RIGGED CONFIGURATIONS AND CATALAN OBJECTS: COMPLETING A COMMUTATIVE DIAGRAM WITH DYCK PATHS AND ROOTED PLANAR TREES

RYAN REYNOLDS

Contents

ABSTRACT. We construct an explicit bijection between rigged configurations and rooted planar trees, which we prove is the composition of the the bijection defined by Kerov, Kirillov, and Reshitikhin between rigged configurations and Dyck paths and the bijection between Dyck paths and rooted planar trees defined by the planar code.

1. INTRODUCTION

Like most ideas in mathematics, the original discovery of the Catalan numbers has multiple stories. Indeed these numbers were named after Eugene Catalan, a nineteenth century mathematician from Belgium; however, two different mathematicians from the eighteenth century have been attributed the discovery of the Catalan numbers. A Chinese mathematician by the name of Antu Ming discovered these numbers by studying trigonometric identities and power series in about 1730, though his book was completed by his students and published in 1839. The connection between the work of Antu to Catalan numbers was discovered by Luo Jianjin in [Luo13]. In 1751, Euler also discovered these numbers while working with triangulations of convex polygons. For more information on the history of the Catalan numbers, see [Kos09, Sta15].

No matter the origins, over the centuries the list of objects found to be enumerated by the Catalan numbers, known as Catalan objects, has grown to over 200 objects. Richard Stanley recently provided an extensive narrative of the properties and applications of the Catalan numbers [Sta15]; he lists 214 combinatorial interpretations of the Catalan numbers as exercises. Some of the interesting exercises are rooted trees with each vertex having either zero or two children (complete binary trees), peaks of height one in all Dyck paths from $(0,0)$ to (n,n), and ballot sequences, which are sequences of length $2n$ of 1's and -1's such that the number of 1's is equal to the number of -1's and every partial sum is non negative.

We focus on two well known families of the Catalan objects, Dyck paths and rooted planar trees. Since these two objects are proven Catalan objects, there exists a bijection between them. We recall a natural bijection between Dyck paths and rooted planar trees which essentially reads Dyck paths as the planar code for rooted planar trees from [Sta99]. Dyck paths are a particularly interesting family of objects since many natural statistics can be used to define the q, t-Catalan polynomials, which can be found in [Hag08].

In this thesis, we describe certain rigged configurations from the special case type $A_1^{(1)}$ of the Kerov-Kirillov-Reshitikhin bijection given in [KKR86] as Catalan objects which surprisingly do not appear in the list given in [Sta15]. One natural statistic on rigged configurations known as cocharge came out of the study of the partition function of the XXZ spin 1/2 Heisenberg spin chain [HKO+02, HKO+99] and the work of Kerov, Kirillov, and Reshetikhin [KKR86, KR86]. This statistic is specifically important in physics and statistical mechanics, and we are interested in them since cocharge corresponds to the statistic major index on Dyck paths through the KKR bijection.

We give an explicit bijection π between rigged configurations and rooted planar trees. We also prove that the bijection π is equivalent to the composition of the two aforementioned bijections. The bijection π constructs the partition ν underlying the rigged configuration a row at a time and immediately sets the rigging of the row. This makes the algorithm defined by π simpler than the algorithm defined by the KKR bijection, since the KKR bijection constructs a rigged configuration (ν, J) based on a Dyck word one step (or at least one box of the Young diagram) at a time and requires multiple computations of the riggings throughout the process. Also, the recursive description of the KKR bijection obscures many properties of the Dyck paths. The main advantage of constructing and proving the new direct bijection π is that these properties will no longer be obscured through the new bijection. As a consequence of our bijection, we can show that many natural statistics on Dyck paths have natural interpretations on rigged configurations.

This thesis is organized as follows. In Section 2 we provide a background on the Catalan numbers. In Section 3, we recall the necessary definitions concerning rigged configurations. Section 4 will be concerned with the concepts of rooted planar trees and the planar code of a rooted planar tree. In Section 5, we recall the Catalan objects Dyck paths and some interesting statistics on these objects. In Section 6, we construct a bijection between rigged configurations and rooted planar trees and prove that this completes a commutative diagram between these three Catalan objects. In Section 7, we define the q-analogue, consider the q, t-Catalan polynomials and how we can interpret these polynomials using area and bounce, describe the fermionic formula, and describe possible future reasearch.

2. Catalan Numbers

Here we give a brief definition, a few interesting properties and a few well-known examples of Catalan numbers found in [Kos09, Sta15]. We define the nth *Catalan number* by

$$(2.1) \qquad C_n := \frac{1}{n+1}\binom{2n}{n},$$

where $\binom{2n}{n}$ is the binomial coefficient. More generally, the binomial coefficient is

$$\binom{m}{n} = \frac{m!}{n!(m-n)!}.$$

The first few Catalan numbers are $1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796$. A Catalan object is an object that is enumerated by the Catalan numbers. A consequence of Euler's work with triangulation of $n+2$-gons is the following recursion satisfied by the Catalan numbers:

$$(2.2) \qquad\qquad C_0 = 1, \ C_n = \frac{4n-2}{n+1}C_{n-1}.$$

We can derive Equation (2.1) by using Equation (2.2):

$$C_n = \frac{4n-2}{n+1}C_{n-1} = \frac{(4n-2)}{(n+1)}\frac{(4n-6)}{n}C_{n-2} = \cdots = \frac{4n-2}{n+1}\frac{4n-6}{n}\frac{4n-10}{n-1}\cdots\frac{6}{3}\cdot\frac{2}{2}C_0$$
$$= 2^n\frac{(2n-1)(2n-3)(2n-5)\cdots 3\cdot 1}{(n+1)!} = \frac{2^n(2n)!}{2^n(n+1)!n!} = \frac{1}{n+1}\binom{2n}{n}.$$

If we use *Sterling's approximation for factorials*, which is $n! \sim (\frac{n}{e})^n\sqrt{2\pi n}$, then we can find an approximation of C_n:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)(n!)^2}$$
$$\sim \frac{(\frac{2n}{e})^{2n}\sqrt{2\pi\cdot 2n}}{(n+1)(\frac{n}{e})^{2n}\cdot 2\pi n} = \frac{2^{2n}}{(n+1)\sqrt{n\pi}}$$
$$\sim \frac{2^{2n}}{n\sqrt{n\pi}} = \frac{4^n}{n^{3/2}\sqrt{\pi}}.$$

Now we give three examples of well-known Catalan objects from [Sta15].

Example 2.1. We describe the triangulations of a convex $n+2$-gon and give some examples. Given a convex $n+2$-gon where n is a positive integer, a triangulation of such a polygon is a division into triangles. These triangulations are Catalan objects.

$$n = 2$$

$$n = 3$$
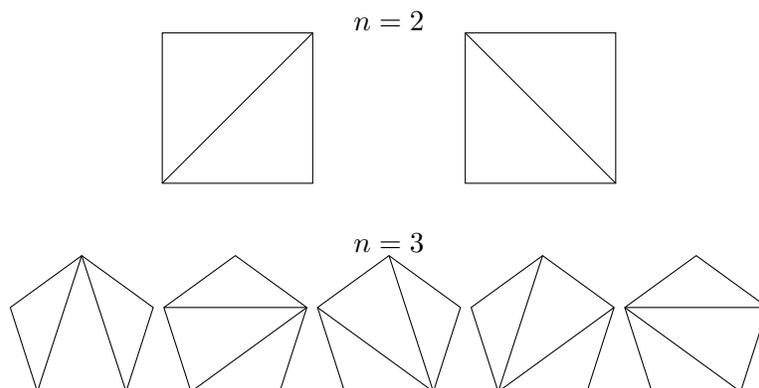
Example 2.2. A sequence of parentheses is said to be valid if for every open parenthesis, "(", there is a close parenthesis, ")", and when considering a part of the sequence, the number of open parentheses is greater than the number of close parentheses. The set of sequences of valid parentheses is a Catalan object.

$$n = 2$$

$$()() \quad (())$$

$$n = 3$$

$$()()() \quad ()(()) \quad (())() \quad (()()) \quad ((()))$$
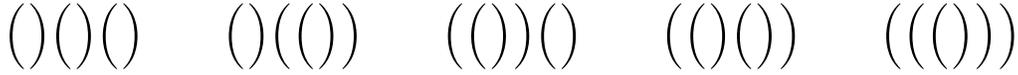
Example 2.3. The number of ways for $2n$ people sitting around a table to shake hands without any arms crossing is also enumerated by the Catalan numbers. This is also equivalent to non-crossing partitions.

$$n = 2$$
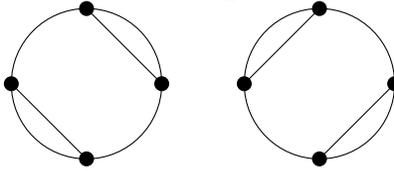
$$n = 3$$
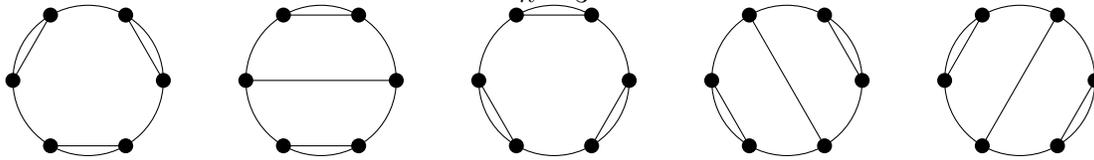
3. RIGGED CONFIGURATIONS

In this section, we describe the Catalan objects which are a special case of rigged configurations of type $A_1^{(1)}$ from [KKR86, Sch03]. For this concept, we need to recall a few definitions from [KSS02, Sta12, Sag01].

A *partition* of a positive integer is a finite sequence of positive integers $\nu = (\nu_1, \nu_2, \ldots, \nu_m)$ such that $\nu_i \geq \nu_{i+1}$. We say ν_k, for some k, is a *part* of ν. The size of the partition $|\nu| = \nu_1 + \nu_2 + \cdots + \nu_m$, and we denote by $\nu \vdash n$ if $|\nu| = n$. The *length* of ν is m and denoted $\ell(\nu)$. These sums can be represented pictorially in many different ways. We represent them in a *Young diagram* as a collection of rows of boxes such that the length of the kth row corresponds to ν_k. We use English convention which has the rows in weakly decreasing order from top to bottom and left aligned. The number of rows of length j in a partition ν is the multiplicity of j which we denote as $m_j(\nu)$ and we simply write m_j when there is no danger of confusion.

Example 3.1. The Young diagram corresponding to the partition of $14 = 5 + 4 + 2 + 2 + 1$.
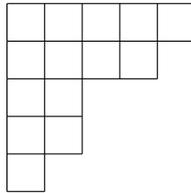
We denote (k^m) as the partition of length m and each row has length k, i.e. an $m \times k$ rectangle. Let ν be a partition. Let μ be a partition such that the Young diagram of μ can sit completely

inside of ν, that is $\ell(\mu) \leq \ell(\nu)$ and $\nu_i \geq \mu_i$ for all $1 \leq i \leq \ell(\mu)$. A *skew partition* ν/μ is the set of boxes in ν which are not in μ.

Example 3.2. Let ν be as in Example 3.1 and let

$$\mu = \;\begin{array}{c}\square\\\square\\\square\\\square\end{array}\;.$$

Then

$$\nu/\mu = \qquad .$$

Definition 3.3. A *rigged configuration* is the multiset of (i, x), where i is the length of a row and x is the corresponding *label* or *rigging*, which we denote as (ν, J), where ν is a partition comprised of the i and J is the set of labels x. Let $\mathrm{RC}(k; w)$, where $k \in \mathbb{Z}_{>0}$ and $w \in \mathbb{Z}_{\geq 0}$, be the multiset of rigged configurations (ν, J) satisfying the following conditions:

(1) for all $(i, x) \in (\nu, J)$, we have $0 \leq x \leq p_i(\nu)$, where $p_i(\nu)$ is the *vacancy number*

$$p_i(\nu) := k - 2 \sum_{j=1}^{\ell(\nu)} \min(\nu_j, i),$$

(2) the *weight* w of (ν, J) is $\mathrm{wt}(\nu, J) := k - 2|\nu| = p_\infty(\nu)$.

Example 3.4. We give an example of a rigged configuration in $\mathrm{RC}(24; 0)$:

$$\begin{array}{l}
0 \\
4 \\
8 \\
14 \\
14
\end{array}
\begin{array}{l}
0 \\
2 \\
3 \\
13 \\
7
\end{array} \qquad .$$

The *corigging* of $(i, x) \in (\nu, J)$ is $p_i(\nu) - x$. A row $(i, x) \in (\nu, J)$ is called *singular* if the rigging is equal to the vacancy number, that is $p_i(\nu) = x$. We can express the vacancy numbers as

$$p_i(\nu) = k - 2 \sum_{j=1}^{\infty} \min(i, j) m_j(\nu).$$

Definition 3.5. Let $(\nu_1, J_1) \in \mathrm{RC}(k_1; 0)$ and $(\nu_2, J_2) \in \mathrm{RC}(k_2; 0)$. We define the *interweaving* $(\nu_1, J_1) \cdot (\nu_2, J_2) \in \mathrm{RC}(k_1 + k_2; 0)$ as the multiset of all $(i, x) \in (\nu_1, J_1)$ and $(i, p_i(\nu_1) + x)$ for $(i, x) \in (\nu_2, J_2)$.

5

Let $(\nu, J) = (\nu_1, J_1) \cdot (\nu_2, J_2)$. We show that the riggings of (ν, J) are given by the coriggings of (ν_2, J_2). Note that $m_j(\nu) = m_j(\nu_1) + m_j(\nu_2)$ for all $j \in \mathbb{Z}_{>0}$, and so

$$p_i(\nu) = k - 2 \sum_{j=1}^{\infty} \min(i,j) m_j(\nu) = k_1 + k_2 - 2 \sum_{j=1}^{\infty} \min(i,j) \big(m_j(\nu_1) + m_j(\nu_2) \big)$$

(3.1)
$$= k_1 + k_2 - 2 \left(\sum_{j=1}^{\infty} \min(i,j) m_j(\nu_1) + \sum_{j=1}^{\infty} \min(i,j) m_j(\nu_2) \right)$$

$$= k_1 - 2 \sum_{j=1}^{\infty} \min(i,j) m_j(\nu_1) + k_2 - 2 \sum_{j=1}^{\infty} \min(i,j) m_j(\nu_2) = p_i(\nu_1) + p_i(\nu_2).$$

Then for a rigging x coming from (ν_2, J_2), we have

$$p_i(\nu) - (p_i(\nu_1) + x) = p_i(\nu_1) + p_i(\nu_2) - p_i(\nu_1) - x = p_i(\nu_2) - x.$$

Thus (ν, J) is constructed essentially by adding the two partitions together and keeping the riggings of (ν_1, J_1) and the coriggings of (ν_2, J_2).

Example 3.6. This is an example of the interweaving of two rigged configurations $(\nu_1, J_1) \in \mathrm{RC}(14; 0)$ and $(\nu_2, J_2) \in \mathrm{RC}(22; 0)$. Let

Then

where the subscript denotes which rigged configuration a particular rigging comes from. Notice that we have $(\nu_1, J_1) \cdot (\nu_2, J_2)$, $(\nu_2, J_2) \cdot (\nu_1, J_1) \in \mathrm{RC}(36; 0)$

Lemma 3.7. *The interweaving operation* $\cdot : \mathrm{RC}(k_1; 0) \times \mathrm{RC}(k_2; 0) \longrightarrow \mathrm{RC}(k_1 + k_2; 0)$ *is well-defined.*

Proof. It is sufficient to check that $(\nu, J) = (\nu_1, J_1) \cdot (\nu_2, J_2)$ is a rigged configuration in $\mathrm{RC}(k_1 + k_2; 0)$. By Equation (3.1), we have $p_i(\nu) = p_i(\nu_1) + p_i(\nu_2) \geq p_i(\nu_1), p_i(\nu_2)$. By construction, we have that $m_i(\nu) = m_i(\nu_1) + m_i(\nu_2)$. Since $\mathrm{wt}(\nu, J) = 0$ and $\mathrm{wt}(\nu, J) = k - 2|\nu|$ by Definition 3.3, we have $k - 2|\nu| = 0$. Since the riggings x for $(i, x) \in (\nu_1, J_1)$ do not change through interweaving, we have $x \geq 0$. Also, since $x \leq p_i(\nu_1) \leq p_i(\nu)$, the riggings are weakly less than $p_i(\nu)$. Since the corigging $p_i(\nu_2) - x$ for $(i, x) \in (\nu_2, J_2)$ change to $p_i(\nu_1) + x$ through the interweaving and since $x \leq p_i(\nu_2)$, then $p_i(\nu) = p_i(\nu_1) + p_i(\nu_2) \geq p_i(\nu_1) + x$. Also, since $x \geq 0$ and $p_i(\nu_1) \geq 0$, then $0 \leq p_i(\nu_1) + x$. Therefore, (ν, J) is a valid rigged configuration in $\mathrm{RC}(k_1 + k_2; 0)$. $\square$

Now we describe a statistic on a partition ν which arose from statistical mechanics called *cocharge*. Cocharge is defined as

$$\mathrm{cc}(\nu) = \sum_{i,j=0}^{\infty} \min(i,j) m_i m_j,$$

where m_i is the multiplicity of i in ν. Now we can define the cocharge on a rigged configuration as

$$\mathrm{cc}(\nu, J) = \mathrm{cc}(\nu) + |J|$$

where $|J|$ is the sum of all riggings in (ν, J).

Example 3.8. In this example, we enumerate the set of rigged configurations $\mathrm{RC}(8; 0)$.
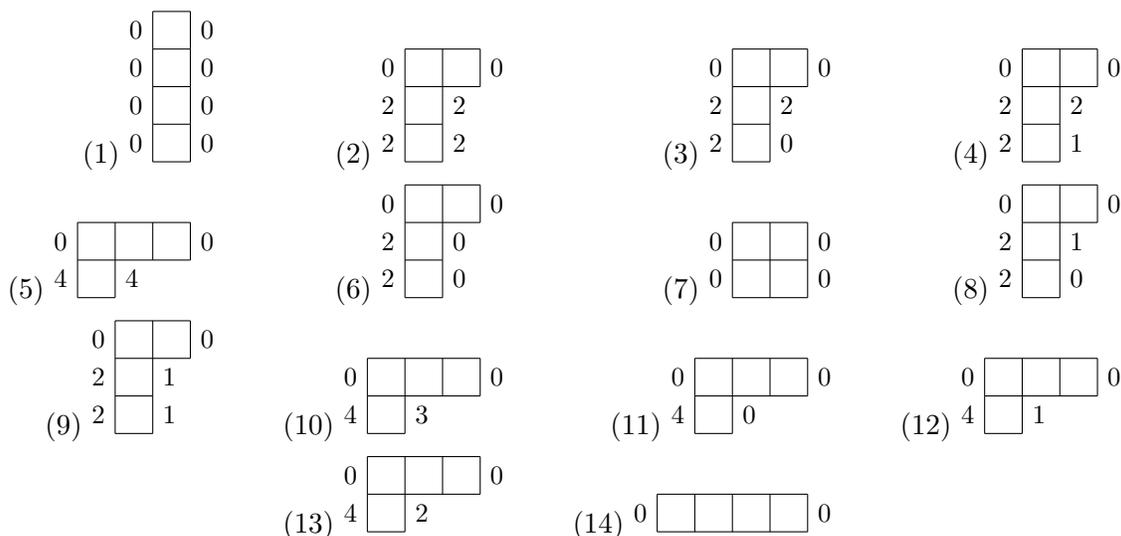
4. Rooted Planar Trees

In this section, we describe the combinatorial objects which we call rooted planar trees, ordered trees or (rooted) plane trees. First we recall some well-known definitions and ideas from graph theory, specifically regarding trees. We follow [Sta12] for the definitions and theorems of rooted planar trees.

A *graph* is a tuple $G = (V, E)$ where V is a set whose elements are called *vertices* and E is a set of pairs of vertices $\{v, u\}$ called *edges*. We call two edges *adjacent* if they share a vertex, and a *path* with *length* ℓ, P_ℓ, is a set of ℓ edges which are adjacent. For the purposes of this thesis, we will not allow multiple edges between adjacent vertices. The *degree* of a vertex u, denoted $\deg(u)$, is the number of vertices v in which there is exactly one edge between u and v.

A *tree* is a graph where there is a unique path between any two vertices. A tree is called *rooted* if there exists a unique minimal vertex which is labeled the *root*.

Let a be a vertex of a tree T. We say a vertex b is a *child* of a if $\{a, b\}$ is an edge of T and the path from the root to a is contained in the path from the root to b. We say a is the *parent* of b. A *leaf* is a node with no children.

A *rooted planar tree* T is a rooted tree where there is a fixed ordering on the children of every node. For convention, we let the root node be the uppermost node on the tree and all children strictly below the root. We let $\mathcal{T}_n^{rp}$ be the set of all rooted planar trees with n edges.

Definition 4.1. The *planar code* of a rooted planar tree is the sequence of 1's and 2's which is a recording of the tracing around the outside of a rooted planar tree starting at the right side of the

root where a 1 denotes a step with exactly one edge length from a parent node to a child node and a 2 denotes a step with exactly one edge length from a child node to a parent node.

Every edge of a rooted planar tree has a 1 and a 2 assigned to it, and the number of 1's is greater than or equal to the number of 2's for every step in the sequence. Also, if they are equal then the planar code brings us back to the root.

Example 4.2. The following is an example of the rooted planar tree in $\mathcal{T}_4^{rp}$ with planar code 11212212.
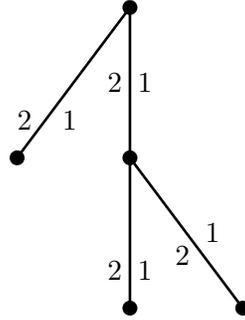
Definition 4.3. A *branch* is a path b in T from a leaf v to a vertex u such that $v \neq u$ and $\deg(u) \neq 2$, and all vertices $a \neq u, v$ of b satisfy $\deg(a) = 2$ in T.

Some statistics on a rooted planar tree T are the *depth* of a rooted planar tree which is the length of the longest embedded path starting at the root and the number of leaves, which we denote as $d(T)$ and $v(T)$ respectively.

Example 4.4. This is an example of a rooted planar tree in $\mathcal{T}_{14}^{rp}$. The paths with dashed lines are branches of length 1, 2, and 3 respectively.
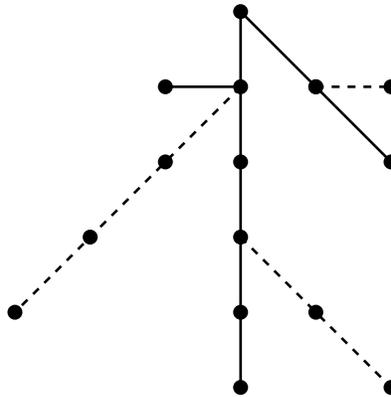
Definition 4.5. Let T_1 and T_2 be two rooted planar trees. We define $T_1 \cdot T_2$ by concatenating T_2 to T_1 at the roots of the trees with the branches of T_1 completely to the right of the branches of T_2.

Example 4.6. This example shows how the definition for $T_1 \cdot T_2$ works. Let

Then we have

$$T_1 \cdot T_2 = \quad,\qquad T_2 \cdot T_1 = \quad.$$
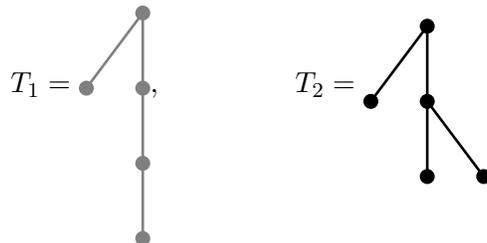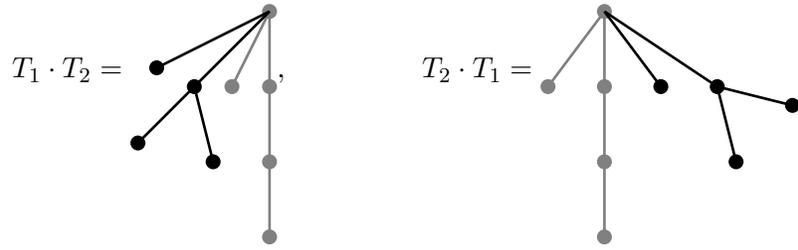
Lemma 4.7. *The planar code of* $T = T_1 \cdot T_2$ *is the sequence* $(a_1, \ldots, a_n, b_1, \ldots, b_k)$ *where* $(a_1, \ldots, a_n)$ *is the planar code of* T_1 *and* $(b_1, \ldots, b_k)$ *is the planar code of* T_2.

Proof. By the definition of $T_1 \cdot T_2$ we get that all the branches of T_1 are completely to the right of the branches of T_2. Then all the elements of the planar code of T_1 will come before all the elements of the planar code of T_2. Thus, if we let the planar codes for T_1 and T_2 be $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_k)$ respectively, then the planar code for T is $(a_1, \ldots, a_n, b_1, \ldots, b_k)$. Therefore, we get the desired sequence for the planar code of T. □

Example 4.8. We enumerate the set of rooted planar trees in $\mathcal{T}_4^{rp}$.
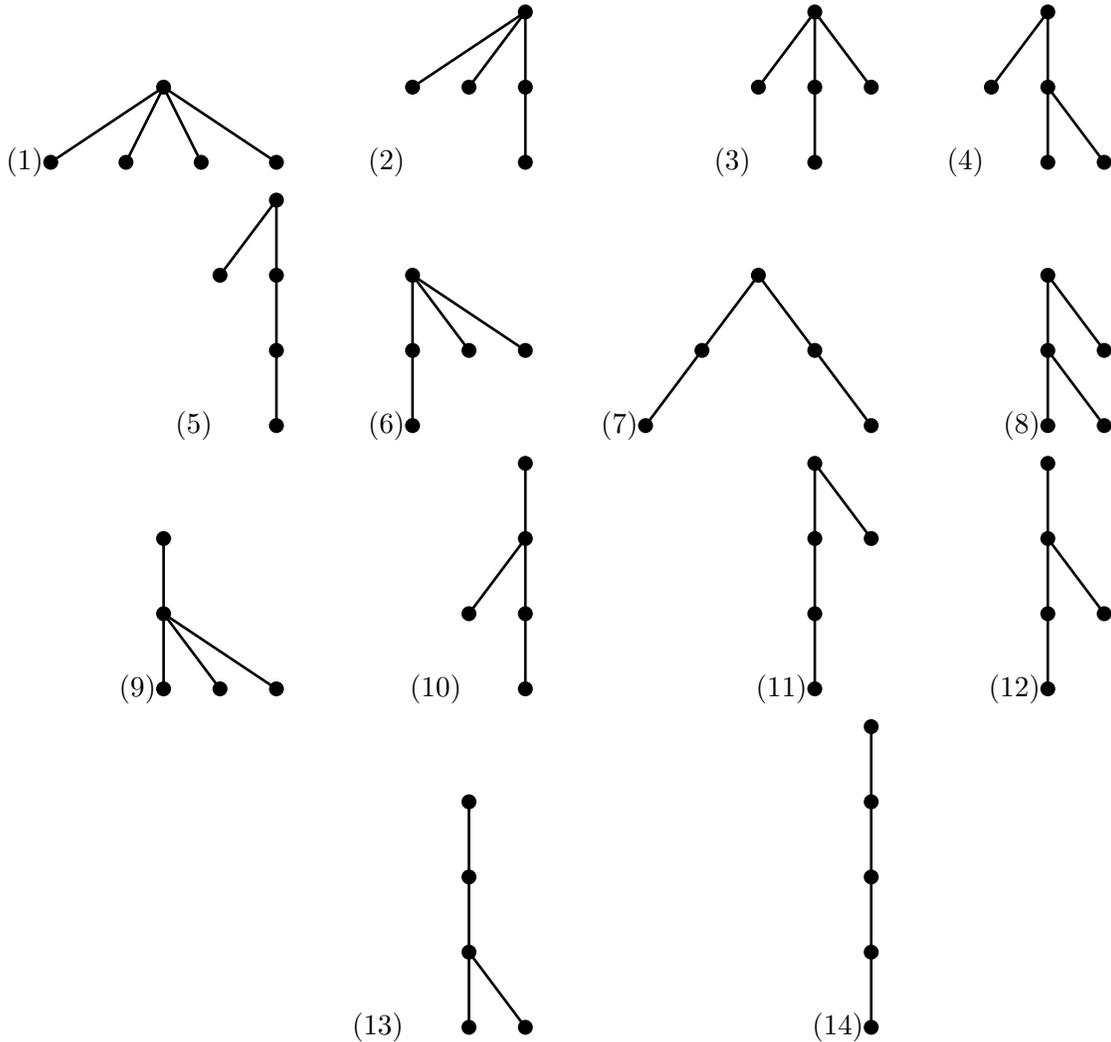
5. Dyck Paths

In this section, we give definitions and statistics surrounding Dyck paths from [Hag08]. Also, we describe the KKR bijection between Dyck paths and rigged configurations, the bijection between Dyck paths and rooted planar trees, and examples of how these bijections work.

Given an $n \times n$ box, the *main diagonal* is the line $y = x$ where $x \in [0, n]$.

Definition 5.1. A *Dyck path* of length $2n$ is a lattice path which starts at $(0,0)$ and ends at (n, n), where each step is either a unit north step or a unit east step and stays weakly above the main diagonal. Given a fixed n, we denote the set of all Dyck paths of length $2n$ by $\mathcal{D}_{2n}$.

From the definition, it is easy to see that in a Dyck path, the number of north steps is equal to the number of east steps. A Dyck paths can also be represented as a *Dyck word*, which is a sequence of n 1's and n 2's such that a 1 represents a unit step north and a 2 is represents a unit step east. We will abuse notation and equate Dyck paths and Dyck words.

Example 5.2. This example is a Dyck path $D \in \mathcal{D}_{16}$ with Dyck word 1211121122212212.
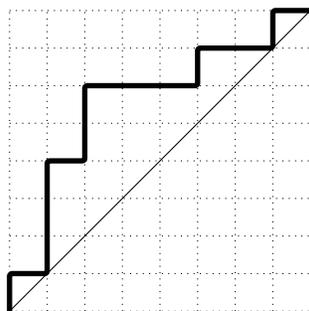
Definition 5.3. Let D_1 and D_2 be two Dyck paths with lengths $2n$ and $2m$ respectively. We define $D_1 \cdot D_2$ as a concatenation of the two Dyck paths such that the first step D_2 follows immediately after the last step of D_1.

This definition is well-defined, since the resulting path stays weakly above the main diagonal and the path starts at $(0,0)$ and ends at $(2n + 2m, 2n + 2m)$.

We abuse notation and define $1 \cdot D \cdot 2$ as an increase of the length of a Dyck path by 2 by adding an north step at the beginning of the path and an east step at the end of the path.

We recall some noteworthy statistics on Dyck paths: area, bounce, dinv, and major index. We define the *area sequence* of a Dyck path D by $a = (a_1, a_2, \ldots, a_n)$, where a_i is the number of boxes between the Dyck path and the main diagonal in row i going from bottom to top. The *area* statistic of a Dyck path is

$$(5.1) \qquad \qquad \text{area}(D) = \sum_{i=1}^{n} a_i$$

where a_i is the ith element in the area sequence.

The *bounce path* starts at (n, n) and travels west along the Dyck path until it turns south, and then travels south to the main diagonal. From the main diagonal the bounce path then travels west until it runs into the Dyck path going south. The bounce path repeats this process until it reaches $(0,0)$. Define the *bounce* statistic as

$$(5.2) \qquad \qquad \text{bounce}(D) = \sum_{i=1}^{k} b_i,$$

where b_i is where the bounce path hits the main diagonal and k is the number of times the bounce path hits the main diagonal after leaving (n, n).

To compute *dinv*, we consider the area sequence $\{a_1, \ldots, a_n\}$. The dinv is the size of the set $\{a_j - a_i \mid i < j, a_j - a_i \in \{0, 1\}\}$.

The *major index* of a Dyck word $D = w_1 \ldots w_{2n}$ is defined as

$$\text{maj}(D) = \sum_{w_i < w_{i+1}} i.$$

The *peaks* of a Dyck path D are the places in the Dyck word where a 1 immediately precedes a 2. The *height* of D is the number of units above the line $y = x$ where the highest peak is. We denote these statistics as $p(D)$ and $h(D)$ respectively.

Example 5.4. Consider D from Example 5.12.
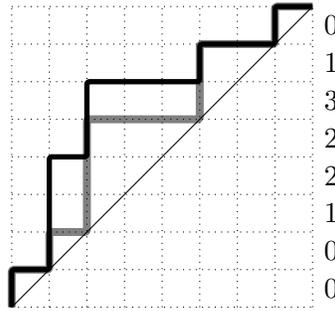
The bounce path of D is in gray, and the area sequence is $\{0, 0, 1, 2, 2, 3, 1, 0\}$. This gives us the following:

$$
\begin{aligned}
\text{area}(D) &= 0 + 0 + 1 + 2 + 2 + 3 + 1 + 0 = 9, \\
\text{bounce}(D) &= 1 + 2 + 5 + 7 = 15, \\
\text{dinv}(D) &= 1 + 2 + 1 + 2 + 2 + 3 + 2 = 13, \\
\text{maj}(D) &= 1 + 5 + 8 + 12 + 15 = 41, \\
p(D) &= 5, \\
h(D) &= 4.
\end{aligned}
$$

Definition 5.5 (KKR bijection). Let $\Phi : \mathcal{D}_{2n} \to \text{RC}(2n; 0)$. Then the algorithm of Φ goes through the Dyck word D of a Dyck path D defined by using a map $\delta : \{1, 2\} \times \text{RC}(k; w) \to \text{RC}(k+1; w+1) \sqcup \text{RC}(k+1; w-1)$. For each 2 in D a box is added to the largest singular string, and the algorithm of Φ computes the vacancy number and makes the string singular again.

The following proposition is a direct result of the KKR bijection from [KKR86, KR86].

Proposition 5.6. $|\text{RC}(2n; 0)| = C_n$.

The following theorem is a result of the KKR bijection and [HKO$^+$02, HKO$^+$99].

Theorem 5.7. *Let D be a Dyck path and $\Phi(D) = (\nu, J)$. Then $\text{maj}(D) = \text{cc}(\nu, J)$.*

Define $* : \mathcal{D}_{2n} \longrightarrow \mathcal{D}_{2n}$ to be the map that exchanges 1's and 2's and reverses the result. This is equivalent pictorially to reversing the Dyck path. Let $\theta : \text{RC}(L; \lambda) \to \text{RC}(L; \lambda)$ be the involution that preserves the partition and sends riggings to coriggings [SS06].

Theorem 5.8 ([SS06]). *Φ intertwines the maps $*$ and θ.*

11

The following is the commutative diagram for Theorem 5.8.

$$\begin{array}{ccc} \mathcal{D}_{2n} & \xrightarrow{\;\Phi\;} & \mathrm{RC}(2n;0) \\ {\scriptstyle *}\downarrow & & \downarrow{\scriptstyle \theta} \\ \mathcal{D}_{2n} & \xrightarrow{\;\Phi\;} & \mathrm{RC}(2n;0) \end{array}$$

Theorem 5.8 shows that $\Phi^{-1}(D^*)$ is the rigged configuration with riggings equal to the coriggings of $\Phi^{-1}(D)$.

Example 5.9. Example of the KKR bijection
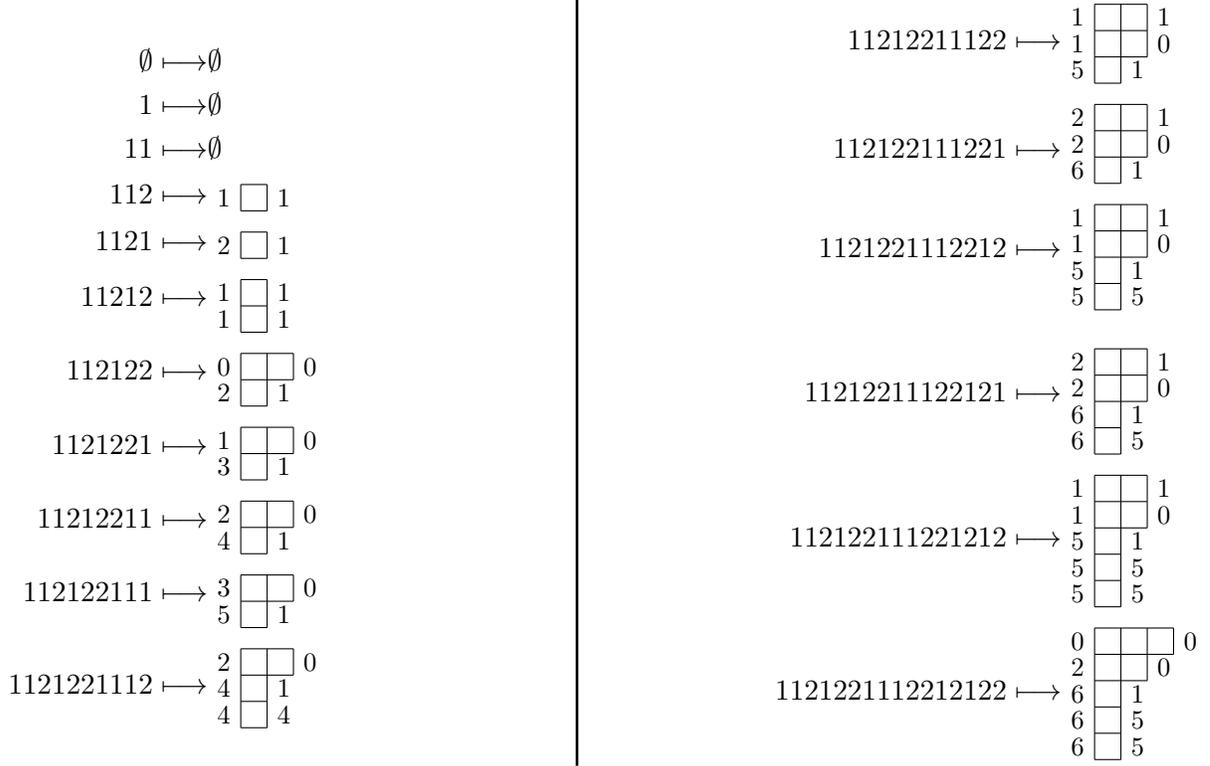
Lemma 5.10. *Let D be a Dyck path which has at least one return to the main diagonal away from the endpoints. Let (ν, J) be the rigged configuration corresponding to D through the KKR bijection. Let s be the step of D which creates a return to the main diagonal. Then the lengths and riggings of the rows of (ν, J) created up to s through the algorithm of the KKR bijection will not change for any step after s.*

Proof. A 2 in D through the algorithm for the KKR bijection adds a box to the longest singular string and makes the row singular. When the path of D returns to the main diagonal, the next step must always be an up step, so a 1. Through the KKR bijection, an up step will always make any singular string non-singular, since an up step increases all of the vacancy numbers by one, but all the riggings stay the same. Thus, when the path does go east and a box is added, no string is singular, so a new row must be created. Let λ be the partition after step s, and let t be a number of steps in D such that $s < t$. It is clear that $t \geq 2 \cdot |\mu|$, where μ is the partition after step t. So, let x be the rigging of an arbitrary row with length k which came from before the return to the diagonal with $p_k(\mu)$ as the vacancy number of this row length. Also we have that the number of

12

boxes added after s is less than or equal to twice the number of steps taken from s to t. Thus we have that

$$p_k(\mu) = t - 2 \cdot \sum_{j=1}^{\ell(\mu)} \min(\mu_j, k) > s - 2 \cdot \sum_{j=1}^{\ell(\lambda)} \min(\lambda_j, k) \geq x.$$

Thus, only when D returns to the main diagonal does a row from λ have a chance at being singular again, but the next step is an up step, so a 1, thus non-singular. Therefore, the claim is proved. $\square$

We now define $\psi : \mathcal{T}_n^{rp} \longrightarrow \mathcal{D}_{2n}$ as the map which reads the planar code from Definition 4.1 as a Dyck word.

Proposition 5.11. *The map $\psi \colon \mathcal{T}_n^{rp} \longrightarrow \mathcal{D}_{2n}$ is a bijection.*

Proof. It follows directly from Definition 4.1 that the number of 1's is greater than or equal to the number of 2's at each intermediate step, since if the are equal, the planar code brings us back to the root node, and from the root node, we cannot take a step closer to the root node. Since a Dyck path is uniquely equivalent as a Dyck word, we only have left to check that a given planar code corresponds to a unique rooted planar tree. This is clear from the definition of a rooted planar tree, since there is a fixed ordering on the children in a rooted planar tree. Therefore ψ is a bijection. $\square$

Example 5.12. This example enumerates the set of all Dyck paths in $\mathcal{D}_8$.
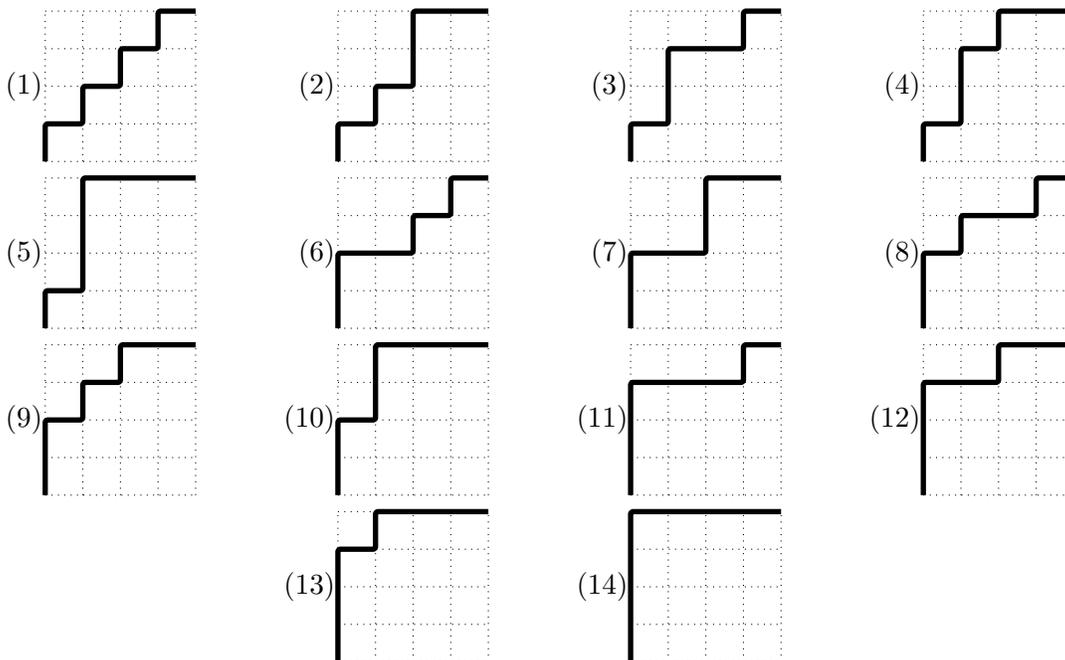
The examples at the end of Sections 3, 4, and 5 are all the $n = 4$ examples of each of the respective combinatorial objects. In particular, the Dyck path (i) of Example 5.12 corresponds to the rigged configuration (i) of Example 3.8 through the KKR bijection, and the Dyck path (i) of Example 5.12 corresponds to the rooted planar tree (i) of Example 4.8 through the bijection ψ given in Proposition 5.11.

6. The Bijection

In this section, we define a map which is algorithmic in nature between rooted planar trees and rigged configurations, and we prove that this map is a bijection. Also, we prove that this map is the composition of the two bijections in the previous section, and we state some interesting corollaries.

Definition 6.1. Let p be a path in a graph of length ℓ with distinguished endpoint v_0. A node a is called k-*valid* if attaching a path q of length k at a by an endpoint of q creates a path from v_0 which has length at most ℓ. We say a node a in a rooted tree T is k-valid if there exists a path p in T from the root to a leaf such that a is k-valid in p with the distinguished node p being the root of T.

Definition 6.2. Let $(\nu, J) \in \mathrm{RC}(2n; 0)$ be a rigged configuration. We define the map π by the following algorithm. The algorithm of π starts with ν_1, that is we start with the longest row of ν, with $T_0 = \emptyset$ and does the following. Consider a row i with length ν_i with the corresponding rigging x_i. Let $\widetilde{T}_{i-1}$ denote the induced tree of all ν_i-valid nodes in T_{i-1}. When tracing around the outside of $\widetilde{T}_{i-1}$ starting at the right side of the root, we count nodes from 0. We call each encounter with a node during this tracing procedure a *position* (note, each node can have multiple positions associated to it). The map π adds a branch of length ν_i to the position on T_{i-1} corresponding to the x_ith position on $\widetilde{T}_{i-1}$. Denote the resulting tree by T_i. Then π repeats with $i + 1$ unless $i = \ell(\nu)$. Then $\pi(\nu, J) = T_{\ell(\nu)}$.

Note that there is no ambiguity in where we add the branch of length ν_i at a node a. This follows from the fact that the paths in $T_{i-1} \setminus (\widetilde{T}_{i-1} \setminus \{a\})$ from a to leaves all have length ν_i, thus correspond to the same rooted planar tree after the branch is added.

Example 6.3. The following is an example of one step of the bijection π. We consider the bottom row of the rigged configuration which has length 2 and rigging 8. This rigged configuration maps to the rooted planar tree where the gray branch corresponds to the bottom row. The numbers around the tree illustrate the positions counted in the tracing procedure.
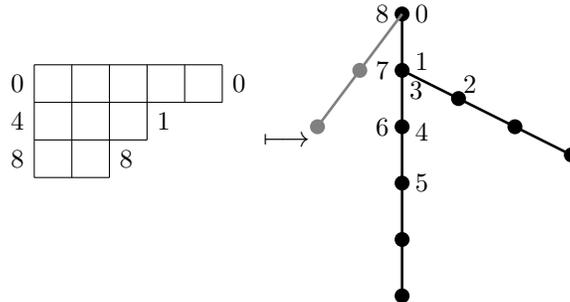
Example 6.4. This is an example of how the algorithm of π executes on a rigged configuration.
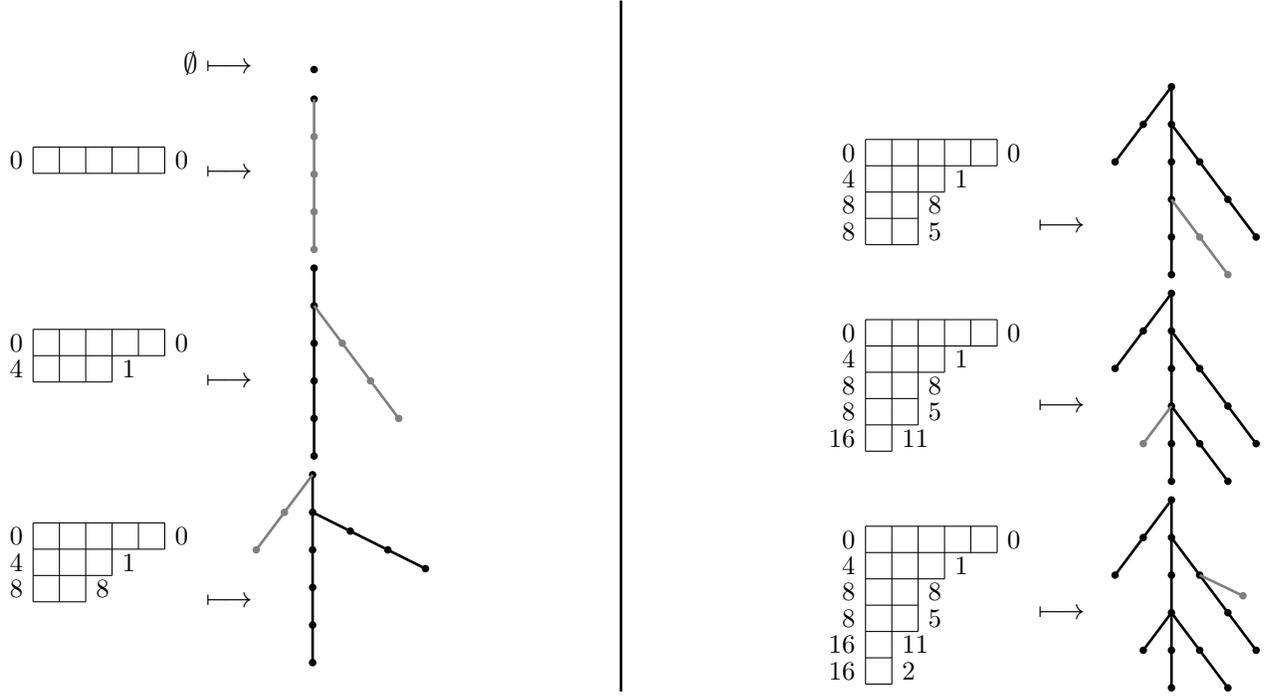
Theorem 6.5. *Let* $\pi : \mathrm{RC}(2n; 0) \longrightarrow \mathcal{T}_n^{rp}$ *be the map given by Definition 6.2. Then* π *is a bijection.*

Proof. Consider a rigged configuration $(\nu, J) \in \mathrm{RC}(2n; 0)$ with the underlying partition ν. We show π is a bijection by induction on $\ell(\nu)$.

The base case is when ν is the empty partition, thus (ν, J) is the empty rigged configuration. This corresponds to a tree with just the root node. Thus, we have $p_i = 0$ for all $i \in \mathbb{Z}_{>0}$. If we add a row of arbitrary length k to (ν, J), the rigging of the row and the vacancy number of the row length are both 0. This corresponds to adding a branch of length k to the tree with just the root node. There is only one way to add this branch through π, and so the base case is proved.

For the inductive step, we assume that (ν, J) is a rigged configuration where all rows of ν have length at least $k \in \mathbb{Z}_{>0}$. Let $T = \pi(\nu, J)$ and note the shortest branch of T has length at least k. We now add a row of length k to ν to show the induction step holds. Let $\mu = \nu/k^{\ell(\nu)}$, and note that μ is an actual partition. Notice that $p_k(\nu) = 2|\mu|$. We need that on each path (not necessarily a branch) b_j corresponding to a row ν_j that there are $2\mu_j$ k-valid nodes. We label the vertices of b_j by $(v_0, \ldots, v_{\nu_j})$ where v_0 is the node at the end of the branch closest to the root node and v_{ν_j} is the leaf node. Note that the vertices $v_0, \ldots, v_{\mu_j}$ are all k-valid with v_0 as the distinguished node since b_j has length $\nu_j = \mu_j + k$.

We consider adding the new branch only to the left side of b_j. The number of nodes on b_j is $\nu_j + 1$, but we skip v_0 as to not over count since v_0 will be counted on the path to which b_j is attached. Also, by construction we have that $\nu_j = \mu_j + k$. Since the resulting tree must have paths with length at most ν_j then, we have counted the number of k-valid nodes on the left side and is exactly $\nu_j - k = \mu_j$. Now we consider the other side of the branch, and this case is similar. However, we skip v_{μ_j} since adding it to v_{μ_j} on this side would not create a distinct tree, but we do count v_0 if $\mu_j \neq 0$ since adding it here creates a distinct tree that was not counted on the path to which b_j is attached. Note if $\mu_j = 0$, then adding it to the left of b_j is the same as adding it to the right of b_j. Therefore, we have counted $2\mu_j$ positions on the jth branch.

15

By adding one for the root node since it is not counted on b_1, the total number of positions are

$$1 + \sum_{j=1}^{\ell(\mu)} 2\mu_j = p_k(\mu) + 1.$$

Now let $\rho : \mathcal{T}_n^{rp} \longrightarrow \mathrm{RC}(2n; 0)$ such that given a rooted planar tree, ρ decomposes the tree starting from the branch of the smallest length and constructs a row (i, x) of a rigged configuration (ν, J) at a time. Consider a tree with shortest branch of length k. By considering how many k-valid nodes there are from the root node to the node where the branch starts, we get the rigging for that row. We have already shown that there are exactly $p_k(\nu) + 1$ k-valid nodes, thus the rigging which we get for each row will be between 0 and $p_k(\nu)$. If there is only one branch of length k we are done. If there are multiple branches of length k, we construct the corresponding rows in the rigged configuration and reorder so that the riggings are weakly decreasing. This can always be done since we count the nodes which any branch of length k is added to once. If we choose a branch with other branches of length k between it and the root nodes, then there are no k-valid nodes on the other length k branches. Thus nothing changes again. Hence we can choose any branches of similar lengths in any order and reorder in the rigged configuration. By the construction of this inverse map, each branch of the resulting tree corresponds with a row of the rigged configuration, and so the map π is surjective. Therefore, $\rho = \pi^{-1}$ since an arbitrary step of the algorithm defined by ρ is the inverse of the complementary step of the algorithm defined by π, and the map π is a bijection. $\qquad\square$

Theorem 6.6. *Let $\Phi : \mathrm{RC}(2n; 0) \longrightarrow \mathcal{D}_{2n}$ be the map defined by the KKR bijection, $\psi : \mathcal{D}_{2n} \longrightarrow \mathcal{T}_n^{rp}$ be the map which interprets the Dyck word as the planar code, and $\pi : \mathrm{RC}(2n; 0) \longrightarrow \mathcal{T}_n^{rp}$ be the bijection from Theorem 6.5. Then the following diagram commutes.*

$$
\begin{array}{ccc}
\mathrm{RC}(2n; 0) & \xrightarrow{\;\;\Phi\;\;} & \mathcal{D}_{2n} \\
& \searrow{\scriptstyle \pi} & \downarrow{\scriptstyle \psi} \\
& & \mathcal{T}_n^{rp}
\end{array}
$$

Proof. To show that these bijections commute, we use induction on n.

First, we consider the base case when $n = 0$. Then we only have the empty Dyck path, which trivially corresponds to the empty rigged configuration and the rooted planar tree with only the root node. Thus our claim holds for the $n = 0$ case.

Now we assume our claim holds for all $k < n$. We fix a $D \in \mathcal{D}_{2n}$, and let $\psi(D) = T \in \mathcal{T}_n^{rp}$ and $\Phi^{-1}(D) = (\nu, J) \in \mathrm{RC}(2n; 0)$. We consider two cases. The first case is if only the endpoints of D touch the main diagonal, and the second case is if D has at least one return away from the endpoints to the main diagonal. This is sufficient since every Dyck path has a return.

<u>Case 1:</u> Assume the first return of D is the endpoint. Consider the Dyck word D and let $D = 1 \cdot D' \cdot 2$. Then we get D' has semi-length $2(n-1)$. Thus by induction hypothesis, $\Phi^{-1}(D') = (\nu', J')$ and $\psi(D') = T'$ correspond under π. So we determine the rigged configuration (ν, J) we get from D through the KKR bijection. Since D has an extra 1 at the beginning, if y is the vacancy number of a row at a step m, then the vacancy number of a row at step $m + 1$ of D is exactly $y + 1$. Thus, at a step of the algorithm of $\Phi^{-1}(D)$ where a box is added to a singular string the rigging of that string will increase by one from a similar step in the algorithm of $\Phi^{-1}(D')$. This shows that all the riggings of the construction of (ν, J) are increased by one from construction of (ν', J') up to the final step. By the algorithm of the KKR bijection, we know that just before the final step

the vacancy number and rigging of the longest row are both 1, thus the row is singular. We now consider the final step, where we must add a box to the longest singular string and again make it singular making both the vacancy number and rigging 0. This row, call it ν_1 will be strictly the longest singular string of (ν, J) with rigging 0. Therefore, the partition of (ν, J) has a box added in the longest row, and the riggings of all but the longest row have increased by one.

Now we determine the rooted planar tree T' we get from D' under the map ψ. The Dyck word of D gives us the planar code for T which has k edges and $k + 1$ nodes. Recall that $D' = 1 \cdot D \cdot 2$, and if we use D' as the planar code for T', then clearly T' has $k + 1$ edges and $k + 2$ nodes. Also, the node which is added to T using D' as the planar code is the new root node, and the edge that is added is attached to the node which was the root of T. Thus we add a new node 1 and and every node other than 0 is increased by one.

Now we check that these are the two objects which correspond to each other through Theorem 6.5. T clearly gives the shape of (ν, J) since the longest path corresponds to the longest row of our rigged configuration. Since the added edge and node is added to the root node, each branch position on the tree increases by one, except the position of the longest branch which stays 0. This increase in branch position corresponds to the rigging of (ν, J). Thus the maps commute for the first case.

<u>Case 2:</u> Let D be a Dyck path which has at least one return to the main diagonal away from the endpoints. Thus we can write $D = D' \cdot D''$, where D' and D'' are non-empty Dyck paths. By the inductive hypothesis, D' and D'' correspond to T' and T'' respectively under ψ and (ν', J') and (ν'', J'') respectively under Φ, and T' and T'' correspond to (ν', J') and (ν'', J'') respectively under π.

Now we show that when we concatenate the Dyck paths the maps still commute. When we join the two Dyck paths, we must preserve order, so we must attach D_2 to the right side of D_1. We show that

$$(6.1) \qquad \Phi^{-1}(D) = \Phi^{-1}(D') \cdot \Phi^{-1}(D'') = \pi^{-1}(\psi(D') \cdot \psi(D'')) = \pi^{-1}(\psi(D)).$$

We first show that $\psi(D') \cdot \psi(D'') = \psi(D)$. This is clear by considering the Dyck path sequence as the planar code. Since $D = D' \cdot D''$ and the last node of D' is the first node of D'', then $\psi(D') \cdot \psi(D'') = \psi(D)$ and $\pi^{-1}(\psi(D') \cdot \psi(D'')) = \pi^{-1}(\psi(D))$.

Now consider (ν', J') and (ν'', J'') with partitions ν' and ν''; also, let $(\nu, J) = (\nu', J') \cdot (\nu'', J'')$. We interweave the rows of (ν'', J'') into (ν', J') according to row length.

Lemma 3.5 shows the rows and riggings coming from (ν', J') in (ν, J), are unchanged. However, the vacancy number $p_i(\nu')$ will increase based on the number of rows of (ν'', J'') which are longer than i. Also, the rows (i, x) which come from (ν'', J'') change to $(i, p_i(\nu') + x)$. It follows from Lemma 5.10 that $\Phi^{-1}(D') = (\nu', J')$. Also, we have $D^* = (D'')^* \cdot (D')^*$. We know that taking a reversed Dyck path through the KKR bijection will give us same partition, but the riggings of the reverse direction will be the coriggings of the forwards direction by applying $(\theta \circ \Phi \circ *)(\mathcal{D}_{2n})$, where θ and $*$ are from Theorem 5.8. So by Lemma 5.10 we know that once $\Phi^{-1}(D'')$ is interweaved back into $\Phi^{-1}(D')$ the riggings supplied by $(i, x) \in \Phi^{-1}(D'')$ will be $p_i(\nu') + x$. Therefore, $\Phi^{-1}(D) = \Phi^{-1}(D') \cdot \Phi^{-1}(D'')$.

Now we show that $\Phi^{-1}(D' \cdot D'') = \pi^{-1}(\psi(D' \cdot D''))$. Recall that by induction hypothesis, the diagram commutes for D', T' and (ν', J') and for D'', T'' and (ν'', J''). Once interweaved together, the rows (i, x) supplied by (ν', J') will be unchanged, and the rows (y, j) supplied by (ν'', J'') will have partition unchanged and riggings $p_i(\nu') + y$. Also, for a branch b' of length ℓ' on T', the number of ℓ'-valid nodes will increase by the number of ℓ'-valid nodes of T''. Analogously, for a branch b'' of length ℓ'' on T'' the number of ℓ''-valid nodes will increase by the number of ℓ''-valid nodes of T'. Thus, $\Phi^{-1}(D') \cdot \Phi^{-1}(D'') = \pi^{-1}(\psi(D') \cdot \psi(D''))$. Therefore, we have shown (6.1).

Finally, since the two cases are sufficient for considering all Dyck paths of a given length, we have shown that $\psi \circ \Phi = \pi$.

$\square$

Now we can see that the examples at the end of Sections 3, 4, and 5 illustrate the commutative diagram for the $n = 4$ case of each of the combinatorial objects. In particular, the Dyck path (i) of Example 5.12 corresponds to the rigged configuration (i) of Example 3.8 through the KKR bijection which corresponds to the rooted planar tree (i) of Example 4.8 through the bijection π given in Theorem 6.5 (i.e

$$
\begin{array}{ccc}
\text{Example } 3.8(i) & \xrightarrow{\ \Phi\ } & \text{Example } 5.12(i) \\
& \searrow{\scriptstyle \pi} & \downarrow{\scriptstyle \psi} \\
& & \text{Example } 4.8(i)
\end{array}
$$

for all i in $\{1, \ldots, 14\}$).

Corollary 6.7. *Let D be a Dyck path, $T = \psi(D)$, and $(\nu, J) = \pi^{-1}(T)$. Then*

$$
\begin{aligned}
h(D) &= d(T) = \nu_1, \\
p(D) &= v(T) = \ell(\nu).
\end{aligned}
$$

Proof. These equalities follow by considering $\Phi(D) = \pi^{-1}(\psi(D))$, and following the construction of the two bijections. Indeed, let D have height $h(D)$. Let r_i and s_i be the number of occurrences of 1 and 2, respectively, before the ith position in the planar code. Let $T = \psi(T)$. It is clear that $h(D)$ is equal to the depth of T. Since the algorithm of π^{-1} makes the lengths of the branches of T the lengths of the rows of (ν, J), $d(T) = \nu_1$.

Let D have $p(D)$ peaks. Since a peak is exactly when a 1 in the Dyck word is followed immediately followed by a 2, and the same criteria must be satisfied for T to have a leaf, then $p(D) = d(T)$. Since the number of leafs is also the number of branches, and π^{-1} makes each branch a row of ν, then $d(T) = \ell(\nu)$. Therefore the equalities are proved.

$\square$

7. Further Research

It was our hope that by giving an explicit commutative diagram that we might be able to interpret the statistics bounce or dinv of Dyck paths in either rooted planar trees or rigged configurations. This might have been helpful proving the symmetry of the q, t-Catalan numbers combinatorially. These ideas have not come to fruition yet; however, in future projects we could consider other Catalan objects and construct bijections from Dyck paths with the desired results. For more information on the q, t-Catalan numbers, see [Hag08].

q-**analogue.** Let $0 < q < 1$, and let $n \in \mathbb{Z}_{\geq 0}$. The q-analogue of n is

$$
(7.1) \qquad\qquad [n] := \frac{1 - q^n}{1 - q} = (1 + q + q^2 + \cdots + q^{n-1}).
$$

The q-analogue of $n!$ is

$$
(7.2) \qquad\qquad [n]! := [n] \cdot [n - 1] \cdots [1].
$$

The q-analogue of binomial coefficients is

(7.3)
$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.$$

One natural q-analogue and equivalence from [Mac04, FH85] of C_n is given by

$$C_n(q) = \frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix} = \sum_{D \in \mathcal{D}_{2n}} q^{\mathrm{maj}(D)}.$$

The fermionic formula from [HKO$^+$02, HKO$^+$99] is

$$M(k, w; q) = \sum_{\substack{\nu \in C(k;w) \\ wt(\nu)=w}} q^{cc(\nu)} \prod_i \begin{bmatrix} m_i + p_i \\ m_i \end{bmatrix},$$

where $C(k; w)$ is the set (without repetition) $\{\nu \mid (\nu, J) \in \mathrm{RC}(k; w)\}$. Note that

$$M(k, w; q) = \sum_{(\nu,J) \in \mathrm{RC}(k;w)} q^{cc(\nu,J)} \prod_i \begin{bmatrix} m_i + p_i \\ m_i \end{bmatrix}.$$

By Theorem 5.7 and the fact that $\begin{bmatrix} m_i + p_i \\ m_i \end{bmatrix}$ is the generating function of partitions in a $p \times m$ box where $p, m \in \mathbb{Z}_{>0}$, we have

$$\sum_{D \in \mathcal{D}_{2n}} q^{\mathrm{maj}(D)} = \sum_{(\nu,J) \in \mathrm{RC}(2n;0)} q^{cc(\nu,J)} \prod_i \begin{bmatrix} m_i + p_i \\ m_i \end{bmatrix}.$$

Thus, we have that

$$C_n(q) = M(2n, 0; q),$$

giving a fermionic interpretation of the Catalan numbers. In particular, when $q = 1$ we have

$$M(2n, 0; 1) = \sum_{(\nu,J) \in \mathrm{RC}(2n;0)} (1)^{cc(\nu,J)} \prod_i \binom{m_i + p_i}{m_i}$$

$$= \sum_{\nu \vdash n} \prod_i \binom{m_i + q_i}{m_i} = \frac{1}{n+1} \binom{2n}{n} = C_n.$$

where $q_i(\nu) = \sum_{j=1}^{\ell(\nu)} \min(\nu_j - i, 0)$ (i.e., the number of boxes strictly to the right of the ith column).

q, t-Catalan Numbers. By considering the statistics area and bounce of Dyck paths, we can get a combinatorial formula for the q, t-Catalan numbers. The formula for the q, t-Catalan numbers denoted $C_n(q, t)$, given and proved in [Mac04], is

(7.4)
$$C_n(q, t) = \sum_{D \in \mathcal{D}_{2n}} q^{\mathrm{area}(D)} t^{\mathrm{bounce}(D)}.$$

Theorem 7.1. *Let $C_n(q, t)$ be the defined as in Equation (7.4). Then $C_n(q, t) = C_n(t, q)$.*

It has been proven algebraically by using diagonal harmonics [Hag03, GH01, GH02]. However, it is an open problem to show this combinatorially.

Open Problem 7.2. *Construct an explicit bijection $\Xi : \mathcal{D}_{2n} \longrightarrow \mathcal{D}_{2n}$ such that $\mathrm{area}(D) = \mathrm{bounce}(\Xi(D))$ and $\mathrm{bounce}(D) = \mathrm{area}(\Xi(D))$.*

It would be interesting to determine an interpretation of area, bounce, and dinv on rigged configurations, especially if the statistics could help in solving Open Problem 7.2.

References

[FH85]   J. Fürlinger and J. Hofbauer. q-Catalan numbers. *J. Combin. Theory Ser. A*, 40(2):248–264, 1985.

[GH01]   A. M. Garsia and J. Haglund. A positivity result in the theory of Macdonald polynomials. *Proc. Natl. Acad. Sci. USA*, 98(8):4313–4316, 2001.

[GH02]   A. M. Garsia and J. Haglund. A proof of the q, t-Catalan positivity conjecture. *Discrete Math.*, 256(3):677–717, 2002. LaCIM 2000 Conference on Combinatorics, Computer Science and Applications (Montreal, QC).

[Hag03]   J. Haglund. Conjectured statistics for the q, t-Catalan numbers. *Adv. Math.*, 175(2):319–334, 2003.

[Hag08]   James Haglund. *The q, t-Catalan Numbers and the Space of Diagonal Harmonics: with an Appendix on the Combinatorics of the Macdonald Polynomials*, volume 41 of *University Lecture*. American Mathematical Society, Providence, Rhode Island, 2008.

[HKO$^+$99]   G. Hatayama, A. Kuniba, M. Okado, T. Takagi, and Y. Yamada. Remarks on fermionic formula. In *Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998)*, volume 248 of *Contemp. Math.*, pages 243–291. Amer. Math. Soc., Providence, RI, 1999.

[HKO$^+$02]   Goro Hatayama, Atsuo Kuniba, Masato Okado, Taichiro Takagi, and Zengo Tsuboi. Paths, crystals and fermionic formulae. In *MathPhys odyssey, 2001*, volume 23 of *Prog. Math. Phys.*, pages 205–272. Birkhäuser Boston, Boston, MA, 2002.

[KKR86]   S. V. Kerov, A. N. Kirillov, and N. Yu. Reshetikhin. Combinatorics, the Bethe ansatz and representations of the symmetric group. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 155(Differentsialnaya Geometriya, Gruppy Li i Mekh. VIII):50–64, 193, 1986.

[Kos09]   Thomas Koshy. *Catalan Numbers with Applications*. Oxford University Press, New York, New York, 2009.

[KR86]   A. N. Kirillov and N. Yu. Reshetikhin. The Bethe ansatz and the combinatorics of Young tableaux. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 155(Differentsialnaya Geometriya, Gruppy Li i Mekh. VIII):65–115, 194, 1986.

[KSS02]   Anatol N. Kirillov, Anne Schilling, and Mark Shimozono. A bijection between Littlewood-Richardson tableaux and rigged configurations. *Selecta Math. (N.S.)*, 8(1):67–135, 2002.

[Luo13]   Jianjin Luo. Ming Antu and his power series expansions. In *Seki, founder of modern mathematics in Japan*, volume 39 of *Springer Proc. Math. Stat.*, pages 299–310. Springer, Tokyo, 2013.

[Mac04]   Percy A. MacMahon. *Combinatory analysis. Vol. I, II (bound in one volume)*. Dover Phoenix Editions. Dover Publications, Inc., Mineola, NY, 2004. Reprint of ıt An introduction to combinatory analysis (1920) and ıt Combinatory analysis. Vol. I, II (1915, 1916).

[S$^+$14]   W. A. Stein et al. *Sage Mathematics Software (Version 6.2)*. The Sage Development Team, 2014. http://www.sagemath.org.

[Sag01]   Bruce E. Sagan. *The symmetric group*, volume 203 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2001. Representations, combinatorial algorithms, and symmetric functions.

[SCc08]   The Sage-Combinat community. Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics, 2008. http://combinat.sagemath.org.

[Sch03]   A. Schilling. Rigged Configurations and the Bethe Ansatz. In T. Lulek, B. Lulek, and A. Wal, editors, *Symmetry and Structural Properties of Condensed Matter*, pages 201–223, July 2003.

[SS06]   Anne Schilling and Mark Shimozono. $X = M$ for symmetric powers. *J. Algebra*, 295(2):562–610, 2006.

[Sta99]   Richard P. Stanley. *Enumerative combinatorics. Volume 2*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1999.

[Sta12]   Richard P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.

[Sta15]   Richard P. Stanley. *Catalan Numbers*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2015.

(Ryan Reynolds) Department of Mathematics, UC Davis, One Shields Ave., Davis, CA 95616-8633, U.S.A.

E-mail address: rnreynolds@ucdavis.edu