The Helly Number of the Prime-coordinate Point Set

By

KEVIN BARRETT SUMMERS

SENIOR THESIS

Submitted in partial satisfaction of the requirements for Highest Honors for the degree of

BACHELOR OF SCIENCE

in

MATHEMATICS

in the

COLLEGE OF LETTERS AND SCIENCE

of the

UNIVERSITY OF CALIFORNIA,

DAVIS

Approved:

Jesús A. De Loera

December 2015

ABSTRACT. In this senior thesis, we study many different properties of symmetric point sets, focusing on points with only prime coordinates. The ultimate goal of this project is to find the Helly number of this prime space. This result is equivalent to finding the largest number of edges that a convex empty polygon is able to have within the space. While there are numeric ways to describe the convex bodies of this space, our approach focused on possible geometric constructions restricted to this point set. Since there is very little information currently availible about this prime-prime point set, we begin with basic observations that explain where convex empty bodies are allowed to exist. Since the distance between two consecutive prime numbers can vary to a large degree, and since it is difficult to check the primality of very large integers, it is difficult to make claims about points with large coodinates. For this reason, examples take advantage of relatively small coordinates where possible. The included observations give a heuristic analysis of where different types of polygons can be both convex and empty, and hueristically show how it becomes increasly difficult to add new edges while remaining convex and empty. While polygons with more than five sides are not distinguished into classes, we discuss how the different classes of pentagons result in different edge expansions. Prior to this report, the largest convex empty polygon constructed in this space was a 12-gon. Using a modified backtrack algorithm implemented in Sage, we were able to find a convex empty 14-gon. The results of this undergraduate thesis were incorporated into the research paper "Helly Numbers of Algebraic Subsets of $\mathbb{R}^d$" [**5**], coauthored by J. A. De Loera, R. N. La Haye, D. Oliveros, and E. Roldán-Pensado, where they discuss Helly numbers of different spaces.

Contents

CHAPTER 1

Primes × Primes

1.1. Convex Geometry

Let's begin with some simple concepts of convex geometry. These will provide context to many of the terms used throughout the rest of the paper.

DEFINITION 1. We call a set, S, *convex* if all points A and B in S are positioned such that the segment AB lies entirely in S.

For example, a crescent moon shape would **not** be convex, because if one were to take a specific pair of points and connected them with a line, the line would lie outside of the crescent moon.

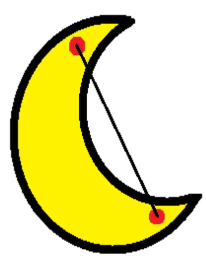

Figure 1: An example of a set which is not convex.

DEFINITION 2. The *boundary* of a set S, is a subset of S. A point, x, is in the *boundary* of S if every circle with center at x has both a point inside S and a point outside S.

DEFINITION 3. The *interior* of a set, S, is a subset of S. A point, x, is in the *interior* of S if x is simultaneously an element of S and not an element of the boundary of S.

DEFINITION 4. We call a polygon *empty* if there are no points in the interior of the polygon. This means the only points of the polygon are its boundary.

The last definition can be confusing when only considering the space $\mathbb{R}^2$. Because $\mathbb{R}^2$ is dense, the space in the interior of a polygon must contain points, and thus no polygon can be empty. However, this paper will be working with a point set that is a subset of $\mathbb{R}^2$, that is not dense in this same way.

An example of an empty polygon can be seen when taking a space that consists of only three points equally spaced from each other in the plane. Connecting these three points creates an empty triangle, T. We know T is empty because there are only 3 points in the space, and there are 3 points in the boundary of T. Thus the interior of T must be the empty set.

1.2. The Problem

Say we are given the set of all points in the two dimensional standard Cartesian coordinate system, $\mathbb{R}^2$, such that both coordinates are prime. *Is it possible, using these points, to construct a convex polygon of any given number of edges, such that no prime coordinate point resides in the interior of the constructed polygon?*

Let $P^2 := \{ (p, q) \mid p, q \text{ are prime}\}$ be the subset of $\mathbb{Z}^2$ such that each point's coordinates are both prime. Then, for all integer $n > 2$ does there exist a convex polygon with n edges with vertices in P^2 such that no points of P^2 lie in the polygons interior? Or, alternatively, does there exist some maximal n, such that for all integer $m > n$ no convex m-gon exists using points in P^2 as vertices with no point in P^2 lying in its interior?

This is an open question in mathematics, meaning currently it has not been solved. However, this paper attempts to gain a better understanding of this question. At present the polygon with the most edges that has been constructed that is both convex and empty has been a 14-gon.

1.2.1. Why is this interesting. There are two crucial limitations being put on the polygons being constructed in this problem. The first is that the polygons are convex. If we were to ignore this criterion it would be quite easy to construct polygons of any given amount of edges using the points of P^2 as vertices. However, there are far fewer polygons with vertices in P^2 that are convex.

The second limitation being imposed is that no interior point of such a polygon lies in P^2. Again, if this criterion were to be ignored then the solution would be near trivial, and it would be easy to show that as long as there are infinitely many primes, one could make a polygon with any given number of edges.

However, imposing these two limitations makes this question far more interesting. The origin of this question helps illuminate why it is being asked and why these limitations are being made. There is a famous theorem dealing with convex bodies known as *Helly's Theorem*.

HELLY'S THEOREM. Suppose K is a family of at least $d+1$ convex sets in affine d-space $\mathbb{R}^d$, and K is finite or each member of K is compact. Then each $d+1$ members of K have a common point, there is a point common to all members of K. [**3**]

Now, this $d+1$ has come to be known as the *Helly number* of $\mathbb{R}^d$. What about for convex bodies in subsets of $\mathbb{R}^d$? For example there is a Helly-type theorem proved by Doignon in 1973, that talks about the existence of intersections over the integer lattice $\mathbb{Z}^d$. It states that a finite family of convex sets in $\mathbb{R}^d$ intersect at a point of $\mathbb{Z}^d$ if every 2^d of members of the family intersect at a point of $\mathbb{Z}^d$ [**5**]. This means that 2^d is the Helly number of $\mathbb{Z}^d$. In other words, Helly's Theorem can be generalized to subsets of $\mathbb{R}^d$ if such a Helly number is found to exist.

This brings us to the problem at hand. If it is indeed possible to create a polygon with any arbitrarily large amount of edges with vertices in P^2 with no interior points in P^2, then this means that there would be no finite Helly number for P^2. This is the result of a theorem relating to the Helly number of subsets of $\mathbb{R}^d$.

THEOREM 1. *Assume $S \subset \mathbb{R}^d$ is discrete, then the Helly number of S is equal to the supremum of the number of vertices of an $S - vertex - polytope$ where we define an $S - vertex - polytope$ as the convex hull of points $x_1, x_2, ..., x_k \in S$ in convex position such that no other point of S is in conv($x_1, ..., x_k$)* [**5**].

If one could construct a polygon of arbitrary number of edges in P^2 then such a supremum would be ∞. This means it would not be possible to generalize Helly's Theorem to the space P^2 containing only prime numbers. This is quite a significant claim that relies completely on the conjecture that such polygons with arbitrary number of edges exist.

1.3. The Summary of the proposed Solution

The method used by this paper in order to provide information towards a solution to our open question is to find out what we can say about each different polygon in P^2 as we increase the number of edges. Starting with triangles, then quadrilaterals, then pentagons, and so on, this paper will explain where such polygons exist in P^2 such that they also satisfy the two limitations outlined in the introduction. It is true, but not trivial, to show that any point in P^2 is a vertex to infinitely many triangles that satisfy the two limitations, but is the same thing true for a polygon like a square, or a pentagon instead? This is the sort of question this paper will discuss.

The goal of this method of approach is to give broad general statements dealing with the polygons of P^2, in order to give a heuristic analysis of the differences between polygons as more and more edges are added. As we discuss polygons with larger numbers of edges, there will be more and more limitations on where such polygons are able to exist in P^2. Such a heuristic analysis can give us information about P^2.

1.4. Technical Details

1.4.1. Further Background Information. In order to work with P^2 as a geometric space, this involves understanding the gaps between prime numbers. When taking P^2 as a subset of $\mathbb{R}^2$ the gaps between points in P^2 correspond to empty space. This empty space is not a concern while working in $\mathbb{R}^2$ because as a space it is dense. As will be seen later, whether or not the space between two lines in P^2 is empty, or if there exist points in P^2 between said lines will be central to our construction of polygons. That being said, there are tools from number theory which can help give insight towards P^2 as a geometric space.

Twin Primes Conjecture. *There exist infinitely many pairs of primes whose difference is 2.*[**4**]

While this conjecture has not been proven, there has recently been headway towards new results, such that there exist infinitely many pairs of primes whose difference is less than 7×10^7 [**6**]. This result is the first proof that there are infinitely many prime gaps smaller than some given constant. P^2 is particularly difficult to work in once the prime coordinates become significantly large, because it is difficult to immediately recognize a seven digit number as prime or not. Knowing there will always exist gaps smaller than 7×10^7 could be significant while working with very large prime numbers in P^2. Also, if we assume the twin prime conjecture to be true, it gives us that there are infinitely many points in P^2 on the line through $(5, 3)$ and $(7, 5)$, as will be discussed later.

de Polignac's Conjecture. *For any positive even integer 2k, there exist infinitely many pairs of consecutive primes that differ by 2k.*[**4**]

The de Polignac Conjecture is very similar to the twin primes conjecture, but is more general. If this conjecture were true it would guarantee that there are infinitely many points on all the lines that are parallel to the main diagonal of P^2. Knowing whether lines continue infinitely in P^2 is a problem that needs to be considered while working in P^2 as a geometric space.

The Green-Tao Theorem. *For any positive integer k, there exists a prime arithmetic progression of length k.*[**4**]

The Green-Tao theorem, proved in 2004, is an extension of Szemerdi's theorem dealing with arithmetic progressions in subsets of integers. An arithmetic progression is a sequence of numbers such that the difference between the consecutive terms is constant. The length k described in the Green-Tao theorem is not this constant difference in terms, but is instead the number of terms in the sequence. The Green-Tao theorem guarantees the existence of progressions of prime numbers with k terms, but does not actually give the progressions. For example, the first known prime arithmetic progression of length 26, involves prime numbers consisting of 17 digits, and was not found until 2010. [**1**] These prime arithmetic progressions, especially when looked at in coordinates of points in two dimensions, could be useful in constructing large polygons in P^2.

1.4.2. Working in P^2. A well-known theorem in number theory that will be of significant use while working in P^2 is that there are infinitely many prime integers. Now define the set $P := \{ p_0, p_1, ..., p_n, ...\}$ to be the infinite set of consecutive prime numbers. If we take $P \times P = \{(p, q) \mid p, q \in P\}$, this is precisely P^2 as defined before. The set P will be useful while looking at both triangles and quadrilaterals in P^2.

While working with polygons with vertices in P^2 the edges of such polygons can be thought of line segments between the vertices. In order to work with such edges it is important to make clear what a line is in P^2.

DEFINITION 5. Let three points in P^2 be *collinear in* P^2 if they are collinear in $\mathbb{R}^2$. A *horizontal line* in P^2 is a set of points of the form $\{ (p_i, q) \mid$ for all $p_i \in P,$ and with fixed $q \in P\}$. A *vertical line* in P^2 is a set of points of the form $\{ (r, p_i) \mid$ for all $p_i \in P,$ and with fixed $r \in P\}$. A *line through two points* X *and* Y in P^2, is the set of all points in P^2 on the line in $\mathbb{R}^2$ through X and Y.

We know that infinitely many points in P^2 lie on any given horizontal line, because we know that P is infinite. The same is true for vertical lines. However, for lines in P^2 that are not horizontal or vertical we must be more careful. For example, if we take the line in P^2 between the two points $(7, 3)$ and $(11, 2)$ this line has a slope of -4, so the next integer valued points on this line would be $(3, 4)$ and $(15, 1)$. However, $4, 15$, and 1 all do not lie in P, so these two points are not on our line in P^2. In fact, $(7, 3)$ and $(11, 2)$ are the only points of P^2 that lie on this line. However, there is a line in P^2 that is not vertical or horizontal that clearly has infinitely many points: the line corresponding to the line $x = y$ in the standard Cartesian coordinates of $\mathbb{R}^2$.

DEFINITION 6. The *main diagonal of* P^2 is the line consisting of the points $\{ (p_i, p_i) \mid p_i \in P\}$

Something interesting about P^2 is that it is symmetric with respect to this main diagonal line. If a point (s, t) is in P^2, then the point (t, s) must also be in P^2, because both $t, s \in P$.

If we assume the twin prime conjecture to be true, then this gives us that the line through $(5, 3)$ and $(7, 5)$ has infinitely many points as well. This is because we know there are infinitely many points (p_{i+1}, p_i) such that $p_{i+1} - p_i = 2$. These infinitely many points would lie on this line. This line is significant, because it is parallel to the main diagonal of P^2. This means that if the twin primes conjecture is true, P^2 has two parallel lines that both have infinitely many points, that are not horizontal and are not vertical. If there are infinitely many points (p_{i+1}, p_i) such that $p_{i+1} - p_i = 4$ then the line through $(7, 3)$ and $(11, 7)$ would also be infinite. This line is also parallel to both the main diagonal of P^2, and to the line through $(5, 3)$ and $(7, 5)$.

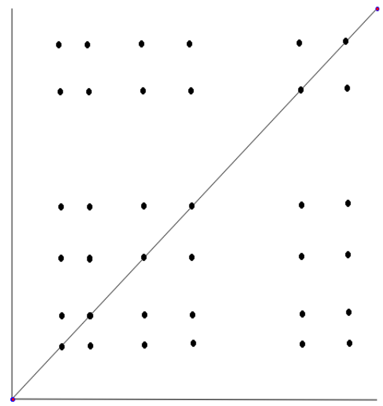

Figure 2: The main diagonal of P^2.

Also, while constructing polygons in P^2 it is useful to take the convex hull of a set of points. The convex hull of a set of points is the smallest convex body that contains said points. For example, a set of six or more points could define different hexagons depending on how the edges defined connect the vertices. However, since we are only concerned with convex polygons anyway, taking the convex hull of the six points mitigates this concern. However, the convex hull of k points does not always define a k-gon. For example the convex hull of three collinear points is simply a line segment, which is not a polygon at all.

If a polygon does not have any points lying in its interior, such a polygon can be referred to as *empty*. If a polygon is both convex and empty then it satisfies both the desired criteria of our problem.

1.4.3. Triangles of P^2. An interesting property of triangles is that a triangle must be convex in all standard Euclidean spaces. This is due a property of convex bodies.

Lemma 1. *If a polygon is convex it will have all interior angles less than 180°*

Lemma 2. *All triangles are convex polygons*

This result is not surprising after a bit of thought. The three angles of a triangle add up to 180° in $\mathbb{R}^2$ as well as in P^2, so by our first lemma the second lemma follows. It is left as an exercise for the reader to try to imagine a triangle which is not convex. This result allows us to know that all triangles contained in P^2 are convex, and thus one of the two criteria stated in the declaration of the problem will always be satisfied for any triangle in P^2.

The remaining criterion for the convex polygons in P^2 is that no point of P^2 lies in the interior of said polygon. It is quite easy to find a triangle where this is the case.

THEOREM 2. *There exists a triangle that is both convex and empty in P^2.*

PROOF. Take the points $a = (2,2)$, $b = (2,3)$, $c = (3,3)$. 2 and 3 are both prime, so $2,3 \in P$. Thus $a,b,c \in P^2$. There exist no prime numbers between 2 and 3, thus there does not exist any point in P^2 in the interior of the triangle formed by the points a, b, c. From our lemmas, we know this triangle is convex. Thus this triangle satisfies both the desired criteria. □

However, not only does one such triangle exist, but there are infinitely many triangles in P^2. In order to prove this easily, we will look at horizontal and vertical lines in P^2.

THEOREM 3. *There exist infinitely many convex empty triangles in P^2.*

PROOF. Let L be the vertical line through $(3, 3) \in P^2$. The points $(3, p_i)$ for all $p_i \in P$ are on L by definition. I claim that the triangle with vertices at $(2, 3), (3, 3)$, and $(3, p_k)$ form a triangle that satisfy both the desired criteria for all $k \in \mathbb{N}$. Because the vertical lines through $(2, 3)$ and $(3, 3)$ are parallel, and 2 and 3 are consecutive primes, no points in P^2 exist in the interior of these two lines. Thus taking the triangle formed by these three points will have no points of P^2 in the interior of the triangle. Also by our previous lemmas, all triangles are convex. Thus this triangle satisfies both criteria. Now, since this is true for infinitely many k, there exist infinitely many triangles that satisfy both criteria. □

However, there is an even stronger theorem that can be proved.

THEOREM 4. *Every point in P^2 is the vertex of infinitely many convex empty triangles in P^2.*

PROOF. Let (s, t) be a given point in P^2. We know $s = p_i$ for some $p_i \in P$ and that $t = p_j$ for some $p_j \in P$. We can take the vertical lines through (p_i, p_j) and (p_{i+1}, p_j). Since these lines are both vertical, they are parallel, and since p_i and p_{i+1} are consecutive primes, no points in P lie between these two parallel lines. As in the previous proof, $(p_i, p_j), (p_{i+1}, p_j)$, and a third point (p_{i+1}, p_k) form a triangle satisfying both criteria, for all points (p_{i+1}, p_k) on the vertical line through (p_{i+1}, p_j). Thus there are infinitely many triangles satisfying both criteria with the given point as a vertex. □

What's more is that the same can be said if one were to take the horizontal line through (p_i, p_j) and the horizontal line through (p_i, p_{j+1}), as well as the horizontal line through (p_i, p_j) and (p_i, p_{j-1}) provided $j \neq 0$. Triangles can also be formed between two parallel lines that are not horizontal or vertical as well, but there is a possibility of getting interior points within such triangles depending on the slopes of the parallel lines. Also, lines which are not vertical or horizontal are not guaranteed to have an infinite number of points in P^2 as discussed previously. For example, any point on the line through $(5, 3)$ and

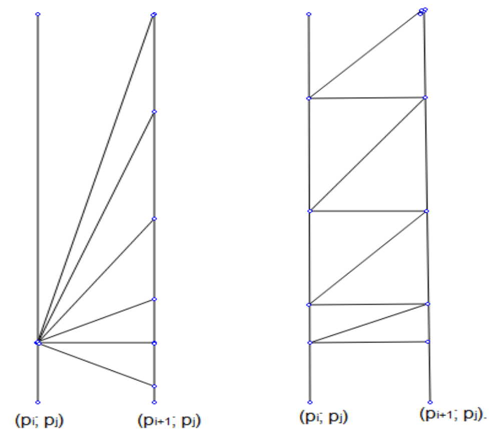

Figure 3: Two ways to visualize infinite triangles between two consecutive vertical lines.

$(7,5)$ in P^2 can create infinitely many triangles, using two points from the main diagonal in P^2, because these two lines are parallel, and we know the main diagonal has infinitely many points. The only point between these two parallel lines is the point (3,2), but this point is actually outside of where any such triangles can be constructed (see figure).

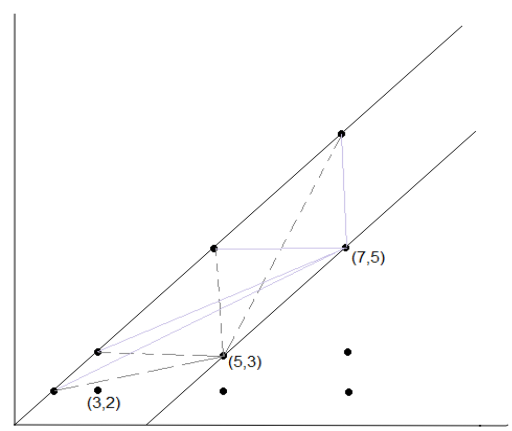

Figure 4: Triangles between the main diagonal and a parallel line.

An interesting consequence of each point in P^2 being the vertex of infinitely many triangles, with no interior points, is that if we are given some area and some vertex, we can find a triangle with no interior points using that vertex, with area greater than the given area. For now, assume the given vertex is not on the line $(2, p)$ for $p \in P$, then it lies on a vertical line, such that a consecutive parallel vertical line in P^2 is a distance of at least 2 away. The area of a triangle is half the base times the height, so if we consider the distance between the parallel lines as the base, we get the (1/2) and the 2 to cancel. Thus to construct a triangle with area greater than the given area, we just pick a prime number that is vertically that distance away from our initial vertex. Because there are infinitely many primes, we can assume there is a prime that is arbitrarily larger than one given. If the point that is given is on $(2, p)$ then the only consecutive vertical line is $(3, p)$ so our base is 1 instead of 2. To construct a triangle with area larger than the given area we just need to take a prime that is twice as far away from our given vertex vertically as the magnitude of the given area. Again, we know this can be done because we can find a prime that is arbitrarily larger than one given. So this tells us that not only are there infinitely many triangles in P^2 with no points of P^2 in their interior, but also that there are infinitely many distinct, incongruent such triangles as well. Simply put, since there is no biggest prime, there is no largest area that can be constructed using convex empty triangles in P^2.

1.4.4. Quadrilaterals in P^2. Much of what was true for triangles is also true for quadrilaterals in P^2. One important difference is that quadrilaterals no longer are required to be convex. However, if two sides of the quadrilateral lie on parallel lines, then the quadrilateral will be convex as long as it does not self-intersect; this follows from the lemma regarding interior angles of convex polygons being less than 180°.

Rather than going through every previous theorem's analogous theorem for quadrilaterals, we will skip to the strongest theorem, which implies the weaker ones.

THEOREM 5. *Every point in P^2 is the vertex of infinitely many convex empty quadrilaterals.*

PROOF. Let (s, t) be a given point in P^2. We know $s = p_i$ for some $p_i \in P$ and that $t = p_j$ for some $p_j \in P$. We can take the vertical lines through (p_i, p_j) and (p_{i+1}, p_j). Since these lines are both vertical, they are parallel, and since p_i and p_{i+1} are consecutive primes, no points in P^2 lie between these two parallel lines. Because of this, any two distinct transversals connecting these two parallel lines will not have any points in P^2 between them. Let lines l and m be two non-intersecting transversals between our vertical lines such that they intersect our vertical parallel lines at points in P^2, one of such points being our given point (s, t). Then let points A and B, be the points where l intersects our two parallel lines, and let points C and D be the points where line m intersects our two parallel lines. Then as long as the four points are distinct, the convex hull of A, B, C, and D, defines a parallelogram in P^2. As already explained since our vertical lines were consecutively parallel, this parallelogram has no points of P^2 in its interior. Since parallelograms must be convex, this parallelogram satisfies both our criteria. Since there

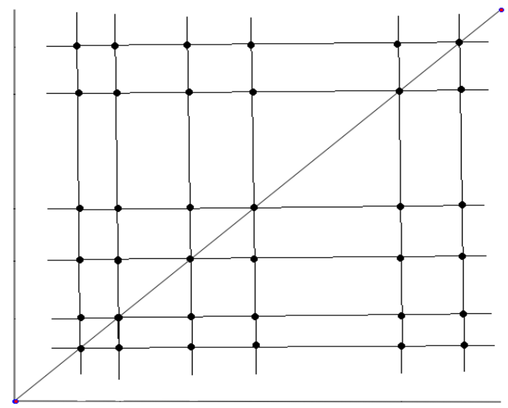

Figure 5: Drawing in all the vertical and horizontal lines of P^2 shows the prevalence of empty rectangles.

are infinitely many points on both our vertical lines, there are infinitely many choices of l and m. This implies that (s, t) is the vertex of infinitely many convex empty quadrilaterals in P^2. □

Also, using the symmetry of P^2 across the main diagonal we can note something interesting.

THEOREM 6. *There exist infinitely many convex empty squares in P^2.*

PROOF. Let d_0 be a point on the main diagonal of P^2. Let d_1 be the next consecutive point on the main diagonal of P^2. $d_0 = (p_i, p_i)$, and $d_1 = (p_{i+1}, p_{i+1})$. Let $n = p_{i+1} - p_i$. Then the points $(p_i, p_i), (p_{i+1}, p_i), (p_{i+1}, p_{i+1}), (p_i, p_{i+1})$ form a square with side lengths n. By the proof of the previous theorem, the vertical lines defined by the points (p_i, p_i), and (p_{i+1}, p_{i+1}), are parallel and consecutive, guaranteeing no points of P^2 lie in the interior of the square. Also it is trivial that squares must be convex. Therefore this square satisfies both criteria. Since d_0 was arbitrary on the main diagonal of P^2, we know there are infinitely many squares that satisfy both criteria in P^2. □

What is more is that any other convex empty square must be congruent to one of these squares along the main diagonal. In order for a square to be empty in P^2 it must be oriented so as to have any pair of vertices that define an edge of the square to lie on either the same vertical line or horizontal line. Let us call a square which is not oriented in this way *diagonally oriented*. No diagonally oriented square in can be empty in P^2, just as no regular pentagon can be empty in P^2, as will be discussed later. This is because the vertices of a diagonally oriented square must have at least three different y-coordinates. The y-coordinate that is between the other two will be the y-coordinates to some interior

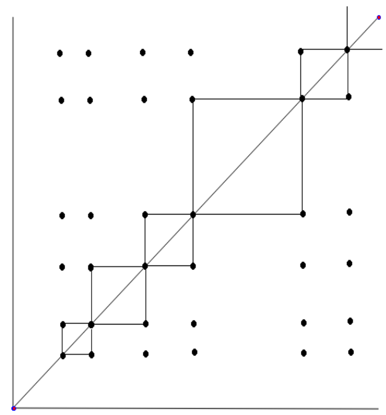

Figure 6: The main diagonal creates the diagonals to infinitely many convex empty squares in P^2.

points. The same can be said for the x-coordinates. Thus the only empty squares are those with vertices that only have two different x-coordinates and two different y-coordinates between them. Any square which is not diagonally oriented, can be transformed linearly in a single direction, either directly up, down, right, or left, so as to lay directly on a square of the main diagonal. Thus any convex empty square in P^2 must be congruent to a square that lies on the main diagonal.

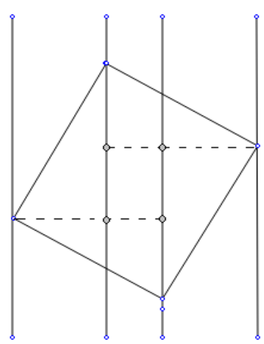

Figure 7: A diagonally oriented square must have these gray interior points.

1.5. Pentagons in P^2

Trying to find polygons with 5 edges in P^2 satisfying the desired criteria is significantly more difficult than polygons with 4 or 3 edges. In contrast to triangles and quadrilaterals, every point of P^2 may not necessarily be a vertex of infinitely many pentagons. For example the point $(2, 11) \in P^2$ is the vertex to at least three pentagons satisfying both criteria, but there is still work to be done to know if it can be the vertex to any other. Also, it is possible that there is a point in P^2 that is not the vertex to any pentagons satisfying the desired criteria but more information is required to know if this is true at this time.

While it is natural to consider the pentagons of P^2 after discussing quadrilaterals, the significant differences between them make pentagons much more useful in looking toward a solution to the overall problem. Say we are given a hexagon with vertices in P^2 that is both convex and does not contain any points of P^2 in its interior, then if we were to remove any vertex of this hexagon, we would be left with a pentagon that is still convex and empty in P^2. We could do this for each individual vertex of such a hexagon, meaning that such a hexagon contains six different pentagons that satisfy our criteria. Any convex empty polygon with an arbitrary number of edges greater than 5 will contain these pentagons. Of course any pentagon contains quadrilaterals and triangles as well, but because pentagons have many more restrictions on them in P^2 focusing on them tells us about where even larger polygons are able to exist.

However, despite having more restrictions than triangles and quadrilaterals in P^2, there are still infinitely many pentagons in P^2 satisfying our criteria, and in order to see this it is useful to once again look at the main diagonal of P^2.

THEOREM 7. *There are infinitely many pentagons in P^2 that are both convex and empty.*

PROOF. We are able to construct infinitely many pentagons satisfying both criteria while keeping 4 of the 5 vertices fixed. Take the points $(2, 2), (3, 2), (5, 3)$, and $(7, 5)$; these will be our fixed vertices. They are set up well to create distinct edges of different slopes in $\mathbb{R}^2$. The edge between $(2, 2)$ and $(3, 2)$ is horizontal, and the edge between $(3, 2)$ and $(5, 3)$ has a slope of 2 in $\mathbb{R}^2$. Our next edge between $(5, 3)$ and $(7, 5)$ has a slope of 1 and is thus parallel to the main diagonal. This allows us to create an edge along the main diagonal starting at $(2, 2)$ and ending at our fifth, not yet defined, vertex. Let us call our last vertex B. For any choice of B that lies on the main diagonal, that is not $(2, 2)$, the edge between B and $(7, 5)$ acts as a transversal between our two parallel edges. This means that the two interior angles created by our fifth edge will be supplementary for any choice of B. This means that we have five distinct points, and five distinct edges, and all of the interior angles are less than 180°. Thus this pentagon is convex. The line between (5,3) and (7,5) is the closest parallel line to the main diagonal, and the only point between the two lines is

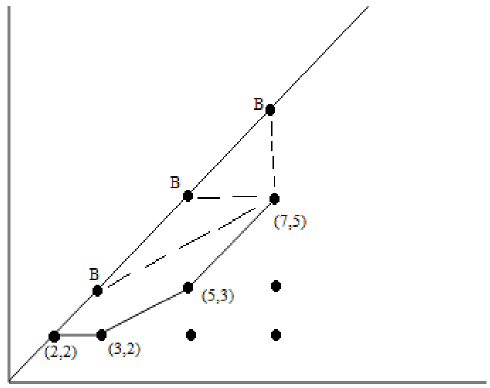

Figure 8: Infinitely many pentagons can be constructed as there are infinitely many choices of B.

the point (3,2). However, since we are using (3,2) as a vertex to our pentagon, it does not lie in our pentagon's interior. Thus since there are infinitely many points B that can be used, there are infinitely many pentagons in P^2 that satisfy both our desired criteria. $\square$

There are a few things to note about this proof before moving forward. We can take (2,2) to be our choice of B, but if we were to take the convex hull of our 5 points, only 4 points would be distinct, leaving us with a convex empty quadrilateral. Another thing to note, is that if we assume the twin primes conjecture to be true, then we can also let our fourth point $(7, 5)$ to not be fixed and choose any point A lying on the line between $(5, 3)$ and $(7, 5)$. The edge between A and B will still be a transversal and the construction will be the same. Lastly, since P^2 is symmetric about its main diagonal we can switch the x-coordinates and y-coordinates of our points and the construction will lead to another set of infinitely many convex empty pentagons in P^2.

This construction shows that infinitely many pentagons can be constructed along the main diagonal of P^2 with four of the five vertices fixed, but where else can pentagons be constructed so as to have no interior points of P^2? To answer this let us look at the structure of a pentagon. In order for five points to define a pentagon, the set of points must have at least three unique x-coordinates, and at least three unique y-coordinates. If one were to take the convex hull of a set of five points that only had two unique x-coordinates (or y-coordinates) then the hull would result in a quadrilateral, or a triangle.

Now, if we were to try to construct a regular pentagon with no interior points in P^2, we run into some immediate problems. Assuming we have points that consist of five unique y-coordinates and three unique x-coordinates, we know that one of those x-coordinates needs to be in between the other two. The vertical line in P^2 corresponding to this middle

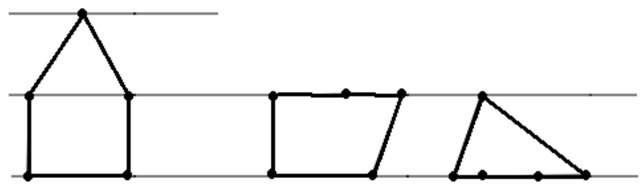

Figure 9: The convex hull of five points will only define a pentagon if the points have distinct enough coordinates.

x-coordinate is what gives our construction trouble. If we have two of the vertices lying on this vertical line, they must be such that their y-coordinates are consecutive primes, or else we are guaranteed to have interior points of P^2 inside our pentagon. Also keep in mind, that we can always switch our results for x-coordinates and y-coordinates, due to the symmetry of P^2 along its main diagonal. This significantly reduces the number of forms that convex pentagons can take in P^2 while still not having interior points.

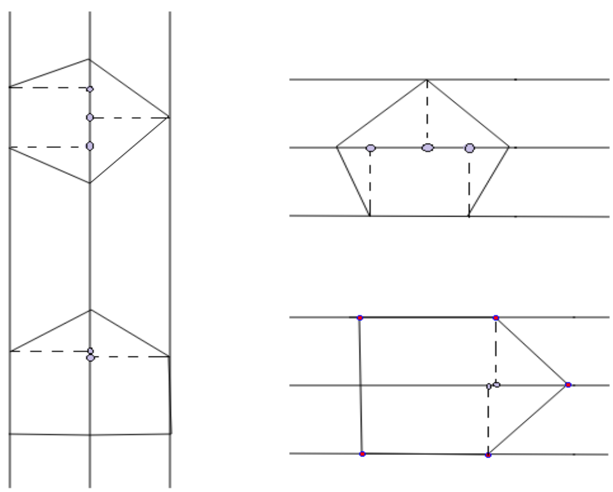

Figure 10: The structure of these pentagons guarantee interior points in P^2, the interior points in gray must be points in P^2 if the vertices are also in P^2.

Pentagons 1,2, and 3 in Figure 11 all are constructed using the same six points that could feasibly exist in P^2. Each one of them illustrates a problem that develops while

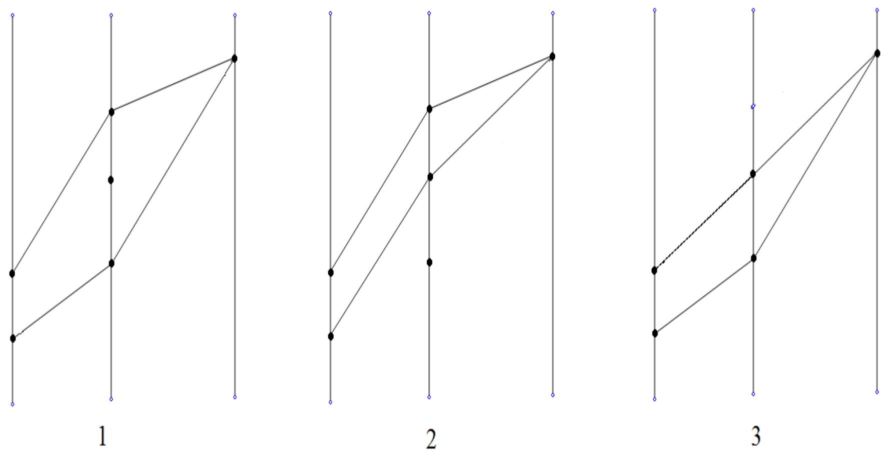

Figure 11: Further ways pentagons can fail the desired criteria.

trying to construct general pentagons in P^2. The problem with pentagon 1 is that, as mentioned before, the two vertices with the middle x-coordinate need to have consecutive y-coordinates as well. Because the y-coordinates are not consecutive primes, we are left with an interior point. Say we see the problem with pentagon 1, and try to construct a new pentagon with consecutive primes, and end up with pentagon 2. Well, pentagon 2 is clearly not convex, since it has an interior angle greater than 180°. Pentagon 2 illustrates a key fact about the pentagons we wish to construct; in order for the pentagon not to have interior points, not only do the two middle vertices need to have consecutive prime y-coordinates, but the edges created by them and the other first two vertices, need to have a point in between them when extended out to our next x-coordinate, otherwise there will not be a possible convex pentagon that uses those first four vertices. Pentagon 3 could possibly work as a convex pentagon with no interior points, but a possible problem may occur in that all five edges may not be unique. In fact if three of the vertices are collinear, then they would only define a quadrilateral, despite the collinear points having all unique coordinates.

These two middle coordinate vertices are key to constructing our pentagons with no P^2 interior points, and therefore are key in constructing even larger polygons. Say we pick two vertices with the same x-coordinate (due to the symmetry along the main diagonal, it is the same to start with two with the same y-coordinates) that we wish to use to construct a pentagon. The next step would be to find these middle coordinate vertices, that must have consecutive y-coordinates (respectively x-coordinates), such that when connected via edges, these edges extended indefinitely would have a point in P^2 between them. However, this last point needs to be close enough to our middle points so as to not include any closer points that would end up as interior points of our pentagon. If any points exist between these two extended edges, then there must exist some closest point, so as to ensure no interior points. However, there could be multiple points in P^2 between these extended edges such that there are no interior points in our pentagon. For example, if the two points share the same x-coordinate, they would not include each other in their respective pentagons. When this happens, this is what allows for polygons with more edges than five. In this example where two points with the same x-coordinate both act as fifth points for respective empty pentagons, then these points could both be taken, so as to construct an empty hexagon. Now, what if we extend the edges between the two original middle vertices of our construction, with these last two vertices of this hexagon? We can again look for a point between these extended edges, and if one exists we would be able to create a heptagon. This sort of inductive algorithm for creating empty polygons in P^2 is quite astounding, but is very difficult to implore in practice. In order for this example heptagon to be empty, we need not only two points between our first extended edges, but we need these points to also have consecutive coordinates, because they too become a new second set of middle vertices.

The problem with trying to use this construction with any pentagon is simply that when we extend our first two edges we are not guaranteed to have any points of P^2 that lie between them. For example, if we take our first two vertices and connect them to points that are on the same horizontal lines of each of them respectively, then when we extend our edges, we get two consecutive parallel lines, which have no points of P^2 between them. So clearly, we cannot form an empty convex pentagon out of any four given points. However, if we are given these first two starting points, and we are able to search for a pair of middle vertices, it is very likely, that there will be some pair of middle vertices that work in the construction. We would be able to find another vertex between the extended edges so as to create an empty pentagon.

Finding the existence of this fifth point comes down to the slopes of the edges we are extending. As the figure of the heptagon construction suggests, the bigger the gap between the y-coordinates (respectively x-coordinates) of our middle vertices are, the more area of P^2 is between their extended edges. Also as the figure suggests, for polygons larger than pentagons, the largest vertical (respectively horizontal) gaps between any vertices on the same vertical (respectively horizontal) lines will tend to be towards the middle vertices of the polygon, and the vertices towards the ends of the polygons will be closer together. More on polygons larger than pentagons will be touched on later. These pentagons, in which two vertices are given, and then the middle vertices are found such that a fifth vertex defines an empty pentagon, are not the only types of pentagons that can be constructed in P^2. For the sake of organization we will call these pentagons Form 1. Due to the symmetry of P^2 along its main diagonal, we can reduce the eight different directions that Form 1 pentagons can be oriented, into only four directions. Taking the axes as cardinal directions, Form 1 pentagons will always be oriented such that their fifth vertex is, loosely speaking,

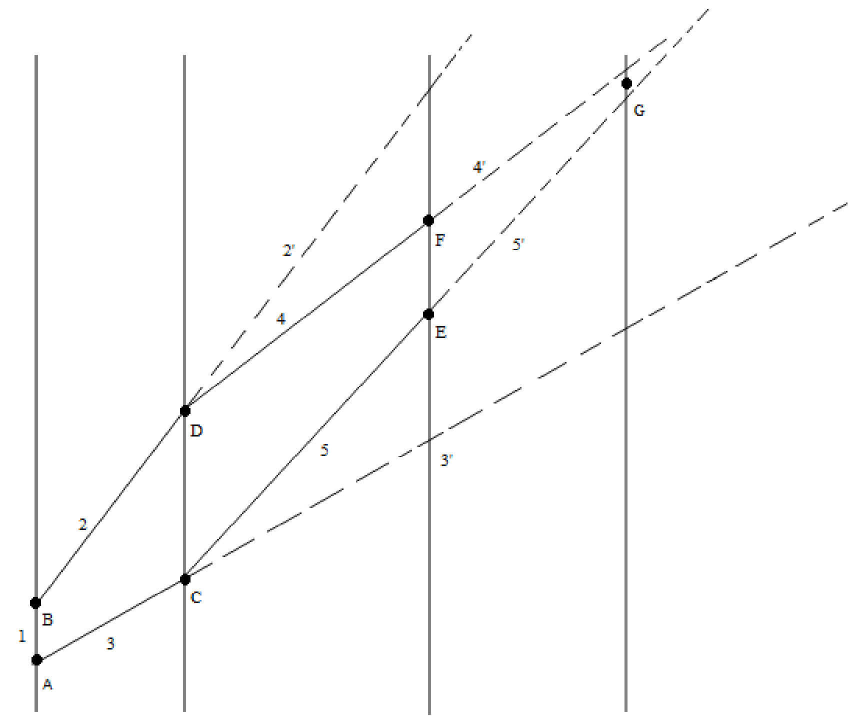

Figure 12: Proposed heptagon construction. Using C and D as our first middle vertices, if our extended edges, 2' and 3', have two points between them, E and F, that are consecutive, we can extend our edges 4 and 5, and search for a possible point G.

either north-east, south-east, south-west, or north-west, from their original two points. They cannot be directly north, south, east, or west, as then they would guarantee interior points, as shown in prior figures.

The next pentagons that will be discussed will be called Form 2. These pentagons, like Form 1, are constructed starting with two initial given points, but instead of having two middle vertices, will only have one middle vertex, and have two endpoint vertices. Form 2 pentagons, are in a way less important to construct than pentagons of Form 1, because for any Pentagon of Form 2, there must be some point that is collinear to the single middle vertex, with which the pentagon can be extended to a convex empty hexagon. Then, such an empty hexagon can be partitioned to as to separate an empty pentagon of Form 1, as any empty convex hexagon contains multiple convex empty pentagons. The only time this is not true, is if the sixth vertex that would define a convex empty hexagon were to be collinear with two of the endpoints on either side of it. In other words, just as the convex hull of five points could define an empty quadrilateral due to collinearities, so too can the

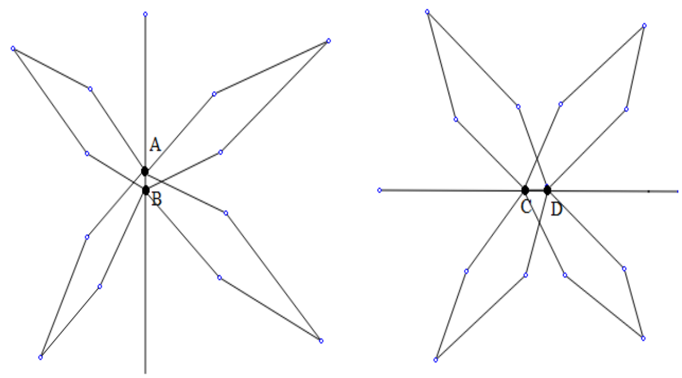

Figure 13: The eight orientations of Form 1 pentagons reduces to four when taking the symmetry of P^2 into account. A and B have the same x-coordinate where C and D have the same y-coordinate.

convex hull of six points define an empty pentagon. As such, Form 2 pentagons are more likely to be problems encountered in constructing convex empty hexagons as opposed to being easier to construct than Form 1 pentagons.

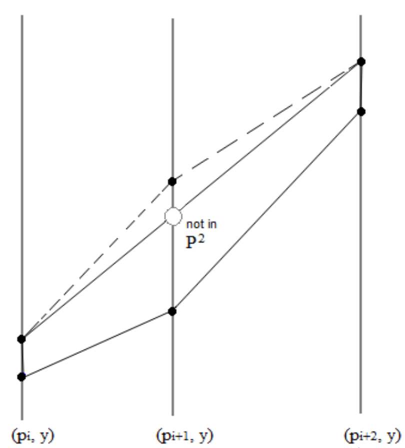

Figure 14: Pentagons of Form 2 can almost always be extended to convex empty hexagons.

The last types of pentagons, Form 3, will be any empty convex pentagons which are not Form 1 and are not Form 2. An example of a Form 3 pentagon has already been discussed;

in the proof of theorem 6, that there are infinitely many empty convex pentagons, all but one of these pentagons is of Form 3. The pentagons defined by the convex hull of the points (2,2), (3,2), (5,3), (7,5), and (3,3) is of Form 1, where (2,2) and (3,2) are our initial two points with the same y-coordinate, and (3,3) and (5,3) are our middle vertices with consecutively prime x-coordinates, leaving (7,5) as our fifth vertex lying in between the extended edges of our other four vertices. However, for the other pentagons constructed using the infinitely many points of the main diagonal, it is not clear which two vertices could be considered our middle two vertices. Because so many points can potentially lie on the main diagonal edge between (2,2) and our chosen last coordinate, the pentagons are left in a form that is not similar to Form 1 or Form 2. We are not able to orient the pentagon into two initial points, and it does not make sense to try and call some vertices middle vertices. This is due to the fact that more than five points of P^2 lie in the convex hull of our vertices. In a sense, just like what was discussed with special Form 2 pentagons being failed constructions of convex empty hexagons due to three vertices being collinear we can think of these Form 3 pentagons from theorem 6, as failures to construct larger polygons that ended up having collinear vertices all along the main diagonal. If a pentagon cannot be oriented to be of Form 1 or of Form 2, then it is likely that it has many collinear points of P^2 lying on one or more of its edges. A way to recognize if a pentagon is of Form 3 is if it is difficult to tell which vertices can be taken to be its middle vertices, and if on both of its ends on either side of its middle section it ends in one isolated point, meaning it doesn't have two points close together with the same x-coordinate or y-coordinate. It is important to check all the vertices in this way, because if two points are close together, and have the same x-coordinate or y-coordinate, it is very possible for this pentagon to be in one of the eight orientations of a Form 1 pentagon, without seeming so at first glance.

The purpose of discussing the different forms that empty convex pentagons of P^2 is to show that the construction discussed in which we found we could extend our pentagon into a hexagon and subsequently a heptagon is not the only way that empty pentagons in P^2 can exist. We have already discussed extending pentagons of Form 1, and it is likely that Form 2 pentagons can be extended to empty convex hexagons, but it is more difficult to discuss trying to extend pentagons of Form 3. However, despite some forms being easier or more difficult to work with, we do know that all such forms exist, and in fact theorem 6 proves there are infinitely many pentagons of Form 3, and infinitely many of differing area. It is suspected that there are infinitely many pentagons of both Form 1, and Form 2, but this will be discussed later.

A theorem that is very relevant to our empty convex pentagons has recently been published and gives insight into where such pentagons can exist in P^2. Adopting the theorem's notation, an empty pentagon in a point set P in the plane is a set of five points in P in strictly convex position with no other point of P in their convex hull. [**2**]

THEOREM 8. *Let P be a finite set of points in the plane. If P contains at least $328l^2$ points, then P contains an empty pentagon or l collinear points.*

This quadratic bound is optimal up to a constant factor since an $(l-1)\times(l-1)$ section of the square lattice has $(l-1)^2$ points and contains neither an empty pentagon nor l collinear points. Also, prior to this theorem, the smallest bound on the number of points required for the point set to contain an empty pentagon or l collinear points was doubly exponential in l. [**2**] Returning to our own notation, we know that P^2 has infinitely many points, so it contains $328l^2$ points for any l. However, we also know that P^2 has many vertical and horizontal lines, as well as an infinite main diagonal. Because of this, it is very likely to that some subset of P^2 with $328l^2$ points will have l collinear points. While this does not rule out the possibility of also having an empty convex pentagon, this theorem cannot be used to guarantee that one exists. However, if we were construct a subset of P^2 with at least $328l^2$ by constructing many specific bounds, and not deleting any points within those bounds, such that we knew there were not l collinear points in our subset, then this theorem would guarantee an empty convex pentagon, that we could then take to be in our original set of P^2 as well. In order to ensure we did not have l collinear points, we would have to bound in such a way as to avoid having l points all lying on the same vertical line, and also avoid having l points on the same horizontal line. It would also be useful to avoid the main diagonal as it could contain l collinear points as well. The resulting bounds would end up being diagonals of differing slopes all on the same side of the main diagonal. Guaranteeing that such a subset of P^2 did not have l collinear points would not be an easy task, and due to P^2 having so many vertical and horizontal lines, giving it a very rectangular nature, depending on the choice of l, such a subset may not even be possible to construct. Also, even if we were to use this theorem to guarantee ourselves a convex empty pentagon, we would not know which of the three forms of pentagons it would be, and thus not know how to go about trying to extend such a pentagon into an empty hexagon. Nevertheless, having another way to construct convex empty pentagons in P^2 could result in significant progress towards constructing empty polygons with even more edges in the future.

1.6. Polygons Larger Than Pentagons

As has already been discussed, if we are given a polygon with vertices in P^2 with more than five edges, that is convex and does not have any points of P^2 in its interior, then any five of its vertices will define a pentagon that is convex that does not have any points of P^2 in its interior as well. Because of this, it does not make sense to look for a larger convex empty polygon, in a place where we know there is not a convex empty pentagon. It has already been discussed how Form 1 and Form 2 pentagons can lead to empty polygons with more edges. However, when we discussed possibly turning a Form 1 pentagon into a convex empty heptagon, we only considered extending the edges between the initial points and the middle vertices. The idea of extending edges works for all the edges of our pentagon, no

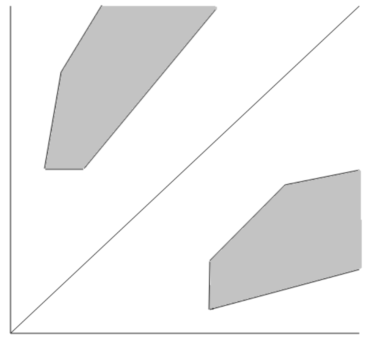

Figure 15: A rough representation of where a $328l^2$ point subset of P^2 could exist without l collinear points.

matter what form the pentagon is. If a point of P^2 lies between these extended edges, it remains to check that when we add it to our set of vertices, the convex hull does not leave us with any interior points. The same is true for larger polygons as well. If we wish to extend a hexagon to a heptagon, we can extend all of its edges, and the regions bounded by those extended edges are the only places where new vertices can be added, that will still result in a convex empty polygon. If we remove and vertex from a convex empty n-gon, then the remaining vertices must define a convex empty $(n-1)$-gon.

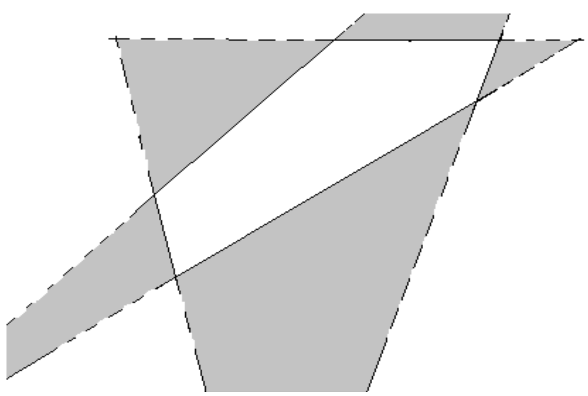

Figure 16: The regions in gray are the only regions where a point can be found that extends a convex empty pentagon to a convex empty hexagon.

A way in which polygons larger than pentagons differ from what we found with pentagons is that while they can be broken up into different forms, these forms don't have properties similar to the three forms of pentagons. For example, it is very likely that a pentagon of Form 2 can be extended to a convex empty hexagon, due to its single middle vertex having a neighboring vertex that can be added while keeping the resulting hexagon convex. However, for larger polygons, this is much more unlikely to happen. On top of the problem where the neighboring point is already collinear with other vertices, resulting in no new edge being constructible, with larger polygons, if we have a middle vertex that is not paired, it is possible that its potential neighbor middle vertex, when added as a vertex, would result in polygon that is no longer convex. This is not an issue with Form 2 pentagons, as Figure 14 pentagons shows, because we only have three different x-coordinates (respectively y-coordinates) for all our vertices. Adding another vertex in between our two extreme values cannot result in a hexagon that is not convex, because if such a point existed, it would have to exist inside the interior of our Form 2 pentagon. However, because we can have more than three different x-coordinates (respectively y-coordinates) for larger empty convex polygons, the same logic does not hold. The fact that neighboring middle vertices do not behave for larger polygons the same way that they do for pentagons shows us that there are definitely differences in polygons as more edges are added. Since convex empty polygons larger than pentagons require a convex empty pentagon to be present, larger polygons have not been categorized to the same extent as pentagons.

The convex empty pentagons of P^2 have given us a lot of insight into what larger polygons are able to look like, and where they are able to exist. However, they have not solved our overarching problem. We also do not know if every point in P^2 can be used as

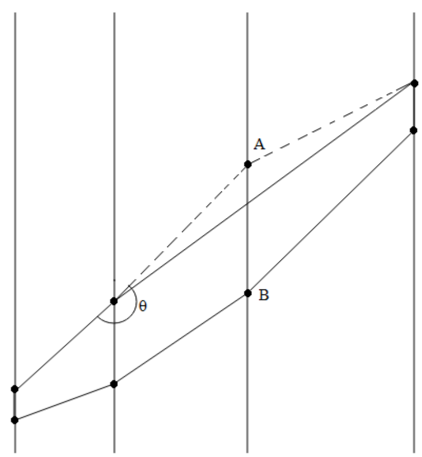

Figure 17: Despite points A and B having consecutively prime y-coordinates, we cannot extend this heptagon to include point A as a vertex of a new octagon, because angle θ would cause such an octagon to no longer be convex.

an initial point for our construction of Form 1 pentagons, or if there exists a point which is not the vertex of any convex empty pentagon. We do not know if there are infinitely many pentagons of Form 1 or of Form 2, or if there are, if we can find one of arbitrary area, like we can for triangles. There are many things that can still be discovered about the convex empty polygons of P^2.

While we are left without a complete solution to our problem, significant progress towards a solution has been made. By studying convex empty pentagons, where they are able to be located, and how they are oriented in P^2, we know that potential larger polygons must be built out on top of them. Also, there are many ideas that have yet to be completely explored that could lead to more interesting results in P^2.

CHAPTER 2

Ideas for Future Progress

2.1. Introduction

In this chapter, we will discuss the multitude of ideas that began to develop after the results from the first chapter had been discovered. After finding the results discussed in the first chapter of this paper I worked alongside my colleague Jianping Pan in order to further develop more specific ideas. While many of these ideas lead to dead ends, and did not provide us further information, these will be included due to their relevance to our overarching question. It is our hope that including under-developed ideas could lead to new results in the future. With this hope in mind, there will be an emphasis on the context behind why such ideas were developed. The reader should be left with an understanding of why we tried what we did, and what results we hoped to receive from these ideas. We will begin with a basic introduction to each of the ideas that will be included in this chapter.

The first experiment we conducted was trying to improve upon the only preexisting method of finding a bound of the Helly number of P^2. The method consisted of constructing, or rather finding, large polygons that had an edge on the main diagonal of P^2. Because the points of the main diagonal are in a one-to-one correspondence with P, it is relatively simple to look at points on the main diagonal that contained large coordinates. Each point of the main diagonal has a single point directly above it, and the space between these vertically consecutive points must be empty. The idea was to take advantage of this empty space. By looking for a pattern of strictly decreasing slopes between these points just above the main diagonal, we could construct a dome-shaped convex empty polygon, with the bottommost edge being a segment of the main diagonal. Because every point on the main diagonal is of the form (p_i, p_i), the vertically consecutive points must be of the form (p_i, p_{i+1}). This is convenient, because the slopes calculated are always between (p_i, p_{i+1}) and (p_{i+1}, p_{i+2}), which makes such calculations simple. Prior to our improvements to this idea, the largest polygon found was a 12-gon. By both improving on both the core of this idea, and by increasing the range of our algorithm to the fifty millionth prime number, we were able to find a 14-gon.

The remaining sections of this chapter will discuss our attempts at constructing large convex empty polygons in P^2 that do not require an edge to be a segment of the main diagonal. We will discuss multiple algorithmic ideas, and the problems associated with each. Due to the infinite size of P^2, we began by trying to find reasonable ways to bound our constructions. However, this too proved difficult.

2.2. Constructing Polygons on the Main Diagonal

Working within P^2 can be tricky due to how arbitrary the gaps between two consecutive primes can be. Despite the symmetries present in P^2, it is not uniform in the distancing between points. When working with particularly large prime numbers, for example $179,426,263$ it can be difficult to quickly know the prime number closest to it, which happens to be $179,426,239$. The smallest prime larger than our example of $179,426,263$ is the integer $179,426,321$. This means the vertical line at $x = 179,426,263$ has verticals lines with a distance of 58 on the right, $x = 179,426,321$, and a distance of 24 on the left, $x = 179,426,239$. This example illustrates how it is difficult to know how different edges with different slopes come together for very large values in P^2. Because of this, it is difficult to paint a picture of how relatively large gaps can interact with each other in P^2.

A way to mitigate the difficulty of working with large coordinates in P^2, was to stay along the main diagonal of P^2. This main diagonal line has many nice properties, which have been mentioned previously. One particularly useful property is that we know there are infinitely many points along the main diagonal. Allowing these collinear points to all lie on a single very large edge allows us to stretch our convex empty polygon to be long and flat, which is beneficial for constructing convex empty polygons in P^2.

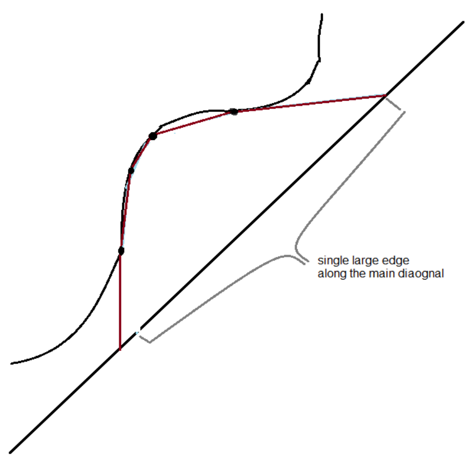

Figure 1: While looking at the curve defined by the points vertically consecutive to the main diagonal, dome shapes of empty space are used to construct polygons.

The main idea of constructing polygons along the main diagonal without looking elsewhere, as mentioned above, was because there was guaranteed empty space above each of these collinear points. Given the right configuration of this empty space, i.e. given there was a specific pattern in the distances between consecutive primes, we could fit a polygon within that empty space, using the points just above the main diagonal as vertices. We know polygons made this way must be empty, so the difficulty in the construction was to find large polygons that were convex.

Looking at the points vertically consecutive to the points of the main diagonal gives us these domed-shaped areas that are empty in P^2. To know where these domed shapes occur we look at the slopes of the lines connecting these points together. The curved line in the figure above is meant to approximate these slopes.

Since convex polygons must have interior angles that are less than $180°$, the slopes used to form the edges of a convex polygon must be strictly decreasing as the x-coordinates of the vertices increase. This has to do with the classifications of pentagons mentioned previously. For polygons with five or more sides, they must point out in diagonal directions. This is because a convex empty pentagon's vertices must have at least three different x-coordinates or three different y-coordinates, and any convex empty polygon with more than 5 sides, can be partitioned to define a convex empty pentagon.

Given we assume this diagonal shape of the polygon we are constructing, we know the bottom edge is going to be along the main diagonal of P^2.

Our improvements to the algorithm allowed for the polygons to be constructed out of points from multiple domes. We found that given very specific circumstances it could be possible to create more edges by skipping over points that lied between vertices. Prior, polygons were only constructed out of a single dome shape, and as soon as the next slope was calculated to have increased no more edges would be considered. The algorithm would stop, say how many points were in the largest dome, and built the polygon there.

We will now look at specific polygons constructed along the main diagonal. As discussed the largest convex empty polygon found in P^2 before our improvements to the algorithm had twelve edges. The coordinates of this polygon are all larger than 3,000,000, which is why discussing the difficulties inherent to large primes was emphasized. This polygon is included in Figure 3 along with the coordinates of its vertices.

Next, Figure 4 shows an example of a convex empty polygon in P^2 that has fourteen edges. A grid showing the horizontal and vertical lines through the points of P^2 has been included. The edges of the polygon are defined in red, while the gray lines show how extending the construction to include the entire dome would lead to problems with convexity. These gray edges appear each time the next consecutively prime point is skipped in order to make a larger polygon, and thus give a visual indication of our improvements. Again the polygon's coordinates are included in the figure.

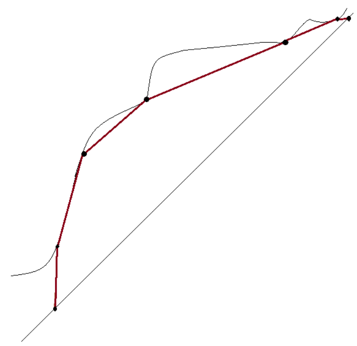

Figure 2: An illustration of how skipping domes can increase the number of edges in our construction.

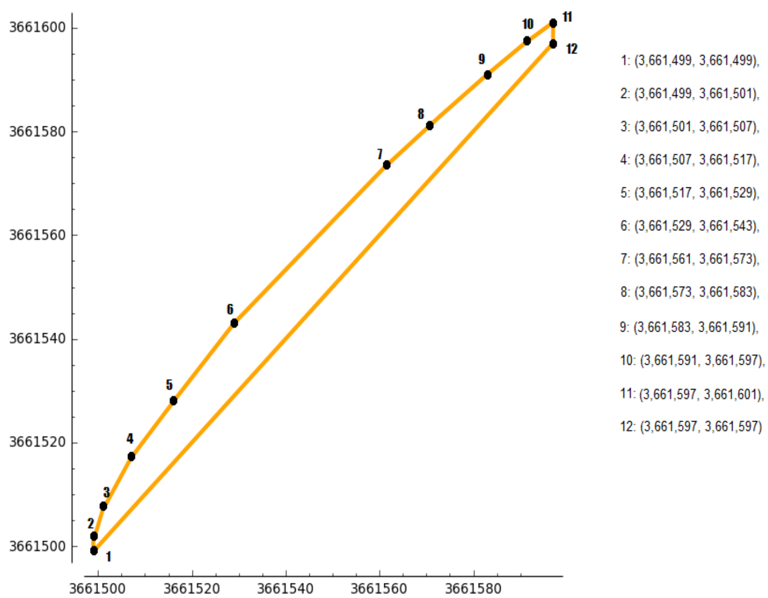

Figure 3: A convex empty 12-gon along the main diagonal of P^2.

The code used to construct these polygons has been included in the Appendix section. However, the code has already been run to construct the largest polygons possible along the main diagonal of P^2 up to the fifty millionth prime. This is because the database we used stored the first fifty million prime numbers. Figure 5 shows a distribution of the largest polygons that were able to be constructed in each interval of one million prime numbers.

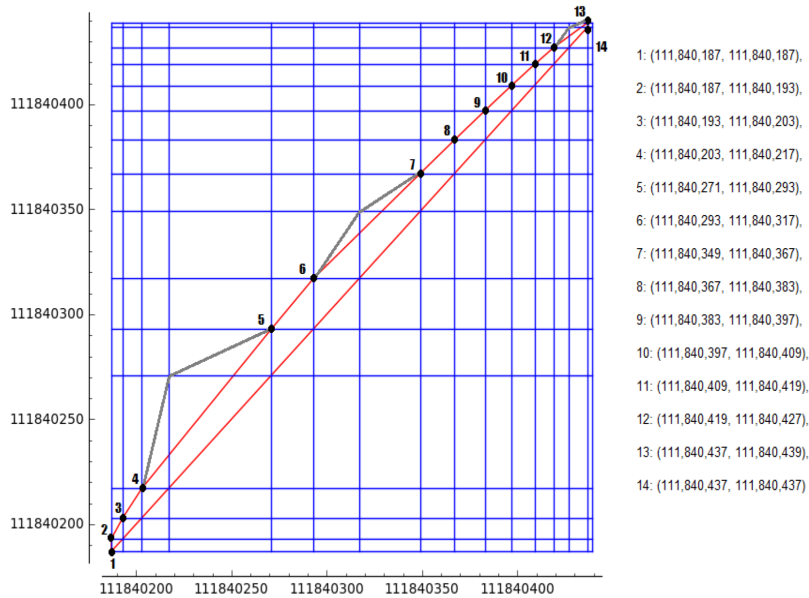

Figure 4: A convex empty 14-gon, in red, along the main diagonal of P^2.

Please note while reading Figure 5 that 1,000,000 is not the millionth prime number, but the millionth prime number is substantially larger than 1,000,000. The millionth prime is 15,485,863, meaning that twelve sides is the largest polygon that can be made along the diagonal of P^2 between 2 and 15,485,863. The example in Figure 3 is one such polygon. The distribution shows that a convex empty 14-gon can only be constructed along the main diagonal of P^2 after seven million primes. The next convex empty 14-gon on the main diagonal is between the thirty-third and thirty-fourth million primes. A 13-gon can be constructed between the fourteenth and fifteenth million primes, but a convex empty 13-gon can also be constructed by partitioning the 14-gon found after only seven million primes. This is why only the largest possible polygons are mentioned in this figure.

2.3. Moving Away from the Main Diagonal

The last section gave specific examples of large convex empty polygons that exist in P^2. However, these discussed have all been special cases of polygons due to how they are connected to the main diagonal. It is likely that many more convex empty polygons with as many edges as these exist in P^2 without requiring this edge along the diagonal. It is possible that much larger convex empty polygons could be found using only the first fifty million primes as coordinates, if we were able to work anywhere in P^2, and did not work solely along its main diagonal.

However, there are many reasons why this is a more difficult task. As already discussed,

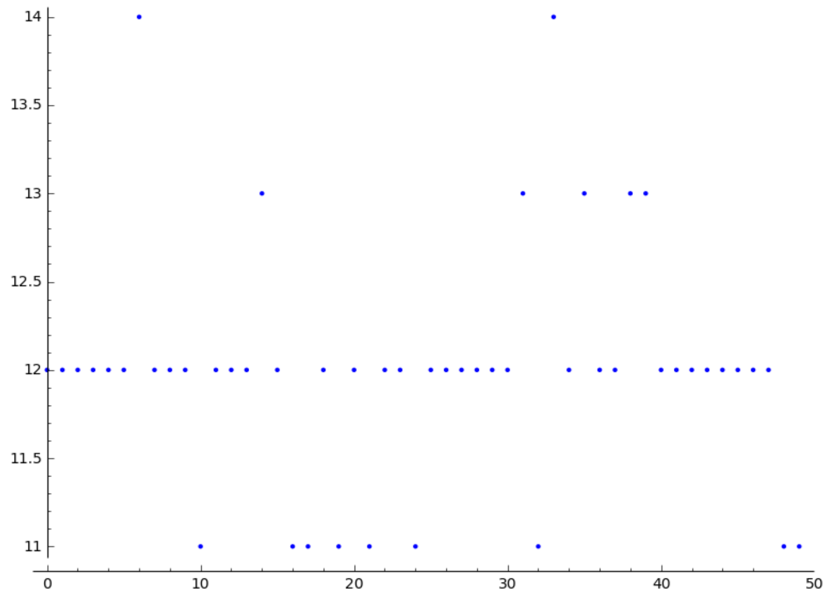

Figure 5: The number of edges of the maximal convex empty polygon along the main diagonal of P^2 in every interval of one million prime numbers, up to fifty million.

the gaps between very large prime numbers can be very difficult to predict. It is also difficult to know when a large integer is prime or not. Really the biggest difficulty in moving away from the main diagonal is how many more options are available during a construction. Because there are so many different points to test as edges, it becomes difficult to define bounds to construct within. This idea will be discussed at length.

2.3.1. Bounds on Construction. The first chapter of this paper discussed how convex empty pentagons in P^2 need to be pointed in diagonal directions. We know this is true of larger polygons as well, since any convex empty n-gon for $n > 5$ contains a convex empty pentagon. These polygons must have two vertices that are the furthest apart from each other. Let the segment connecting these points be considered the *diagonal* of the polygon.

Using the lemma that the polygon's interior angles must be less than 180°, we get the following lemma.

LEMMA 3. *The slopes of the edges of a convex empty polygon in P^2 above the polygon's diagonal must be either strictly decreasing or strictly increasing. The slopes of the edges of the polygon below its diagonal must then be strictly the opposite.*

If we apply this lemma to the 12-gon shown in the examples of the last section, we see that the edges above the polygon's diagonal are all strictly decreasing, and the edges below the polygon's diagonal are only the edges connecting vertex 11 and vertex 12 and the edge connecting vertex 1 and vertex 12 (the main diagonal of P^2). In reality this main diagonal edge is the many bottom edges of the 12-gon that are all collinear with each other. The same is true when applying this lemma to the 14-gon example.

This shows how working along the main diagonal results in only special cases of convex empty polygons. Polygons constructed along the main diagonal will always have these collinear bottom edges. This concept of a polygon's diagonal will be discussed further in the next section.

The long $x = y$ bottom edge inherent to how polygons were constructed along the main diagonal allowed the strictly decreasing slopes of the top edges to eventually meet again with the main diagonal, naturally bounding the construction. However, when trying to construct large polygons without using the main diagonal we are left without such natural bounds. This is the heart of the problem of proving the following conjecture.

CONJECTURE 1. *For every fixed point $p \in P^2$, there is some $n \in \mathbb{N}$ such that for all $N > n$, p is not the vertex of a convex empty N-gon of P^2.*

This conjecture is saying that every point in P^2 is the vertex to some largest convex empty polygon, and is not the vertex to any larger convex empty polygon. Ideally, if this conjecture were proved to be true, then this maximal n value could be found for every point in P^2, and the supremum of all these n for every point in P^2 would have to be the Helly number of P^2. Of course, since there are infinitely many points in P^2, explicitly finding this n value for each point would not be possible.

Nevertheless, we attempted to prove this conjecture. Using what we know about convex empty pentagons in P^2 we can simplify the proof to be near trivial in all but one of the cases.

If $p \in P^2$ cannot be the vertex of a convex empty pentagon, then we know $n < 5$. However, for p where $n > 4$ we can take a convex empty pentagon with p as a vertex, and partition P^2 around p into quadrants. Because convex empty 5-gons and larger must have the diagonal characteristics mentioned in the first chapter, the pentagon containing p must only be able to be expanded further into the I and III quadrants, or expanded into the II and IV quadrants (see figure).

Let us begin by looking at the pentagon that points toward the II and IV quadrants. The top most edge of our polygon can be extended to a line that defines a half-space of P^2. We know that in order to expand this pentagon into a polygon with more than 5 edges, any vertex that is added must be on the same bottom side of this half-space as the rest of the vertices. We know this from the results in the first chapter. If a point were added on the other side of the half-space, it would need to be below the line defined by the pentagon's horizontal edge, and to the left of the line defined by the pentagon's vertical edge, as Figure 16 in chapter 1 shows. However, if a point existed in this area, it would guarantee an interior point between the pentagon's two middle vertices.

Next, we know that the line defined by this top-most edge must hit both the y-axis and

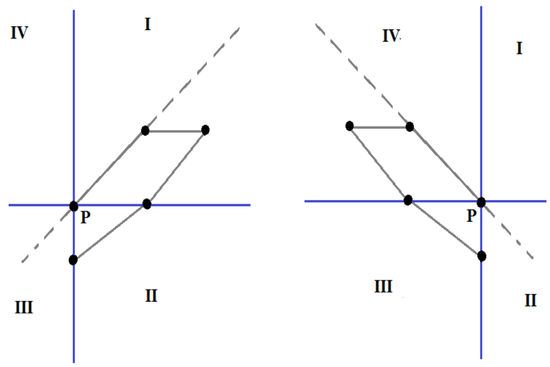

Figure 6: Pentagons are broken down into two categories, depending on which quadrants they point in.

x-axis of our coordinate system. Since P^2 does not include any negative integers, this line and the axes form a large triangle. We know any points added as vertices to our pentagon that result in a convex empty polygon must be within this large triangle. We also know that there are finitely many points in P^2 within this triangular area. This means there are only finitely many points that can possibly be added as vertices to our pentagon. *Thus any pentagon that points towards the II and IV quadrants cannot be expanded indefinitely.* However, the same conclusion cannot be made for the other type of pentagon.

If we look at the top-most edge of the pentagon that points in the I and III quadrants, this edge does not form the same large triangle with the axes of P^2. As the proof to Theorem 7 shows, the top-most edge of this type of pentagon could be the main diagonal, which we know infinitely extends further into P^2. We are left without a bounded area to work within, and thus cannot make a similar claim for this type of pentagon.

We tried different ideas to try and bound the I and III quadrant type pentagons inside of an area in order to prove that they could not be expanded indefinitely. The idea that seemed the most promising had to assume the Twin Prime Conjecture. The smallest gaps in P^2 occur between twin primes, and we tried to make use of these small gaps find boundaries on how polygons could be expanded.

Unfortunately, these twin primes did not completely bound where the polygons could be expanded. We were able to conclude that if an edge passes between two sets of twin primes (between two of the blue boxes in Figure 7), than another edge must also pass between the same twin primes. In other words, a polygon would not be able to include one of these twin prime boxes inside its interior, because the expanded polygon would no longer be empty. Despite not being able to prove our conjecture, this conclusion is significant. We tried to zoom our focus further and further out, and contain our polygon between more

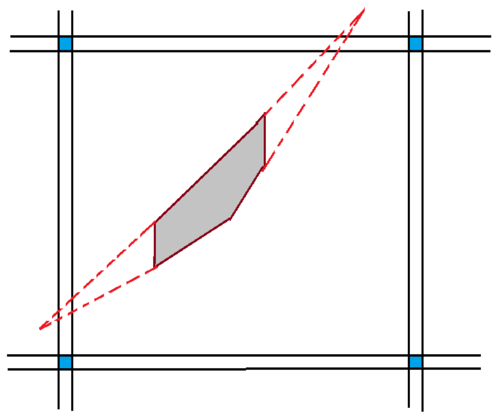

Figure 7: Even assuming the existence of twin primes surrounding our polygon, they do not completely bound expansion.

and more twin prime boxes, but this resulted in many cases that had be considered. It is possible that considering every possible configuration of expansion within enough twin prime boxes could result in a proof of our conjecture.

However, this method still relies on assuming the Twin Prime Conjecture. If it is ever proven that there is some largest pair of twin primes, then the expansion of polygons with coordinates larger than these largest twin primes could not be bounded in this way.

We found we could not prove that the I and III quadrant type pentagons could not be theoretically expanded indefinitely. It is for this reason that we could not finish writing an algorithm to construct polygons away from the main diagonal.

2.3.2. Construction Algorithm Ideas. Both Jianping and I tried to create our own algorithms in order to try and construct the largest possible convex empty polygon given a single vertex input. The hope was to run the algorithm for as many points in P^2 as possible, in order to see if any polygons with more than 14 edges could be found. However, both algorithms had problems that we could not overcome.

My idea for an algorithm to construct large convex empty polygons was very similar to what was discussed in chapter 1. The idea was to take the given point, and construct a convex empty pentagon with it as a vertex. The pentagon would then be expanded diagonally, either in the I and III quadrants or in the II and IV quadrants, depending on the shape of the pentagon. This expansion would take place by testing pairs of middle vertices. The edges connecting to these middle vertices would be extended to bound where further vertices could be added (as shown in Figure 12 of Chapter 1). When two extended edges intersect we would know that no points past that intersection could be included, or else the resulting polygon would have some interior angle larger than 180°. Each pairs of potential edges would be tested, in order to find the most edges that it is possible to add.

There are numerous problems associated with trying to implement this construction idea. The first problem is that there are very many potential pairs of middle vertices to be used when making the convex empty pentagon in the first step. In order for the maximally large (in terms of number of edges) convex empty polygon with the given point as a vertex, each of these pentagons would need to be expanded. Another problem is that expansion in the direction of the I quadrant could possibly never stop. The new pairs of edges added at each step could possibly never intersect, and never produce a closed polygon. This algorithm would also end up producing the exact same polygons multiple times using of the same vertices as it tested vertices that are close to each other.

Jianping's idea for an algorithm to construct large convex empty polygons relied on examining a polygon's diagonal. This method would take the given vertex, and connect it to another point around it, and assume that this segment was a convex empty polygon's diagonal. It would then use Lemma 3, and construct as many edges above the polygon's diagonal and below polygon's the diagonal that would result in a convex empty polygon.

One problem with this method is that there are infinitely many choices of points to use to define an infinite number of diagonals. We thought if we were going to implement this method, in order to deal with this problem we would put some maximum length that a diagonal could extend to. Another problem with this method is that many of the resulting polygons would be very small, and would be unhelpful but necessary to record. A single point would have potentially infinitely many (depending on if the length of the diagonal is bounded) convex empty triangles that would need to be recorded alongside any larger polygons.

With improvements, both of these ideas have potential for constructing convex empty polygons without using the main diagonal of P^2.

2.4. P^2 as a Symmetric Space

Something remarkable we realized very late while working on this problem is that none of our geometric results were unique to P^2. Rather, any set of integers, S, crossed with itself would be symmetric about a main diagonal $x = y$. As long as S has infinite cardinality, this main diagonal, as well as all vertical lines and horizontal lines must have an infinite number of points. Every result from the first chapter can be applied to $S \times S = S^2$.

For this reason, further research should be conducted in other less complicated symmetric point sets. Insight that is discovered in simpler spaces could be helpful when looking at P^2.

2.4.1. Integers with P^2 Removed. One such symmetric set that is closely related to P^2 is its complement within the Integers. There is a theorem that relates the Helly number of a set to the Helly numbers of its subsets [**5**]. Since $P^2 \cup (\mathbb{Z}^2 \setminus P^2) = \mathbb{Z}^2$ we get the following theorem.

THEOREM 9. *$h(\mathbb{Z}^2) = h(P^2 \cup (\mathbb{Z}^2 \setminus P^2)) \leq h(P^2) + h(\mathbb{Z}^2 \setminus P^2)$ where $h(S)$ denotes the Helly number of a set.*

This theorem comes directly from plugging P^2 and $\mathbb{Z}^2 \setminus P^2$ into Proposition 2.3 of "Helly Numbers of Subsets of $\mathbb{R}^2$ and Sampling Techniques in Optimization" [**5**]. Rearranging the inequality we get:

$$h(P^2) \geq h(\mathbb{Z}^2) - h(\mathbb{Z}^2 \setminus P^2)$$

Thus it is possible to gain information about the Helly number of P^2 by looking at its compliment, the integers with P^2 removed. This set is all points with coordinates that are composite numbers or the number 1.

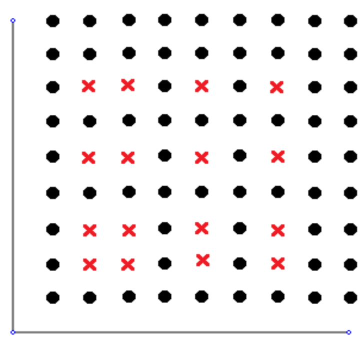

Figure 8: The integer lattice with points of P^2 marked in red.

We know that $h(\mathbb{Z}^d) = 2^d$, so for our case $h(\mathbb{Z}^2) = 4$. We also know that $\mathbb{Z}^2 \setminus P^2$ will behave exactly like $\mathbb{Z}^2$ in areas that do not contain any prime coordinates. This means that the largest convex empty polygons that can be made in such areas are quadrilaterals (in red in Figure 9).

For this reason it only remains to observe $\mathbb{Z}^2 \setminus P^2$ where prime points have been removed. Since 2 is the only even prime number, this means all prime points with both x-coordinates and y-coordinates greater than 5 will be surrounded by 8 composite coordinate points. The largest convex empty polygon that can be constructed within these 8 composite points is a diagonal hexagon (in brown in Figure 9). This means that for $\mathbb{Z}^2 \setminus P^2$ as we move further away from the axes towards larger coordinates, there will never be enough empty space to construct anything larger than a convex empty hexagon.

However, since 2 is the only even prime number, the gaps created by removing prime points with coordinates 2 and 3 become special cases in $\mathbb{Z}^2 \setminus P^2$. The largest empty space of $\mathbb{Z}^2 \setminus P^2$ is thus where the points $(2,2),(2,3),(3,2)$, and $(3,3)$ have been removed. The largest convex empty polygon that can be constructed within this space is an octagon (in blue in Figure 9). Thus we arrive at the following conclusion.

THEOREM 10. *The Helly number of $\mathbb{Z}^2 \setminus P^2$ is 8.*

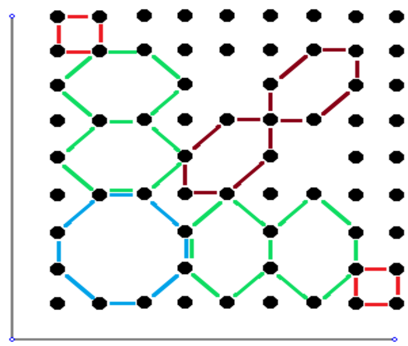

Figure 9: The largest convex empty polygons of $\mathbb{Z}^2 \setminus P^2$.

This means we have enough information to fill in our the inequality:

$$\begin{gathered} h(P^2) \geq h(\mathbb{Z}^2) - h(\mathbb{Z}^2 \setminus P^2) \\ h(P^2) \geq 4 - h(\mathbb{Z}^2 \setminus P^2) \\ h(P^2) \geq 4 - 8 \\ h(P^2) \geq -4 \end{gathered}$$

Unfortunately this conclusion is not at all useful. Helly numbers cannot be negative, so we already know that $h(P^2) > 0$. Also from our previous work we already had found a better lower bound on $h(P^2)$. Since we have found a convex empty 14-gon in P^2, we know that $h(P^2) \geq 14$.

The next step we tried was to look at the same situation in three dimensions. Even when considering $\mathbb{Z}^3 \setminus P^3$ the largest empty space would need to be where the points $(2,2,2),(2,2,3),(2,3,2),(3,2,2),(2,3,3),(3,3,2),(3,2,3)$, and $(3,3,3)$ have been removed. The largest convex empty polygon that can be created in that empty space is a body with 26 facets (18 square facets and 8 triangular facets). The resulting inequality in three dimensions results in another negative lower bound that was even further away from 0. We thus assumed that higher dimensions would also not yield results.

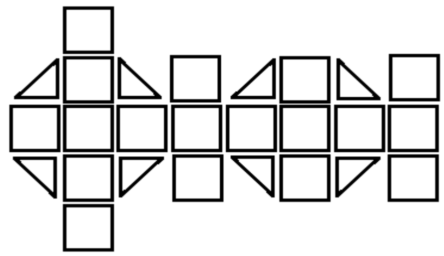

Figure 10: All the facets of the largest convex empty polygon of $\mathbb{Z}^3 \setminus P^3$.

APPENDIX A

Algorithm for Constructing Convex Empty Polygons Along the Main Diagonal

Details

The following code is what was used to find the largest possible convex empty polygons along the main diagonal of P^2. It was found that loading a list of the prime numbers from a database signficantly reduced the run time of the code, as the computer would not need to conduct any primality testing. The code was prepared for use in Sage which uses the Python programming language. The # symbol denotes comments in the code, and a single hyphen at the start of a line denotes when the previous line of code is too long to appear in a single line of text.

```
#------------------------------------------------------------
# Name: DiameterSurfer
# Function: find the polygon w/ maximum number of edges
# Author: Jianping Pan
# Date: August 25rd, 2015
#------------------------------------------------------------
# Global Variables
#------------------------------------------------------------
PG=[] #  prime number list
maxlen = 0 # maxmum number of vertex
LPath = [] # longest path up
depth_cutoff = 5 # const
#------------------------------------------------------------
# Pre-load prime data base
#------------------------------------------------------------
with open('/Users/Panda/Dropbox/Helly/PrimeData/primes1.txt','r') as infile:
     for i in infile:
          for num in i.split(" "): # split the whole line based upon " "
               if num and num != '\r\n':
                    PG.append(ZZ(int(num)))
#------------------------------------------------------------
```

```
def initialize():
      maxlen = 0
      LPath = []
#-----------------------------------------------------------
# Calculate Slope between two points pt1 and pt2, where pt1=[x1,y1], pt2=[x2,y2]
#-----------------------------------------------------------
def f(pt1,pt2):
     x1 = PG[pt1[0]]
     x2 = PG[pt2[0]]
     y1 = PG[pt1[1]]
     y2 = PG[pt2[1]]
     # print 'x1 and x2 :',x1,x2
     return ZZ(y2-y1) / ZZ(x2-x1)
#-----------------------------------------------------------
def background(x0,xl,y0,yl,candidate1,candidate2,candidate3):
     whs = line([])
     s += line([(PG[x0],PG[y0]),(PG[xl],PG[yl])])
     x0 = max(x0-1,0)
     y0 = max(y0-1,0)
     for i in range(max(y0-1,0), yl + 1):
          s += line([(PG[x0],PG[i]),(PG[xl],PG[i])])
     for i in range(x0, xl +1):
          s += line([(PG[i],PG[y0]),(PG[i],PG[yl])])
     s.show()
     for i in candidate1:
          s += point([PG[i[0]],PG[i[1]]],color = 'red', size=20)
     for i in candidate2:
          s += point([PG[i[0]],PG[i[1]]],color = 'blue', size = 20)
     for i in candidate3:
          s += point([PG[i[0]],PG[i[1]]],color = 'orange',size=20)
     s.show(title = 'Grid background with lists bolded')
#-----------------------------------------------------------
def polyonback(x0,xl,y0,yl,candidate1,candidate2,candidate3,poly):
     s = line([])
     s += line([(PG[x0],PG[y0]),(PG[xl],PG[yl])])
     x0 = max(x0-1,0)
     y0 = max(y0-1,0)
     for i in range(max(y0-1,0), yl + 1):
          s += line([(PG[x0],PG[i]),(PG[xl],PG[i])])
     for i in range(x0, xl +1):
          s += line([(PG[i],PG[y0]),(PG[i],PG[yl])])
```

```
    s.show()
    for i in candidate1:
        s += point([PG[i[0]],PG[i[1]]],color = 'red', size=20)
    for i in candidate2:
        s += point([PG[i[0]],PG[i[1]]],color = 'blue', size = 20)
    for i in candidate3:
        s += point([PG[i[0]],PG[i[1]]],color = 'orange',size=20)
    s += polygon(poly,color = 'yellow')
    s.show(title = 'Grid background with lists bolded')
#------------------------------------------------------------
def check_conditions(prev,cur,next,candidate,mark):
    cn_slope = f(candidate[cur], candidate[next])
    if f(candidate[prev], candidate[cur]) * mark >= cn_slope * mark:
        return all(f(candidate[cur], candidate[u]) * mark >= cn_slope * mark for u in
-range(cur+1, next))
    return False
#------------------------------------------------------------
# clear lattices on the edges, avoid colinearity
#------------------------------------------------------------
def cleanpath(path,candidate,mark):
    newpath = []
    newpath.append(path[0])
    for i in range(len(path)-2):
        if f(candidate[path[i+1]],candidate[path[i+2]]) * mark <
-f(candidate[path[i]],candidate[path[i+1]]) * mark:
            newpath.append(path[i+1])
    newpath.append(path[-1])
    return newpath
#------------------------------------------------------------
# Main Function: backtrack algorithm, find the longest path from candidate list
# starting from current path recursively
#------------------------------------------------------------
def backtrack(current,candidate,mark):
    global maxlen
    global LPath
    # candidate: all possible 2-dimensional points
    found = False
    for np in range(current[-1]+1, min(current[-1] + depth_cutoff, len(candidate))):
        if check_conditions(current[-2], current[-1], np, candidate,mark):
            found = True
            current.append(np)
```

```
                backtrack(current, candidate,mark)
                temp = current.pop()

     if not found:
          if len(current) > maxlen:
               path = cleanpath(current, candidate,mark)
               if len(path) > maxlen:
                    maxlen = len(path)
                    LPath = path # Make a copy
          return

     if len(current) == 2 and temp == min(current[-1] + depth_cutoff, len(candidate)):
          return # if it tried all the possible ways, return to the primitive one
#------------------------------------------------------------
def combinepath(left,right,candidate1,candidate2):
     for i in reversed(right):
          if  not candidate1[left[-1]][0] >= candidate2[right[-1]][0] and
-f(candidate1[left[-2]],candidate1[left[-1]]) < f(candidate1[left[-1]],candidate2[i]):
               right.pop()
          poly = []
          for i in reversed(left):
               poly.append(candidate1[i])
          for i in right:
               poly.append(candidate2[i])
          return poly
#------------------------------------------------------------
def DrawDue(left,right,candidate1,candidate2,polypoly):
     s = line([])
     poly = []
     x0 = max(min(candidate1[left[0]][0],candidate2[right[0]][0])-1,0)
     y0 = max(min(candidate1[left[0]][1],candidate2[right[0]][1])-1,0)
     xl = max(candidate1[left[-1]][0] + 1,candidate2[right[-1]][0] + 1)
     yl = max(candidate1[left[-1]][1] + 1,candidate2[right[-1]][1] + 1)
     head = min(left[0],right[0])
     tail = max(left[-1],right[-1])
     print 'head,tail=',head,tail
     for i in range(x0,xl+1):
          s += line([(PG[i],PG[y0]),(PG[i],PG[yl])])
     for i in range(y0,yl+1):
          s += line([(PG[x0],PG[i]),(PG[xl],PG[i])])
     for i in range(head,tail+1):
```

```
          s += point([PG[candidate1[i][0]],PG[candidate1[i][1]]],color = 'red',size = 20)
          s += point([PG[candidate2[i][0]],PG[candidate2[i][1]]],color = 'blue',size = 20)
     for i in polypoly:
          poly.append([PG[i[0]],PG[i[1]]])
     s += polygon(poly,color = 'yellow')
     s.show(title = 'Polygon inside the hull.')
     return poly
#------------------------------------------------------------
# Starting_Ending Function: input starting and ending point, find the polygon
# w/ maximum vertex!
#------------------------------------------------------------
def Start_Ending(pt1,pt2):
     global maxlen
     global LPath
     maxlen = 0
     LPath = []
     Ulist = [] # list of candidate points above diameter
     Olist = [] # list of candidate points on diameter
     Dlist = [] # list of candidate points below diameter
     left = []
     right = []
     maxlensum = 0
     leftmax = []
     rightmax = []
     poly = []
     polypoly = []
     maxpoly = []
     x0 = pt1[0]
     y0 = pt1[1]
     xl = pt2[0]
     yl = pt2[1]
     se_slope = f([x0,y0],[xl,yl])
     Ulist.append([x0,y0+1])
     Dlist.append([x0,y0-1])
     for i in range(x0,xl+1):
          ver = se_slope * (PG[i]-PG[x0]) + PG[y0]
          for j in range(y0-1,yl+2):
               if ver > PG[j] and ver < PG[j+1]:
                    Ulist.append([i,j+1])
                    Dlist.append([i,j])
               if ver == PG[j] and not j == y0 and not j == yl:
```

```
                Olist.append([i,j])
                Ulist.append([i,j+1])
                Dlist.append([i,j-1])
print 'LIST generation complete.'
print 'Olist = ',Olist
background(x0,xl,y0,yl,Ulist,Dlist,Olist)

if Olist: # if there is lattice on diagonal
     print 'Something inside!'
     for i in range(xl-x0-depth_cutoff):
          for k in range(i+1, i + depth_cutoff):
               current = [i,k]
               backtrack(current,Ulist,1)
     print 'Upper part maxlen=',maxlen + 2
     left = DrawPoly(x0,y0,xl,yl,LPath,Ulist,1,'Upper Polygon')
     print 'Upper polygon complete.'

     maxlen = 0
     LPath = []
     for i in range(xl-x0-depth_cutoff):
          for k in range(i+1, i + depth_cutoff):
               current = [i,k]
               backtrack(current,Dlist,-1)
     print 'Lower part maxlen=',maxlen + 2
     right = DrawPoly(x0,y0,xl,yl,LPath,Dlist,-1,'Lower Polygon')
     print 'Lower polygon complete.'

else:
      print 'Empty inside!'
     for i in range(xl-x0-depth_cutoff):
          for k in range(i+1,i+depth_cutoff):
               left = []
               right = []
               initialize()
               backtrack([i,k],Ulist,1)
               left = LPath
               initialize()
               backtrack([i,k],Dlist,-1)
               right =  LPath
               polypoly = combinepath(left, right, Ulist,Dlist)
               if len(polypoly) > maxlensum:
```

```
                    maxlensum = len(polypoly)
                    maxpoly = polypoly
                    leftmax = left
                    rightmax = right
          print 'maxlen=',maxlensum
          poly = DrawDue(leftmax,rightmax,Ulist,Dlist,maxpoly)
          polyonback(x0,xl,y0,yl,Ulist,Dlist,Olist,poly)
#------------------------------------------------------------
```

Bibliography

1. Jens Kruse Andersen, *Primes in arthimetic progression records*, 2005-2015, http://primerecords.dk/aprecords.html.
2. Janos Barat, Vida Dujmovic, Gwenael Joret, Michael S. Payne, Ludmila Scharf, Daria Schymura, Pavel Valtr, and David R. Wood, *Empty pentagons in point sets with collinearities*, Discrete Mathematics **29** (2015), 198–209.
3. Ludwig Danzer, Brank Grunbaum, and Victor Klee, *Helly's theorem and its relatives*, Proc. Sympos. Pure Math, vol. VII, Amer. Math. Soc., Providence, R.I., 1963.
4. Joshua D. Laison, Colin Starr, and Andrea Walker, *Finite prime distance graphs and 2-odd graphs*, Discrete Mathematics **313** (2013), 2281–2291.
5. Jesus Antonio De Loera, Reuben Neamiah La Haye, Deborah Oliveros, and Edgardo Roldn-Pensado, *Helly numbers of subsets of* $\mathbb{R}^d$ *and sampling techiniques in optimization*, 2015, http://arxiv.org/pdf/1504.00076.pdf.
6. Yitang Zhang, *Bounded gaps between primes*, Annals of Mathematics **179** (2014), 1122.