum_{c,d,i,j} \eta^{ci}\eta^{dj}\eta(e_a, e_c e_d)\gamma_{0,2}(e_i, e_b)\gamma_{1,1}(e_j) \\ &= \sum_{c,d,i,j} \eta^{ci}\eta^{dj}\eta(e_a, e_c e_d)\epsilon(e_i e_b)\gamma_{1,1}(e_j) \qquad \text{by Prop. 3.3,} \end{aligned} \tag{B.3}$$

while if $I = \emptyset$, ECA 2-2 reads

$$\begin{aligned} \gamma_{1,2}(e_a, e_b) &= \sum_{c,d,i,j} \eta^{ci}\eta^{dj}\eta(e_a, e_c e_d)\gamma_{0,1}(e_i)\gamma_{1,2}(e_j, e_b) \\ &= \sum_{c,d,i,j} \eta^{ci}\eta^{dj}\eta(e_a, e_c e_d)\epsilon(e_i)\gamma_{1,2}(e_j e_b) \qquad \text{by Prop. 3.3} \\ &= \sum_{c,d,j} \eta^{c1}\eta^{dj}\eta(e_a, e_c e_d)\gamma_{1,2}(e_j e_b) \\ &= \sum_{d,j} \eta^{dj}\eta(e_a, e_d)\gamma_{1,2}(e_j e_b) \\ &= \sum_{d,j} \eta^{dj}\eta_{ad}\gamma_{1,2}(e_j e_b) \\ &= \sum_{j} \delta_a^j \gamma_{1,2}(e_j e_b) \\ &= \gamma_{1,2}(e_a e_b), \end{aligned}$$

[4] Although $\mathbb{C}[Q] \ncong \mathbb{C}[D_4]$, where Q denotes the quaternion group, which is the other non-abelian group of order 8, it turns out that $Z(\mathbb{C}[Q]) \cong \mathbb{Z}(\mathbb{C}[D_4])$.

which is vacuously true. Therefore, for all cases below we only need to check that (B.1), (B.2) and (B.3) give the same values.

For $G = S_3$, the basis elements are

$$e_1 = 1, \qquad e_2 = (1,2,3) + (1,3,2), \qquad e_3 = (1,2) + (1,3) + (2,3).$$

The multiplication table

	e_1	e_2	e_3
e_1	e_1	e_2	e_3
e_2	e_2	$2e_1 + e_2$	$2e_3$
e_3	e_3	$2e_3$	$3 + 2e_2$

gives

$$[\eta_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \qquad [\eta^{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

From these we use (3.8) to compute $\gamma_{1,1}$:

v	e_1	e_2	e_3
$\gamma_{1,1}$	3	3	0

By direct computation, we find that (B.1), (B.2) and (B.3) all give

$\gamma_{1,2}$	e_1	e_2	e_3
e_1	3	3	0
e_2	3	9	0
e_3	0	0	15

For $G = D_4 = \langle x, y | x^4 = y^2 = (xy)^2 = 1 \rangle$, the basis elements are

$$e_1 = 1, \qquad e_2 = x^2, \qquad e_3 = x + x^3, \qquad e_4 = y + x^2 y, \qquad e_5 = xy + x^3 y.$$

The multiplication table

	e_1	e_2	e_3	e_4	e_5
e_1	e_1	e_2	e_3	e_4	e_5
e_2	e_2	e_1	e_3	e_4	e_5
e_3	e_3	e_3	$2e_1 + 2e_2$	$2e_5$	$2e_4$
e_4	e_4	e_4	$2e_5$	$2e_1 + 2e_2$	$2e_3$
e_5	e_5	e_5	$2e_4$	$2e_3$	$2e_1 + 2e_2$

gives

$$[\eta_{ij}] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \qquad [\eta^{ij}] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

In this case, all three formulas give

$\gamma_{1,2}$	e_1	e_2	e_3	e_4	e_5
e_1	5	3	0	0	0
e_2	3	5	0	0	0
e_3	0	0	16	0	0
e_4	0	0	0	16	0
e_5	0	0	0	0	16

For $G = Q = \langle x, y | x^7 = y^3 = 1, y^{-1}xy = x^2 \rangle$, the basis elements are

$$e_1 = 1, \qquad e_2 = x + x^2 + x^4, \qquad e_3 = x^3 + x^5 + x^6, \qquad e_4 = \sum_{i=0}^{6} x^i y, \qquad e_4 = \sum_{i=0}^{6} x^i y^2.$$

The multiplication table

	e_1	e_2	e_3	e_4	e_5
e_1	e_1	e_2	e_3	e_4	e_5
e_2	e_2	$e_2 + 2e_3$	$3e_1 + e_2 + e_3$	$3e_4$	$3e_5$
e_3	e_3	$3e_1 + e_2 + e_3$	$2e_2 + e_3$	$3e_4$	$3e_5$
e_4	e_4	$3e_4$	$3e_4$	$7e_5$	$7(e_1 + e_2 + e_3)$
e_5	e_5	$3e_5$	$3e_5$	$(e_1 + e_2 + e_3)$	$7e_4$

gives

$$[\eta_{ij}] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 7 & 0 \end{pmatrix}, \qquad [\eta^{ij}] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{7} \\ 0 & 0 & 0 & \frac{1}{7} & 0 \end{pmatrix}.$$

In this case, all three formulas give

$\gamma_{1,2}$	e_1	e_2	e_3	e_4	e_5
e_1	5	8	8	0	0
e_2	8	24	31	0	0
e_3	8	31	24	0	0
e_4	0	0	0	147	0
e_5	0	0	0	0	147

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