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Garnir Polynomial and their Properties

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In this paper, we are studying the polynomials in n variables which behave nicely under permutation of variables. The study in n variables polynomials which could be widely applied in many areas including representation theory, algebra and physics. We start with some basic notions like symmetric polynomials which do not change the value under permutations (for example: $x_1^2 + x_2^2$), and anti-symmetric polynomials which only change the sign of the value under permutations (for example: $x_1^2 - x_2^2$) to illustrate how permutations work as well as some other properties of these polynomials. Then we will focus on more interesting examples like Garnir polynomials which can be expressed in terms of Young diagrams. Garnir polynomials are polynomials of the form $(x_i - x_j) \cdot ... \cdot (x_s - x_t)$, they have many interesting properties. In this paper, we mainly care about Garnir polynomials which only involve four variables at a time, and study the dimension of their span.

1 Symmetric polynomials

Symmetric polynomials have the same value at any permutations.

Definition 1.0.1. A polynomial $f(x_1, ..., x_n)$ is a symmetric polynomial if for all permutations ϕ, $f(x_1, ...x_n) = f(x_{\phi(1)}, ...x_{\phi(n)})$.

We define the polynomials of e_k as follows:

$$e_k(x_1, x_2, ..., x_n) = \sum_{1 \leq j_1 < j_2 < ... < j_k \leq n} x_{j_1} ... x_{j_k}$$

For k from 1 to n, we can define the following elementary symmetric polynomials:

$e_1(x_1, x_2, ..., x_n) = \sum_{1 \leq j \leq n} x_j$
$e_2(x_1, x_2, ..., x_n) = \sum_{1 \leq j < k \leq n} x_j x_k$
...

$e_n(x_1, x_2, ..., x_n) = x_1 x_2 ... x_n$

Clearly, e_k is symmetric for all k and n.

Theorem 1.0.2. *Every symmetric function in $x_1 \ldots x_n$ is a polynomial in the elementary symmetric polynomials e_k .*

Proof. We first define an order on monomials that $x_1^{a_1} x_2^{a_2} ... x_n^{a_n} < x_1^{b_1} x_2^{b_2} ... x_n^{b_n}$

if $a_n < b_n$;

or $a_n = b_n$ and $a_{n-1} < b_{n-1}$;

or $a_n = b_n, a_{n-1} = b_{n-1}$ and $a_{n-2} < b_{n-2}$;

and so on...

For any $s = x_1^{a_1} x_2^{a_2} ... x_n^{a_n}$ and $t = x_1^{b_1} x_2^{b_2} ... x_n^{b_n}$, s and t are always comparable, so either $s = t$, or $s > t$ or $s < t$.

For a given symmetric polynomial, we can always find a maximal monomial $x_1^{j_1} x_2^{j_2} ... x_n^{j_n}$, and we can always find a product of elementary symmetric polynomials e_k such that $x_1^{j_1} x_2^{j_2} ... x_n^{j_n}$ is also the maximal in it.

For example, given $x_1^2 x_2^3 x_3^4$ is the maximal element, we can construct a polynomial by multiplying $e_3(x_1, x_2, x_3) \cdot e_3(x_1, x_2, x_3) \cdot e_2(x_1, x_2, x_3) \cdot e_1(x_1, x_2, x_3)$=$(x_1 x_2 x_3)^2 \cdot (x_1 x_2 + x_1 x_3 + x_2 x_3) \cdot (x_1 + x_2 + x_3)$. Multiplying the leading term in each elementary symmetric polynomial, we get $(x_1 x_2 x_3)^2 \cdot (x_2 x_3) \cdot (x_3) = x_1^2 x_2^3 x_3^4$, which gives the maximal element as we desired.

For a symmetric polynomial $f(x_1,...,x_n)$, if $x_1^{a_1}x_2^{a_2}...x_n^{a_n}$ is the leading term, we know that $a_1 \leq a_2 \leq a_3 \leq ... \leq a_n$. Since f is symmetric, we can always make this inequality hold.
If we subtract this product from the original symmetric polynomial, it is obvious that the maximal element in the new polynomial is strictly smaller than the maximal element in the old one. Since the number of elements smaller than the maximal element is finite, by subtracting these polynomials again and again, the original polynomial will eventually become 0 in a finite number of steps. Hence, all symmetric polynomials are polynomials in e_k. $\square$

2 Antisymmetric polynomials

Antisymmetric polynomials change the sign and keep the value at any permutations. In other words:$f(x_1,...,x_i,...,x_j,...,x_n)$=$-f(x_1,...,x_j,...,x_i,...,x_n)$.

Theorem 2.0.1. $W = \prod_{i<j}(x_i - x_j)$ *is antisymmetric.*

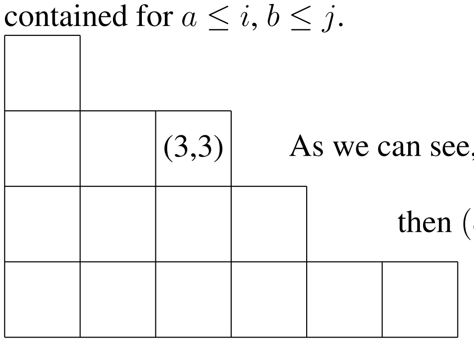
Proof. Let $f(x_1,...,x_s,...,x_t,...,x_n)$=$\prod_{i<j}(x_i - x_j)$ and we permute x_s and x_t. Denote the new polynomial by $f(x_1,...,x_t,...,x_s,...,x_n)$. We can find out that the two polynomials have some common factors which do not involve x_s, x_t, and those which have one of them but the other variable is before both of x_s, x_t or after both of them. Denote them as C_1= $\prod_{i,j\neq s,t}(x_i - x_j)$ and C_2= $\prod_{m=s \text{ or } t, n>s,t}(x_m - x_n)$, C_3= $\prod_{n=s \text{ or } t, m<s,t}(x_m - x_n)$ Replace the common factors and we get $f(x_1,...,x_s,...,x_t,...,x_n)$=$C_1 \cdot C_2 \cdot C_3 \cdot [(x_s - x_{s+1})...(x_s - x_{t-1})][(x_{s+1} - x_t)...(x_{t-1} - x_t)](x_s - x_t)$ and $f(x_1,...,x_t,...,x_s,...,x_n)$=$C_1 \cdot C_2 \cdot C_3 \cdot [(x_t - x_{s+1})...(x_t - x_{t-1})][(x_{s+1} - x_s)...(x_{t-1} - x_s)](x_t - x_s)$. There are exactly $(t - s + 1)$ factors in the first bracket and second bracket, and they are negative value of each other, this leaves us the last factor $(x_s - x_t)$ and $(x_s - x_t)$.
So,$f(x_1,...,x_s,...,x_t,...,x_n)$=$(-1)^{2(t-s+1)+1}f(x_1,...,x_t,...,x_s,...,x_n)$.
It gives $f(x_1,...,x_s,...,x_t,...,x_n)$=$-f(x_1,...,x_t,...,x_s,...,x_n)$ as we designed to have. $\square$

Theorem 2.0.2. *Every antisymmetric polynomial f can be written as $f(x_1, x_2, ..., x_n) = W \cdot g(x_1, x_2, ..., x_n)$ where W is define in Theorem 2.0.1 and g is symmetric.*

Proof. First, we want to prove that f is divisible by $(x_i - x_j)$ for all $i < j$.
Assume that $x_i = x_j$, then $f(x_1, ..., x_i, ..., x_j, ..., x_n) = f(x_1, ..., x_j, ..., x_i, ..., x_n)$.
In addition f is antisymmetric, we have: $f(x_1, ..., x_i, ..., x_j, ..., x_n) = -f(x_1, ..., x_j, ..., x_i, ..., x_n)$.
Combining above two equations gives $f(x_1, ..., x_i, ..., x_i, ..., x_n) = 0$. So f is divisible by $(x_i - x_j)$ and therefore divisible by W.
We rewrite the equation as $\frac{f(x_1,x_2,...,x_n)}{W} = g(x_1, x_2, ..., x_n)$, since both $f(x_1, x_2, ..., x_n)$ and W are antisymmetric, we know that $g(x_1, x_2, ..., x_n)$ does not change the value under permutations.
Hence g is symmetric. ☐

3 Young Diagrams

A Young diagram is a combinatorial object useful in representation theory.
A Young diagram is a subset of $(\mathbf{Z}^+)^2$, such that if (i, j) is contained in it, then all (a, b) are contained for $a \leq i$, $b \leq j$.



As we can see, since (3,3) is contained in this Young diagram, then (a, b) is also contained for $a \leq 3$, $b \leq 3$.

Definition 3.0.1. A Young tableau is a way to fill the diagram with numbers from 1 to n (n is the number of boxes).

Definition 3.0.2. A standard Young tableau is a Young tableau such that the numbers in the same row and column are in increasing order from bottom to top and left to right.

The following is an example of Young diagram (4,3,1) and one of its Young tableau and one of its standard Young tableau.

$$\begin{array}{|c|c|c|c|}\cline{1-1} \\ \cline{1-3} & & \\ \hline & & & \\ \hline \end{array}$$

Young diagram

1			
4	3	7	
5	8	2	6

Young tableau

$$\begin{array}{ccccccc} 8 & & & & & & \\ \vee & & & & & & \\ 3 & < & 5 & < & 7 & & \\ \vee & & \vee & & \vee & & \\ 1 & < & 2 & < & 4 & < & 6 \end{array}$$

standard Young tableau

4 Garnir Polynomials

Suppose that T is a standard Young tableau.

Definition 4.0.1. We define Garnir polynomial by the equation $G_T = \prod_{i \text{ above } j} (x_i - x_j)$.

In the following case, $G(a, b, c, d) = (x_b - x_a)(x_d - x_c)$

b	d	
a	c	e

In the following theorem, we only consider Young diagrams with rows of length $(n-2, 2)$.

Theorem 4.0.2. *If we permute the variables* $G(a, b, c, d) \to G(a', b', c', d') \Rightarrow$ *linear span* V_n *of all* $G(a, b, c, d)$ *is preserved by permutations. (so* V_n *is a representation of* S_n*).*

Theorem 4.0.3. *The basis in* V_n *is given by* $G(a, b, c, d)$ *such that the corresponding Young tableaux are standard. (so* $a < b < d$*,* $a < c < d$*,* $c < e$*, and we can have several* $e's$*)*

Theorem 4.0.4. *The number of standard Young tableaux is equal to* $\frac{n(n-3)}{2}$.

Theorem 4.0.2 is clear. Let's prove Theorem 4.0.3:

Proof. In order to prove Theorem 4.0.3, we want to prove that the Garnir polynomials we generated from any four numbers in the first 2 by 2 square can be written as the span of Garnir polynomials which correspond to standard Young tableaux.

We have the following equivalence relations:

1.Exchange numbers in the same column is antisymmetric.

$$\begin{array}{|c|c|c} \hline b & d & \\ \updownarrow & & \\ \hline a & c & \multicolumn{1}{|c|}{e} \\ \hline \end{array}\; T \qquad <---> \qquad \begin{array}{|c|c|c} \hline a & d & \\ \hline b & c & \multicolumn{1}{|c|}{e} \\ \hline \end{array}\; T' \qquad G_T = -G'_T$$

2.Exchange the entire column is symmetric.

$$\begin{array}{|ccc|c|c} \hline b & \leftrightarrow & d & & \\ \hline a & \leftrightarrow & c & \multicolumn{1}{|c|}{e} \\ \hline \end{array}\; T \qquad <---> \qquad \begin{array}{|c|c|c} \hline d & a & \\ \hline c & b & \multicolumn{1}{|c|}{e} \\ \hline \end{array}\; T' \qquad G_T = G'_T$$

By using these two relations, we could at least ensure $a < b$, $c < d$ and $a < c$ in the following Young tableau.

$$\begin{array}{ccccc} b & ? & d & & \\ \vee & & \vee & & \\ a & < & c & & e \end{array}$$

If $b < d$, then this Young tableau is standard; we need to discuss the case $b > d$.

We construct a relation as follow:

$$\begin{array}{|c|c|c} \hline b & d & \\ \hline a & c & \multicolumn{1}{|c|}{e} \\ \hline \end{array} + \begin{array}{|c|c|c} \hline c & b & \\ \hline a & d & \multicolumn{1}{|c|}{e} \\ \hline \end{array} - \begin{array}{|c|c|c} \hline d & b & \\ \hline a & c & \multicolumn{1}{|c|}{e} \\ \hline \end{array} = 0$$

$(x_b - x_a)(x_d - x_c) + (x_c - x_a)(x_b - x_d) - (x_d - x_a)(x_b - x_c) = 0$

Since we know that $b > d > c > a$, one can check that the second and the third Young tableaux are standard.

From the previous steps, we could always make $d > b > a$ and $d > c > a$.

$$\begin{array}{ccccc} b & < & d & & \\ \vee & & \vee & & \\ a & < & c & ? & e \end{array}$$

Last, we need to check whether $c < e$. If $c < e$, it is standard, so we need consider the case that $e < c$. We construct a relation as follow:

$$\begin{array}{|c|c|c|} \cline{1-2} b & d & \multicolumn{1}{c}{} \\ \hline a & c & e \\ \hline \end{array} - \begin{array}{|c|c|c|} \cline{1-2} b & d & \multicolumn{1}{c}{} \\ \hline a & e & c \\ \hline \end{array} + \begin{array}{|c|c|c|} \cline{1-2} b & c & \multicolumn{1}{c}{} \\ \hline a & e & d \\ \hline \end{array} = 0$$

$(x_b - x_a)(x_d - x_c) - (x_b - x_a)(x_d - x_e) + (x_b - x_a)(x_c - x_e) = 0$

Since we know that $d > b > a, d > b > c > e$, one can check that the second and the third Young tableaux are standard.

Hence, Theorem 4.0.3 is proved. $\square$

Next, we want to prove Theorem 4.0.4:

Proof. We will prove it by induction. First, we check the base case that only has 4 variables. As shown below, there are 2 standard Young tableaux of 4 variables.

$$\begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}$$

Assume there are $\frac{n(n-3)}{2}$ standard Young tableaux for n variables. We now check the number of standard Young tableaux of $(n+1)$ variables.

It is true that the biggest number $(n+1)$ can only be placed on the rightmost blank, so we divide it into two cases.

		$\cdots$	$n+1$

case 1

x	$n+1$		
1		$\cdots$	

case 2

For case 1, the number of standard Young tableaux are the same as the case for n variables, which is $\frac{n(n-3)}{2}$.

For case 2, the smallest number 1 is definitely filled in the left bottom corner, and we only need to choose the number for x. As long as number x is chosen, the remaining numbers will automatically filled in the remaining blanks in a line in an increasing order. Hence, it is $(n-1)$ choices.

Sum up these two cases, $\frac{n(n-3)}{2} + (n-1) = \frac{(n+1)(n-2)}{2}$.

We get $\frac{(n+1)(n-2)}{2}$ standard Young tableaux as we desired to have. $\square$

5 Hook Length

Before we introduce Hook Length formula, we define $a_\lambda(i,j)$ and $l_\lambda(i,j)$ as follow:

Suppose that (i,j) is the box in a Young diagram,

$a_\lambda(i,j)$=number of boxes to the right of (i,j);

$l_\lambda(i,j)$=number of boxes above (i,j).

Definition 5.0.1. The Hook Length formula for the Young tableau is express by:

$H_\lambda = \frac{n!}{\prod h_\lambda(i,j)}$, where $h_\lambda(i,j) = a_\lambda(i,j) + l_\lambda(i,j) + 1$.

Example 5.0.2. For the following Young diagram (6,4,3,1), we have the $h_\lambda(i,j)$ in the boxes as follow:

1					
4	2	1			
6	4	3	1		
9	7	6	4	2	1

Then the Hook Length formula $H_\lambda = \frac{n!}{\prod h_\lambda(i,j)}$

$= \frac{14!}{2^2 \cdot 3 \cdot 4^3 \cdot 6^2 \cdot 7 \cdot 9}$

$=50050$

Theorem 5.0.3. *The number of standard Young tableaux of shape λ is equal to H_λ.*

This theorem is known in general, here we prove it for any two-row Young diagram.

Proof. Consider a Young diagram with row lengths k and l $(k < l)$.In the Pic 5.1, $n = k+l$, the product of hook length for the first row is $\prod h_\lambda(2,j) = k!$ and the product of the hook length for the second row is $\prod h_\lambda(1,j) = \frac{(l+1)!}{l-k+1}$. By calculation,

$$H_\lambda = \frac{(k+l)! \cdot (l-k+1)}{k!(l+1)!}$$

.

Let $\dim(k,l)$ be the number of standard Young tableaux with row lengths k and l.

Knowing that the largest number can only be filled in the rightmost coordinate for standard Young tableaux, we can get the relation that: $\dim(k,l) = \dim(k-1,l) + \dim(k,l-1)$.

Check the relation by replacing by Hook Length formula,

Right Hand Side= $\frac{(k+l)! \cdot (l-k+1)}{k!(l+1)!}$

Left Hand Side= $\frac{(k+l-1)! \cdot (l-k+2)}{(k-1)!(l+1)!}$+$\frac{(k+l-1)! \cdot (l-k)}{k!(l)!}$

Now checking RHS-LHS, first we multiply each factor by $\frac{(k-1)!l!}{(k+l-1)!}$, then

$\frac{(k-1)!l!}{(k+l-1)!}$(RHS-LHS)=$\frac{(k+l)(l-k+1)}{k(l+1)} - \frac{l-k+2}{l+1} - \frac{l-k}{k} = 0$

Since $\frac{(k-1)!l!}{(k+l-1)!} \neq 0$, we have RHS=LHS.

Last, we check the base case $k = 1$:

RHS=$\frac{(l+1)! \cdot (l-1+1)}{(l+1)!} = l$

LHS=$\frac{(1+l-1)!\cdot(l-1+2)}{0!(l+1)!}$+$\frac{(1+l-1)!\cdot(l-1)}{1!(l)!}$ $=\frac{(l+1)!}{(l+1)!}$+$\frac{l!\cdot(l-1)}{(l)!}=1+(l-1)=l$

It also gives RHS=LHS, which shows this relation is correct. $\square$

k	$\cdots$	1		
$l+1$	$\cdots$	l-k+2	$\cdots$	1

Pic 5.1

Corollary 5.0.4. *For the Young diagram with row length* $(l_1, l_2, ..., l_n)$,*(*$l_i > l_j$ *for* $i < j$*), the number of standard Young tableaux is*

$$\frac{(\sum_{i=1}^{n} l_i)! \prod_{i<j}(l_i - l_j + j - i)}{\prod_{i=1}^{n}((l_i + n - i)!)}$$

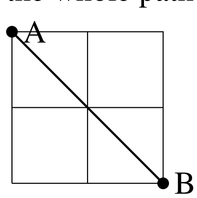
by applying the Hook Length Formula.

One can apply Hook Length Formula to prove the above corollary.

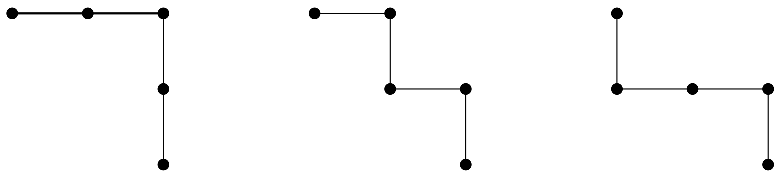
6 Application and Related Questions

Here is a question that many of us are familiar.

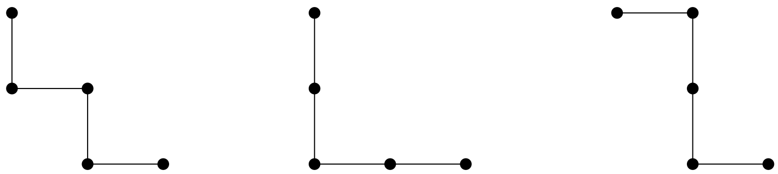
Suppose there is a 2 by 2 square and we denote the top-left corner A, bottom-right corner B, (a) how many ways to walk from A to B? (b) How many ways to walk from A to B such that the whole path is under the diagonal AB?



For question(a), we have the following 6 ways:



(1) (2) (3)



(4) (5) (6)

For question (b), we have 2 ways, they are (4) and (5) in question (a).

How is this problem related to Young tableaux?

Consider the 2 by 2 Young diagram, fill the numbers 1,2,3,4. Then we find each number from 1 to 4, if the number is in the top row, we move right, and if the number is in the bottom row, we move down. Since we are moving 2 steps right and 2 steps down, we will eventually goes to B.

Following are the corresponding Young tableaux to question (a). (There are more than one corresponding Young tableau to each path in (a), but we arrange the order, so that each row is in increasing order from left to right.

1	2
3	4

(1)

1	3
2	4

(2)

2	3
1	4

(3)

2	4
1	3

(4)

3	4
1	2

(5)

1	4
2	3

(6)

We can find that (4),(5) is standard.

Theorem 6.0.1. *For an n by n square, the path from top-left corner to bottom-right corner under the diagonal is corresponding to the 2 by n standard Young tableau.*

Proof. We denote each vertex (a, b) for a is the number of boxes to the left and b is the number of boxes above it. Hence, we have top-left vertex as (0,0), and bottom-right vertex as (n, n), while moving from (0,0) to (n, n) takes n steps right and n steps down, totally $2n$ steps.

As we want the whole path under the diagonal, the vertices (a, b) we pass should have the property that $a \leq b$.

In the standard Young tableaux, in the same column, the top number is always larger than the bottom number. So the number of steps we move right cannot be larger than the number of steps we move down, this is equivalent to $a \leq b$. $\square$