PARAMETRIZED AND SHIFTED NUMERICAL SEMIGROUPS

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Abstract. A numerical semigroup S is an additive subgroup of the non-negative integers. Previous works have developed the shifted numerical semigroup family M_n which comes from adding n to each generator of S and appending n to the list of generators. We further generalize this family of semigroups through the development of a parametrized and shifted numerical semigroup P_n obtained from applying weights to n before adding it to each generator of S. This paper generalizes several results of shifted numerical semigroups, and by doing so, we also develop a weighted factorization length and weighted delta set.

Contents

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1. Introduction

Numerical semigroups are additive semigroups of $\mathbb{Z}_{\geq 0}$, which are often written in terms of a generating set, e.g.

$$S = \langle r_1, \ldots, r_k \rangle = \{z_1 r_1 + z_2 r_2 + \cdots z_k r_k \mid z_1, \ldots, z_k \in \mathbb{Z}_{\geq 0}\}$$

As we have studied in algebra, groups have an amazing habit of finding their way into multiple branches of mathematics. Numerical semigroups, although more restricted than general groups, are no different. They appear when solving multi-variable homogeneous and non-homogeneous linear equations, they arise naturally in optimization as "knapsack problems," and they are relevant when ordering chicken McNuggets from McDonald's, to name a few applications.

In order to better understand the object, and to explain the last application, consider what is known as the "McNugget Semigroup." The fast food restaurant McDonald's used to sell McNuggets in sets of $6, 9$ or 20. Thus, if we do not want to overbuy, we can only buy amounts of McNuggets that can be generated by these values. What if we wanted 53 McNuggets? Is there a way to get exactly this amount, and if so, how many of each pack size do we need to purchase? To answer these questions, we can consider the numerical semigroup $S = \langle 6, 9, 20 \rangle$. From this, we can see that

$$53 = 4(6) + 1(9) + 1(20) = 1(6) + 3(9) + 1(20)$$

which gives us two ways to obtain our desired number of McNuggets: $(4, 1, 1)$ and $(1, 3, 1)$. We can then use these "factorizations" and cost information to minimize the overall price.

We study parametrized families of numerical semigroups of the form

$$P_n = \langle n, w_1 n + r_1, w_2 n + r_2, \ldots, w_k n + r_k \rangle$$

Our goal is to express semigroups of interest for large n. This generalizes families considered in [5, 7] as well as several central properties. In the process, we introduce the notion of weighted length and generalize results from [2].

To do so, each section builds upon its predecessor allowing us to work with slightly more complicated families of numerical semigroups until we eventually arrive at the parametrized numerical semigroup. Section 2 provides basic definitions and some initial examples to create familiarity with our object in both theory and application. Section 3 introduces the concept of shifted numerical semigroups. Additionally, it introduces theorems proven in [7] as well as some example applications of these properties. Section 4 is where we cover the central discovery of our paper, namely weighted factorization length. This concept is at the heart of nearly every generalization in subsequent sections. Also, this is where we demonstrate the relationship between the generators of our semigroup and an independent weight vector. Section 5 is where we present our definition of a parametrized and shifted numerical semigroup. More importantly, though, we offer an algorithm for taking any numerical semigroup and structuring it as a parametrized and shifted numerical semigroup. Section 6 gives more generalizations of proven applications. Section 7 offers suggestions for further proving applications of weighted length and parametrized numerical semigroup family.

The main takeaways from this paper are (i) the development of weighted factorization lengths and (ii) an algorithm for viewing arbitrary numerical semigroups as parametrized numerical semigroups.

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2. Background

Numerical semigroups are powerful tools in linear algebra and optimization. Before we start working with the objects themselves, we give several fundamental definitions, concepts, and notation that will be used throughout the paper. For a thorough introduction to numerical semigroups, see [8].

Definition 2.1. A *numerical semigroup* S is an additive subgroup of $\mathbb{Z}_{\geq 0}$. Typically we denote S by a sequence of generators $r_1 < r_2 < \cdots < r_k$ where

$$S = \langle r_1, r_2, \ldots, r_k\rangle = \left\{z_1r_1 + z_2r_2 + \cdots + z_kr_k \middle| z_1, z_2, \ldots, z_k \in \mathbb{Z}_{\geq 0}\right\}$$

We say S is *primitve* if $\gcd(r_1, r_2, \ldots, r_k) = 1$.

Remark 2.2. In general, there is no reason why each generator needs to be distinct. This becomes especially apparent when considering applications of numerical semigroups. For our purposes, if there are multiple $r_i = r_{i+1} = \cdots = r_j$, then we can consider grouping them together as a single generator to help with calculations.

Definition 2.3. For an element $s \in S$, a *factorization* of s is an expression of the form

$$s = z_1r_1 + z_2r_2 + \cdots + z_kr_k$$

for $z_i \geq 0$. We define the *set of factorizations of* s by

$$\mathsf{Z}(s) = \left\{s = z_1r_1 + z_2r_2 + \cdots + z_kr_k \middle| z_1, z_2, \ldots, z_k \in \mathbb{Z}_{\geq 0}\right\}$$

Generally, a particular factorization is denoted as a k-tuple

$$z = (z_1, z_2, \ldots, z_k) \in \mathsf{Z}(s)$$

The *length of* z is

$$|z| = z_1 + z_2 + \cdots + z_k$$

and *the set of factorization lengths of* s is

$$\mathsf{L}_S(s) = \{|z| : z \in \mathsf{Z}(s) \text{ with respect to } S\} = \{\text{factorization lengths of } s \text{ respecting } S\}$$

Example 2.4. Let $S = \langle 6, 9, 20\rangle = \{6a + 9b + 20c : a, b, c \in \mathbb{Z}_{\geq 0}\}$. Then,

$$26 = 6(1) + 9(0) + 20(1) \in S$$

As we can see, this is the only possible factorization of $26 \in S$. Alternatively,

$$67 = 6(3) + 9(1) + 20(2) = 6(0) + 9(3) + 20(2) \in S$$

Thus, for $z = (3, 1, 2)$ and $z' = (0, 3, 2)$, we have $z, z' \in \mathsf{Z}(67)$ with

$$|z| = 3 + 1 + 2 = 6 > 5 = 0 + 3 + 2 = |z'|$$

In this example, there were two factorizations of differing lengths. This is not always the case, though. Also, note that there is no restriction on the number of factorizations an element $s \in S$ can have. This is entirely dependent on the number of linear combinations of the generators that equal s. Hence, it is possible for there to be exactly 1 or a multitude of factorizations. Moreover, there is no restriction on the factorization lengths, so there can be several different factorizations of the same length, exactly one at a particular length, or none with some length.

Example 2.5. Let $S = \langle 3, 5, 7\rangle$. Then,

$$\mathsf{Z}(27) = \{\underbrace{(9,0,0)}_{\text{length 9}}, \underbrace{(4,3,0),(5,1,1)}_{\text{length 7}}, \underbrace{(0,4,1),(1,2,2),(2,0,3)}_{\text{length 5}}\}$$

For the purposes of this paper, we need to look at properties of a particular type of element, namely Betti elements. These are defined in terms of factorization graphs [5]. We will provide the same definition here, as well as a more intuitive notion through examples.

Definition 2.6 ([5, Definition 3.1])**.** For a fixed numerical semigroup S and an element $s \in S$, the *factorization graph* of s, denoted ∇_s has vertex set $\mathsf{Z}(s)$, and two vertices $z, z' \in \mathsf{Z}(s)$ are connected by an edge whenever they have at least one generator in common.

Example 2.7. Consider $S = \langle 3, 5, 7\rangle$. From Example 2.5, the factorization graph of $27 \in S$ is included as Figure 1 at the top of this page. We can see that the factorizations $(9, 0, 0)$ and $(0, 4, 1)$ are not connected directly.

We can think of the edges as relations between the generators. In this particular case, we can see that there are the following minimal relations between the generators.

$$10 = 2(5) = 1(3) + 1(7) \Rightarrow ((0, 2, 0), (1, 0, 1))$$
$$12 = 4(3) = 1(5) + 1(7) \Rightarrow ((4, 0, 0), (0, 1, 1))$$
$$14 = 2(7) = 3(3) + 1(5) \Rightarrow ((0, 0, 2), (3, 1, 0))$$

Notice that these relations have a special property where they do not "overlap" in their factorizations. By this we mean that if we have two factorizations z and z', then if some $z_i > 0$, then we are guaranteed that the corresponding $z'_i = 0$. Thus these relations

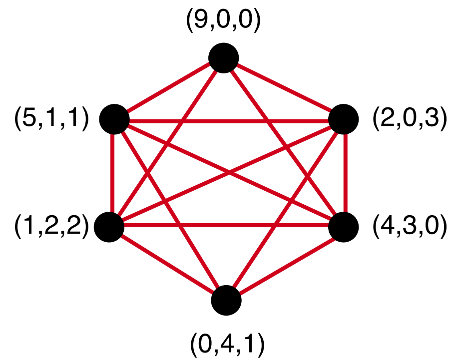

FIGURE 1. For $S = \langle 3, 5, 7\rangle$, this is the Factorization Graph of 27 from Example 2.7.

FIGURE 2. For $S = \langle 3, 5, 7\rangle$, these are the Factorization Graphs of 10 (left), 12 (middle), and 14 (right) from Example 2.9.

are minimal in a sense, allowing us to move between factorizations. In fact, the special elements whose factorizations encode minimal relations called "Betti elements."

Definition 2.8. For a fixed numerical semigroup S, $s \in S$ is a *Betti element* of S if its factorization graph ∇_s is disconnected. We define

$$B(S) = \{s \in S \mid s \text{ is a Betti Element of } S\}.$$

Example 2.9. From Example 2.7, the Betti elements of $S = \langle 3, 5, 7\rangle$ are

$$B(S) = \{10, 12, 14\}$$

and the factorization graphs are included above in Figure 2. Moreover, the missing edges are precisely the minimal relations from Example 2.7.

Example 2.10. The Betti elements of $S = \langle 6, 9, 20\rangle$ are $B(S) = \{18, 60\}$. The factorization graphs are depicted in Figure 3.

Now that we have some basic mechanisms to discuss numerical semigroups as a whole, we can start to examine them a bit more carefully. By definition, any numerical semigroup S is an infinite set. One non-intuitive property, though, is that $\left|\mathbb{Z}_{\geq 0} - S\right|$ is finite when $\gcd(r_1, \ldots, r_k) = 1$. The next couple of definitions and examples relate to this fact.

Definition 2.11. Define the *Apéry set* of $x \in S$ as

$$\mathrm{Ap}(S; x) = \{s \in S : s - x \in \mathbb{Z} \setminus S\}$$

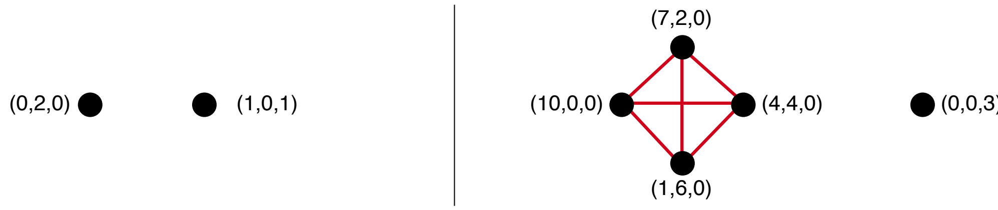

FIGURE 3. For $S = \langle 6, 9, 20 \rangle$, these are the Factorization Graphs of 18 (left) and 60 (right) from Example 2.10.

The *Apéry set of* S is $\text{Ap}(S) = \text{Ap}(S; r_1)$ where r_1 is the smallest generator of S.

Definition 2.12. The *Frobenius number* of a primitive numerical semigroup S, denoted $F(S)$, is the largest integer not in S, i.e.

$$F(S) = \max\left(\mathbb{Z}_{\geq 0} \setminus S\right).$$

For a non-primitive numerical semigroup $T = dS$ with $d \geq 1$, define $F(T) = d \cdot F(S)$.

Remark 2.13. Notice the relationship between the Apéry Set of S and its Frobenius number. Thus, by solving for the former, we immediately know the latter. This is not always easy, though. In many cases, solving for the Apéry Set is quite computationally expensive. Intuitively, we can understand that the larger the generators are, the larger the "gaps" between each of the initial values in S. This means we need to compute more values until these gaps disappear and we eventually obtain every value larger than $F(S)$. We will discuss runtimes and methods to decrease these times in the following section. For now, we can look at some examples.

Example 2.14. Let $S = \langle 6, 9, 20 \rangle$. Now we can solve for $\text{Ap}(S) = Ap(S; 6)$ using the definition. First note that

$$S = \left\{ \begin{array}{l} 0, 6, \mathbf{9}, 12, 15, 18, \mathbf{20}, 21, 24, 26, 27, \mathbf{29}, 30, 32, 33 \\ 35, 36, 38, 39, \mathbf{40}, 41, 42, 44, 45, 46, 47, 48, \mathbf{49}, \ldots \end{array} \right\}$$

From definition of $S \subseteq \mathbb{Z}_{\geq 0}$, we know that $0 \in Ap(S)$ because if we subtract any generator, we get a value that is not in S. Now note that, $1, 2, 3, 4, 5 \notin S$ by minimality of $6 \in S$. Since we are looking for values $s \in S$ such that $s \in S$ and $s - 6 \notin S$, we can iteratively add 6 to each value until we eventually land in S. We know this is possible because $\left|\mathbb{Z}_{\geq 0} - S\right|$ is finite, so we are guaranteed to get values in S if we do this for long enough.

We then get Table 1 on the following page. As we can see, we have obtained

$$\text{Ap}(S) = \{0, 9, 20, 29, 40, 49\}$$

which also tells us that $F(S) = 49 - 6 = 43$.

x mod 6	Values
1	$1, 7, 13, 19, 25, 31, 37, 43, \mathbf{49}, \mathbf{55}, \mathbf{61}, \ldots$
2	$2, 8, 14, \mathbf{20}, \mathbf{26}, \mathbf{32}, \ldots$
3	$3, \mathbf{9}, \mathbf{15}, \mathbf{21}, \ldots$
4	$4, 10, 16, 22, 28, 34, \mathbf{40}, \mathbf{46}, \mathbf{52}, \ldots$
5	$5, 11, 17, 23, \mathbf{29}, \mathbf{35}, \mathbf{41}, \ldots$

Table 1. Iterative method for Example 2.14, bolded values lie in $S = \langle 6, 9, 20\rangle$

Remark 2.15. This example gives us a basic method to compute Apéry sets. Moreover, we can see that there is a direct correspondence between elements of an Apéry set and modular equivalence classes. This implies $|Ap(S; s)| = s/\gcd(S)$ [8].

3. Shifted Semigroups

Using the objects defined in the previous section, we now develop the concept of "shifting" a numerical semigroup which comes directly from [5, 7].

Definition 3.1. Let $S = \langle r_1, r_2, \ldots, r_k\rangle$ be a numerical semigroup and let $n \in \mathbb{N}$. The *shifted numerical semigroup* M_n is

$$M_n := \langle n, n + r_1, n + r_2, \ldots, n + r_k\rangle$$

In this case, we will call n the *shift parameter*.

Remark 3.2. We can consider this shifting as a mapping where $(S, n) \mapsto M_n$.

Notice that our shifted semigroup M_n includes one additional generator n. This is to ensure every numerical semigroup lies in exactly one shifted family.

Example 3.3. Let $S = \langle 6, 9, 20\rangle$, $n = 400$, $S' = \langle 5, 8, 19\rangle$, and $n' = 401$. In order to demonstrate the above point, suppose we do not add the shifting value to our list of shifted generators. Notice

$$(S, 400) \mapsto N_{400} = \langle 406, 409, 420\rangle$$
$$(S', 401) \mapsto N'_{401} = \langle 406, 409, 420\rangle$$
$$N_{400} = N'_{401}$$

Since N_{400} can be created from different initial generators and shifting values, we cannot define a unique bijection between the semigroup and its shifted counterpart. On the other hand, by adding the shifted value as a generator as in Definition 3.1, we ensure the existence of such a bijection.

$$(S, 400) \mapsto M_{400} = \langle 400, 406, 409, 420\rangle$$
$$(S', 401) \mapsto M'_{401} = \langle 401, 406, 409, 420\rangle$$

As we discussed in Section 2, running computations becomes more expensive as the generators become larger and as the number of generators increases. Hence, if the shifting value is significantly large, directly calculating properties of M_n (such as its Apéry set) becomes computationally expensive. Immediately one can wonder about the relationship between the original numerical semigroup S, the shifting value n, and the resulted shifted numerical semigroup M_n. Again, this further justifies the importance of including the shifting value as a generator of M_n. If we find ourselves in the scenario of the above example, then it becomes very unclear as to which set of generators and shifting values are important.

First, we will examine Apéry sets of shifted numerical semigroups. As presented in [7], there is a crucial connection between the Apéry set of S and that of M_n.

Theorem 3.4 ([7, Theorem 3.3])**.** *If $n \in S$ satisfies $n > r_k^2$, then*

$$\mathrm{Ap}(M_n; n) = \{i + \mathsf{m}_S(i) \cdot n | i \in \mathrm{Ap}(S; dn)\}$$

where $d = gcd(S)$ and m_S denotes the minimum factorization length in S. Moreover,

$$L_{M_n}(i + \mathsf{m}_S(i) \cdot n) = \{\mathsf{m}_S(i)\}$$

for each $i \in \mathrm{Ap}(S; dn)$.

Remark 3.5. At the heart of Theorem 3.4 is [2, Theorem 4.2] stating that $m(n+r_k) = m(n) + 1$ for $n \gg 0$ for the minimum factorization length function m_S. This result will be generalized for weighted lengths in Theorem 4.13.

Example 3.6. To see this relation, consider $S = \langle 3, 5, 7\rangle$ and $n = 50$. Notice, $gcd(S) = 1$ and $n = 50 > 49 = 7^2 = r_3^2$. By definition, $|\mathrm{Ap}(M_{50}; 50)| = 50$, so we will walk through the calculation of the first four elements. Using the GAP package *numericalsgps* [6], we know that

$$\{0, 51, 52, 3\} \subset \mathrm{Ap}(S; 1 \cdot 50)$$

and the associated minimum factorizations are $0, 9, 8$, and 1. Applying Theorem 3.4,

$$0 + 0 \cdot 50 = 0 \in \mathrm{Ap}(M_{50}; 50)$$
$$51 + 9 \cdot 50 = 501 \in \mathrm{Ap}(M_{50}; 50)$$
$$52 + 8 \cdot 50 = 452 \in \mathrm{Ap}(M_{50}; 50)$$
$$3 + 1 \cdot 50 = 53 \in \mathrm{Ap}(M_{50}; 50)$$

Remark 3.7. By definition, each $a \in \mathrm{Ap}(S; n)$ corresponds to the minimum value in an equivalence class modulo n. We can see that the resulting value corresponds to the same equivalence class. Let $a \in \mathrm{Ap}(M_n; n)$, then we can see that this is because

$$a \bmod n \equiv (i + \mathsf{m}_S(i) \cdot n) \bmod n \equiv i \bmod n$$

Therefore, because GAP orders the Apéry Set based on the equivalence class's value (i.e. $1 \bmod n$, $2 \bmod n$, $\ldots$, $(n-1) \bmod n$), these four values correspond exactly to the first four values outputted for $\mathrm{Ap}(M_{50}; 50)$.

Proposition 3.8 ([2, Theorem 4.2])**.** *If $dn \in S$ and $dn > F(S)$, then* $\mathrm{Ap}(S; dn) = \{a_0, \ldots, a_{n-1}\}$*, where*

$$a_i = \begin{cases} di & \text{if } di \in S \\ di + dn & \text{if } di \notin S \end{cases}$$

and $d = gcd(S)$. In particular, this holds whenever $n > r_k^2$ as in Theorem 3.3.

Remark 3.9. Let us briefly walk through the intuition behind Proposition 3.8. In order to do so, there are a couple cases to consider.

First, suppose $di \in S$. Since $i < n$ and $d > 0$, we know $di < dn$. Thus, $di - dn < 0$ which tells us that $di - dn \notin S$. Hence, by definition, we must have that $di \in \mathrm{Ap}(S; dn)$.

Second, suppose that $di \notin S$. We must have that $di + dn \in S$ because $di + dn > dn > F(S)$. Also, we know that $(di + dn) - dn = di \notin S$, so $di + dn \in \mathrm{Ap}(S; n)$ by definition.

Example 3.10. Let $S = \langle 7, 11, 16 \rangle$, so $\mathrm{Ap}(S) = \{0, 22, 16, 38, 11, 33, 27\}$ and $F(S) = 31$. Let $n = 40$. Notice that S is primitive, so $d = 1$. Now we can calculate $\mathrm{Ap}(S; 40)$.

0	1	2	3	4	5	6	**7**	8	9
10	**11**	12	13	**14**	15	**16**	17	**18**	19
20	**21**	**22**	**23**	24	**25**	26	**27**	**28**	**29**
30	31	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**
40	**41**	**42**	**43**	**44**	**45**	**46**	47	**48**	**49**
50	51	**52**	**53**	54	**55**	56	**57**	58	**59**
60	61	62	63	**64**	65	**66**	67	68	69
70	**71**	72	73	74	75	76	77	78	79

In the above presentation, the bold values are those in $\mathrm{Ap}(S; 40)$. Hence, we can see the relationship between values in $\mathrm{Ap}(S; 40)$ and those that are not.

Example 3.11. Let $S = \langle 6, 9, 20 \rangle$, so $\mathrm{Ap}(S) = \{0, 49, 20, 9, 40, 29\}$ and $F(S) = 43 = 49 - 6$. Let $n = 58$. Notice that S is primitive, so $d = 1$. By definition, $|\mathrm{Ap}(S; n)| = 58$, so we will calculate a few elements.

$$0 \in S \Rightarrow a_0 = 1 \cdot 0 = 0$$

$$1 \notin S \Rightarrow a_1 = 1 \cdot 1 + 58 = 59$$

$$2 \notin S \Rightarrow a_2 = 1 \cdot 2 + 58 = 60$$

$$\vdots$$

$$56 \in S \Rightarrow a_{56} = 1 \cdot 56 = 56$$

$$57 \in S \Rightarrow a_{57} = 1 \cdot 57 = 57$$

n	M_n	GAP [6]	Algorithm Times
50	$\langle 50, 56, 59, 70\rangle$	1 ms	1 ms
200	$\langle 200, 206, 209, 220\rangle$	30 ms	30 ms
400	$\langle 400, 406, 409, 420\rangle$	170 ms	170 ms
1000	$\langle 1000, 1006, 1009, 1020\rangle$	3 sec	1 ms
5000	$\langle 5000, 5006, 5009, 5020\rangle$	17 min	1 ms
10000	$\langle 10000, 10006, 10009, 10020\rangle$	3.6 hr	1 ms

TABLE 2. This table comes directly from [7, Table 1]. In this example, $S = \langle 6, 9, 20\rangle$.

Theorem 3.4 and Proposition 3.8 provide powerful algorithms for calculating Apéry sets of shifted numerical semigroups with large shift parameter. Rather than needing to do calculations with large generators of M_n, we can compute from those of S. This drastically decreases the computation time necessary to find Apéry sets. Table 1 demonstrates the computational benefits of these algorithms. Note that in the table's example semigroup S, $r_k^2 = 20^2 = 400$. In order to apply Theorem 3.4, we must have that $n > 400$. This explains why the $n = 400$ case is not improved since $400 \not> 400$. Moreover, since we already have the exact Apéry set, if follows that we can calculate the Frobenius number of these "larger" sets faster using these algorithms.

4. Weighted Factorization Lengths

Thus far, we have only been considering semigroups of the form

$$S = \langle r_1, r_2, \ldots, r_k\rangle \qquad M_n = \langle n, n + r_1, n + r_2, \ldots, n + r_k\rangle$$

This form is rather restrictive, so one may ask what would happen if we apply "weights" to our list of generators. In particular, if we have a vector of weights

$$w = (w_1, w_2, \ldots, w_k) \in \mathbb{Z}^k,$$

we get

$$S = \langle r_1, r_2, \ldots, r_k\rangle \qquad P_n = \langle n, w_1n + r_1, w_2n + r_2, \ldots, w_kn + r_k\rangle$$

We will discuss the parametrized semigroups P_n in the following section. This section will set the foundation for a generalization of future theorems by establishing a weighted ordering (which we will refer to as a w-ordering) of the generators of S. The main property shown here is that the minimum weighted length function m_w upholds a generalization of Remark 3.5.

Definition 4.1. Consider the numerical semigroup $S = \langle r_1, r_2, \ldots, r_k\rangle$ and weight vector $w = (w_1, w_2, \ldots, w_k) \in \mathbb{Z}^k$. Given $s \in S$ and $a = (a_1, a_2, \ldots, a_k) \in \mathsf{Z}(s)$, the *weighted length of* a is

$$|a|_w = a_1w_1 + a_2w_2 + \cdots + a_kw_k$$

and *the set of weighted factorization lengths* of s respecting S is

$$\mathsf{L}_{S,w}(s) = \{|z|_w : z \in \mathsf{Z}(s) \text{ with respect to } S\}$$

Moreover, we define the following two maps

$$\mathsf{M}_w : S \mapsto \mathbb{N} \qquad \mathsf{m}_w : S \mapsto \mathbb{N}$$

where M_w maps $s \in S$ to its maximum weighted factorization length and m_w maps $s \in S$ to its minimum weighted factorization length.

Remark 4.2. We can think of the standard length $|\cdot|$ as $|\cdot|_w$ where $w = (1, 1, \ldots, 1)$ since

$$\begin{array}{rcccccccccc} |\vec{a}| & = & a_1 & + & a_2 & + & \cdots & + & a_k & = & \\ & = & a_1(1) & + & a_2(1) & + & \cdots & + & a_k(1) & = & \\ & = & a_1w_1 & + & a_2w_2 & + & \cdots & + & a_kw_k & = & |\vec{a}|_w \end{array}$$

Thus, we can think of this weighted length as a egeneralization of the usual factorization length. Because of this, all of the previously proven properties still hold under this notion with an assumed weight vector of $w = (1, 1, \ldots, 1)$.

Example 4.3. Let $S = \langle 5, 9, 13 \rangle$ and $w = (2, 2, 3)$. Consider $60 \in S$ which has factorizations

$s \in \mathsf{Z}(60)$	$\lvert s\rvert_w$
$(12, 0, 0)$	12(2) + 0(2) + 0(3) = 24
$(3, 5, 0)$	3(2) + 5(2) + 0(3) = 16
$(4, 3, 1)$	4(2) + 3(2) + 1(3) = 17
$(5, 1, 2)$	5(2) + 1(2) + 2(3) = 18

Remark 4.4. At this time, we do allow our weights to be unrestricted in $\mathbb{Z}$, so factorizations can have negative weighted length.

Example 4.5. Let $S = \langle 5, 9, 13 \rangle$ and $w' = (1, -2, 3)$. We then get a different set of factorization lengths for $60 \in S$.

$s \in \mathsf{Z}(60)$	$\lvert s\rvert_{w'}$
$(12, 0, 0)$	12(1) + 0(-2) + 0(3) = 12
$(3, 5, 0)$	3(1) + 5(-2) + 0(3) = -7
$(4, 3, 1)$	4(1) + 3(-2) + 1(3) = 1
$(5, 1, 2)$	5(1) + 1(-2) + 2(3) = 9

Before proving any properties of this weighted length, we need several tools.

Definition 4.6. Let $S = \langle r_1, r_2, \ldots, r_k \rangle$ and $w = (w_1, w_2, \ldots, w_k)$. Define the following

$$\Pi_i := \frac{w_i}{r_i}$$

Additionally, we will define the following notation

$$r_i \leq_w r_j \text{ if and only if } \Pi_i \geq \Pi_j$$

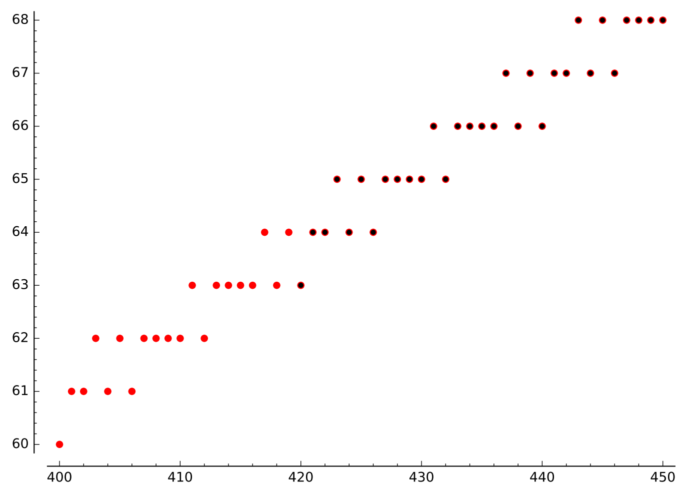

FIGURE 4. For Example 4.8, minimum factorization length plot with shifted values overlayed

$$r_i <_w r_j \text{ if and only if } \Pi_i > \Pi_j$$

When this kind of ordering is used, we will call it the *w-ordering*.

Remark 4.7. If every $w_i = 1$, then this is the usual ordering on $\mathbb{Z}$.

It is important to note that when the w-ordering is necessary, it will be specified in the property statement. Now, we can examine what this ordering tells us about the numerical semigroup.

Example 4.8. Let $S = \langle 9, 6, 20 \rangle$ and $w = (2, 1, 3)$.
(Note that the generators and weight vector have been reordered based on w-ordering.)

$$\Pi_1 = 6 \cdot 2 \cdot 20 = 240$$

$$\Pi_2 = 1 \cdot 9 \cdot 20 = 180$$

$$\Pi_3 = 6 \cdot 9 \cdot 3 = 162$$

Then for $400 \leq n \leq 500$, we get the minimum weighted length plots in Figure 4 on the following page. For each point in the original weighted length graph (x, y), we can map it to its corresponding shifted point $(x + 20, y + 3)$. Thus, it satisfies the relationship $\mathsf{m}_w(x) = \mathsf{m}_w(x - 20) + 3$.

Lemma 4.9. *Fix a weight vector $w = (w_1, \ldots, w_k) \in \mathbb{N}^k$ and a numerical semigroup $S = \langle r_1, \ldots, r_k \rangle$ which is minimally generated by $r_1 <_w r_2 \leq_w \cdots \leq_w r_k$. If $a, b \in \mathsf{Z}(s)$ for some $s \in S$ where $b = (b_1, 0, 0, \ldots, 0)$ and $a = (0, a_2, \ldots, a_k)$, then $|b|_w > |a|_w$.*

Proof: Consider the above setup where $b = (b_1, 0, 0 \ldots, 0)$ and $a = (0, a_2, a_3, \ldots, a_k)$. Hence, $b_1 r_1 = a_2 r_2 + \cdots + a_k r_k$. Notice that

$$|a|_w = \sum_{i=2}^{k} a_i w_i = \sum_{i=2}^{k} a_i \Pi_i r_i < \Pi_1 \sum_{i=2}^{k} a_i r_i = \Pi_1 \cdot b_1 r_1 = b_1 w_1 = |b|_w$$

Therefore, we can conclude $|b|_w > |a|_w$. □

Lemma 4.10. *Given a weight vector $w = (w_1, \ldots, w_k)$ and a numerical semigroup $S = \langle r_1, \ldots, r_k \rangle$ which is minimally generated by $r_1 \leq_w r_2 \leq_w \cdots \leq_w r_{k-1} <_w r_k$. If $a, b \in \mathsf{Z}(s)$ for some $s \in S$ where $b = (0, 0, \ldots, 0, b_k)$ and $a = (a_1, a_2, \ldots, a_{k-1}, 0)$, then $|b|_w < |a|_w$.*

Proof: Consider the above setup where $b = (0, 0, 0 \ldots, b_k)$ and $a = (a_1, a_2, a_3, \ldots, a_{k-1}, 0)$. Hence, $b_k r_k = a_1 r_1 + \cdots + a_{k-1} r_{k-1}$. Notice that

$$|a|_w = \sum_{i=1}^{k-1} a_i w_i = \sum_{i=1}^{k-1} a_i \Pi_i r_i > \Pi_k \sum_{i=1}^{k-1} a_i r_i = \Pi_k b_k r_k = b_k w_k = |b|_w$$

Therefore, we can conclude $|b|_w < |a|_w$. □

Finally, there is one lemma relating to modular arithmetic that we will need in the proof of Theorem 4.12 and Theorem 4.13.

Lemma 4.11 ([2, Lemma 4.1])**.** *Let $k \geq 0$, and fix $c_1, c_2, \ldots, c_r \in \mathbb{Z}$ with $r \geq k$. There exists $T \subsetneq \{1, \ldots, r\}$ satisfying $\sum_{i \in T} c_i \equiv \sum_{i=1}^{r} c_i \bmod k$.*

Now we can examine properties of the weighted length just as we did with the original notion of factorization lengths. The following results generalize from [2, Theorem 4.2, Theorem 4.3].

Theorem 4.12. *Given a weight vector $w = (w_1, \ldots, w_k)$ and a numerical semigroup $S = \langle r_1, \ldots, r_k \rangle$ which is minimally generated by $r_1 <_w r_2 \leq_w \cdots \leq_w r_k$, the maximal weighted length function $\mathsf{M}_w : S \mapsto \mathbb{N}$ satisfies*

$$\mathsf{M}_w(n) = \mathsf{M}_w(n - r_1) + w_1$$

for all $n > (r_1 - 1) r_k$.

Proof. Fix $n \in S$ and fix $a = (a_1, \ldots, a_k) \in \mathsf{Z}(n)$, and suppose that $a_2 + \ldots + a_k \geq r_1$. We will find a factorization with strictly larger weighted length. By definition of a factorization, $a_1 r_1 + a_2 r_2 + \ldots + a_k r_k = n$. Notice,

$$-a_1 r_1 = (a_2 r_2 + \ldots + a_k r_k) - (a_1 r_1 + a_2 r_2 + \ldots + a_k r_k) = (a_2 r_2 + \ldots + a_k r_k) - n,$$

So $r_1 \mid [(a_2 r_2 + \ldots + a_k r_k) - n]$. Hence, $a_2 r_2 + \ldots + a_k r_k \equiv n \mod r_1$. Viewing this sum as $a_2 + \cdots + a_k$ integers from $\{r_2, \ldots, r_k\}$, Lemma 4.11 guarantees the existence of $b_2, \ldots, b_k \geq 0$ such that (i) $b_i \leq a_i$ for each $i > 1$, (ii) $\sum_{i=2}^{k} a_i > \sum_{i=2}^{k} b_i$, and (iii)

$b_2r_2 + \cdots + b_kr_k \equiv n \mod r_1$. This implies $b_2r_2 + \cdots + b_kr_k < a_2r_2 + \cdots + a_kr_k$, so there exists $b_1 > 0$ so that $b = (b_1, \ldots, b_k) \in \mathsf{Z}(n)$. Notice,

$$b_1r_1 + b_2r_2 + \cdots + b_kr_k = n = a_1r_1 + a_2r_2 + \cdots + a_kr_k$$
$$(b_1 - a_1)r_1 = (a_2 - b_2)r_2 + \cdots + (a_k - b_k)r_k$$

We can then apply Lemma 4.9 to the factorizaions $(b_1 - a_1, 0, 0, \ldots, 0)$ and $(0, a_2 - b_2, a_3 - b_3, \ldots, a_k - b_k)$ which implies

$$(b_1 - a_1)w_1 > (a_2 - b_2)w_2 + \cdots + (a_k - b_k)w_k$$

meaning

$$b_1w_1 + b_2w_2 + \cdots + b_kw_k > a_1w_1 + a_2w_2 + \cdots + a_kw_k$$

so $|b|_w > |a|_w$.

Now suppose that $a_2 + \ldots + a_k < r_1$. Since $n > (r_1 - 1)r_k$, we have $a_1 > 0$. This means $a - e_1 \in \mathsf{Z}(n - r_1)$, so

$$\mathsf{M}_w(n - r_1) \geq |a - e_1|_w = (a_1 - 1)w_1 + a_2w_2 + \ldots a_kw_k = |a|_w - w_1,$$

and since a was maximal length, $\mathsf{M}_w(n - r_1) = |a| - w_1$ is of maximal length. □

Theorem 4.13. *Given a weight vector $w = (w_1, \ldots, w_k)$ and a numerical semigroup $S = \langle r_1, r_2, \ldots, r_k \rangle$ minimally generated by $r_1 \leq_w r_2 \leq_w \cdots \leq_w r_{k-1} <_w r_k$, the minimal weighted factorization length $\mathsf{m}_w : S \mapsto \mathbb{N}$ satisfies*

$$\mathsf{m}_w(n) = \mathsf{m}_w(n - r_k) + w_k$$

for all $n > (r_k - 1)r_{k-1}$.

Proof. Fix $n \in S$ and fix $a = (a_1, \ldots, a_k) \in \mathsf{Z}(n)$, and suppose that $a_1 + \ldots + a_{k-1} \geq r_k$. We will find a factorization with strictly shorter weighted length. By definition of a factorization, $a_1r_1 + a_2r_2 + \ldots + a_kr_k = n$. Notice,

$$-a_kr_k = (a_1r_1 + \ldots + a_{k-1}r_{k-1}) - (a_1r_1 + a_2r_2 + \ldots + a_kr_k) = (a_2r_2 + \ldots + a_kr_k) - n,$$

So $r_k \mid [(a_1r_1 + \ldots + a_{k-1}r_{k-1}) - n]$. Hence, $a_1r_1 + \ldots + a_{k-1}r_{k-1} \equiv n \mod r_k$. Viewing this sum as $a_1 + \cdots + a_{k-1}$ integers from $\{r_1, \ldots, r_{k-1}\}$, Lemma 4.11 guarantees the existence of $b_1, \ldots, b_{k-1} \geq 0$ such that (i) $b_i \leq a_i$ for each $i < k$, (ii) $\sum_{i=1}^{k-1} a_i > \sum_{i=1}^{k-1} b_i$, and (iii) $b_1r_1 + \cdots + b_{k-1}r_{k-1} \equiv n \mod r_k$. This implies $b_1r_1 + \cdots + b_{k-1}r_{k-1} < a_1r_1 + \cdots + a_{k-1}r_{k-1}$, so there exists $b_k > 0$ so that $b = (b_1, \ldots, b_k) \in \mathsf{Z}(n)$. Notice,

$$b_1r_1 + b_2r_2 + \cdots + b_kr_k = n = a_1r_1 + a_2r_2 + \cdots + a_kr_k$$
$$(b_k - a_k)r_k = (a_1 - b_1)r_1 + (a_2 - b_2)r_2 + \cdots + (a_{k-1} - b_{k-1})r_{k-1}$$

We can then apply Lemma 4.10 to the factorizations $(0, 0, \ldots, 0, b_k - a_k)$ and $(a_1 - b_1, a_2 - b_2, a_3 - b_3, \ldots, a_{k-1} - b_{k-1}, 0)$ which implies

$$(b_k - a_k)w_k < (a_1 - b_1)w_1 + \cdots + (a_{k-1} - b_{k-1})w_{k-1}$$

meaning

$$b_1w_1 + b_2w_2 + \cdots + b_kw_k < a_1w_1 + a_2w_2 + \cdots + a_kw_k$$

so $|b|_w < |a|_w$.

Now suppose that $a_1 + \ldots + a_{k-1} < r_k$. Since $n > (r_k - 1)r_{k-1}$, we must have $a_k > 0$. This means $a - e_k \in \mathsf{Z}(n - r_k)$, so

$$\mathsf{m}_w(n - r_k) \geq |a - e_k|_w = a_1 w_1 + a_2 w_2 + \ldots (a_k - 1) w_k = |a|_w - w_k$$

and since a was maximal length, $\mathsf{m}_w(n - r_k) = |a| - w_k$ is of minimal length. □

Corollary 4.14 follows immediately from the first portion of the proof of Theorem 4.13. This is important for the proof of Lemma 5.5(a).

Corollary 4.14. *Let $w = (w_1, \ldots, w_k)$ be a weight vector and $S = \langle r_1, r_2, \ldots, r_k \rangle$ be a numerical semigroup minimally generated by $r_1 \leq_w r_2 \leq_w \cdots \leq_w r_{k-1} <_w r_k$. Suppose $n > (r_k - 1)r_{k-1}$ and $a = (a_1, \ldots, a_k) \in \mathsf{Z}(n)$. If $a_1 + \cdots + a_{k-1} \geq r_k$, then $\exists b \in \mathsf{Z}(n)$ such that $|b|_w < |a|_w$ with $b_k > 0$.*

Thus, we have explained the relationship demonstrated in Example 4.8. One immediate question we can ask is, for the w-ordering, what happens when multiple Π_i values are equal? Although it has not been proven, it leads to the following conjecture.

Conjecture 4.15. *In Theorem 4.13, the enforcement of a strictly larger r_k generator is unnecessary, and if there exists two maximal $r_i =_w r_j$, then*

$$\mathsf{m}_w(n) = \mathsf{m}_w(n-1) + 1.$$

We will address this conjecture through the following example.

Example 4.16. Let $S = \langle 3, 5, 7 \rangle$ and $w = (3, 5, 8)$. Then,

$$\Pi_1 = 3 \cdot 5 \cdot 7 = 105$$
$$\Pi_2 = 3 \cdot 5 \cdot 7 = 105$$
$$\Pi_3 = 3 \cdot 5 \cdot 8 = 120$$

This scenario is quite different than the previous example since $\Pi_1 = \Pi_2$. When this occurs, we have not distinguished between the orderings of $S = \langle 3, 5, 7 \rangle = \langle 5, 3, 7 \rangle$. Thus, if the conjecture holds, then Theorem 4.13 tells us that the minimum weighted factorization length of our n value satisfies two equations.

$$\mathsf{m}_w(n) = \mathsf{m}_w(n-3) + 3 \text{ and } \mathsf{m}_w(n) = \mathsf{m}_w(n-5) + 5$$

So, with $x = n - 3$, we get

$$\mathsf{m}_w(n-3) + 3 = \mathsf{m}_w(n-5) + 5$$
$$\mathsf{m}_w(x) = \mathsf{m}_w(x-2) + 2$$

Now we have a third equation that our minimum factorization must satisfy. So, with $y = n - 2$, we have

$$\mathsf{m}_w(n-2) + 2 = \mathsf{m}_w(n-3) + 3$$
$$\mathsf{m}_w(y) = \mathsf{m}_w(y-1) + 1$$

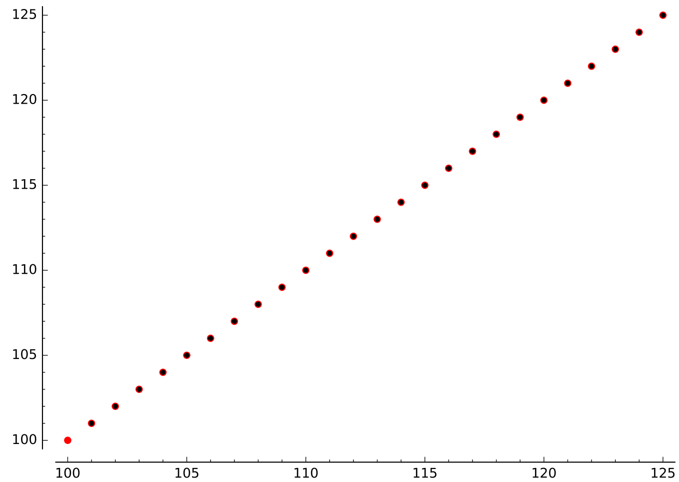

FIGURE 5. For Example 4.16, minimum factorization length plot with shifted values overlayed

Therefore, up to renaming variables, we must have that $\mathsf{m}_w(n) = \mathsf{m}_w(n-1)+1$ which explains the minimum weighted length plots in Figure 5. For each point in the original weighted length graph (x, y), we can map it to its corresponding shifted point $(x+1, y+1)$. Thus, it satisfies the relationship $\mathsf{m}_w(x) = \mathsf{m}_w(x-1)+1$.

Remark 4.17. Notice that a similar method of cancellation can be used for any case where multiple Π_i values are minimal. Later we will restrict that the minimal Π_i value must be uniquely achieved, so this kind of a scenario will not be too big of a concern to us.

5. Parametrized Numerical Semigroups

We have now built our way to the notion of parametrized semigroups. This "parametrization" comes from weighting our shifting value. This will be become clear through the next few definitions and examples.

Definition 5.1. Let $S = \langle r_1, r_2, \ldots, r_k\rangle$ be a numerical semigroup, let $w = (w_1, w_2, \ldots, w_k) \in \mathbb{Z}^k_{>0}$ be a weight vector, and let $n \in \mathbb{N}$. The *parametrized numerical semigroup* P_n is

$$P_n := \langle n, w_1 n + r_1, w_2 n + r_2, \ldots, w_k n + r_k\rangle$$

Remark 5.2. Note that we are restricting our weights to be only positive integers. If we do not, then it is possible that we can have negative generators for P_n. This restriction simply ensures we stay within the realm of semigroups.

Observation. With this format in place, we can algorithmically restructure any numerical semigroup into this style. Let $\Phi = \langle g_0, g_1 \ldots, g_k\rangle$ be an arbitrary numerical semigroup.

1) Order and relabel the generators such that $g_0 < g_1 < \cdots < g_\ell$ for $1 \leq \ell \leq k$. If there exists some $g_i = g_j$ where $i \neq j$, then we can disregard the redundant generator g_j.
2) For each $i \neq 1$, divide g_i by g_1 using the Division Algorithm yielding some $w_i \in \mathbb{Z} > 0$ and $r_i \in \mathbb{Z} \geq 0$ such that
$$g_i = w_i g_1 + r_i$$
If $r_i = 0$, then g_i must be a multiple of g_1, so we can disregard g_i because every value it contributes to can be replaced with multiples of g_1.
3) Then for $1 \leq t \leq \ell \leq k$, we can rewrite Φ as
$$\Phi = \langle g_0, w_1 g_0 + r_1, w_2 g_0 + r_2, \ldots, w_t g_0 + r_t\rangle$$

Therefore, each numerical semigroup lies in exactly one parametrized family. Notice that this is similar to what was discussed in Remark 3.2.

Example 5.3. Let $\Phi = \langle 37, 892, 355, 452\rangle$. Then,

1) $g_1 = 37 < g_2 = 355 < g_3 = 452 < g_4 = 892$
2) $r_1 = 355 \mod 37 = 22$, $w_1 = \frac{355-22}{37} = 9$
$r_2 = 452 \mod 37 = 8$, $w_2 = \frac{452-8}{37} = 12$
$r_3 = 892 \mod 37 = 4$, $w_3 = \frac{892-4}{37} = 24$
3) So,
$$\Phi = P_{37} = \langle 37, 9(37) + 22, 12(37) + 8, 24(37) + 4\rangle$$

which then tells us that $S = \langle 4, 8, 22\rangle$ and $w = (24, 12, 9)$ are the related semigroup and weight vector written with the Π ordering.

Since we now have a way to associate weights with our generating values, this offers an opportunity to use our concept of weighted lengths and the previously proven properties. Hence, we have developed a method to "map down" any given numerical semigroup to one with fewer generators and a related weight vector.

For the remainder of this section, we will focus on generalizing [5, Theorem 3.4]. This theorem is a technical result that sits at the center of the results proven in [5, 7], and it will serve as a key property for generalizations in the following section.

In order to prove Theorem 5.7, we need the following lemma and notation.

Notation 5.4. Consider the weight vector $w = (w_1, \ldots, w_k)$ and numerical semigroup $S = \langle r_1, \ldots, r_k\rangle$. Let
$$W = \max\{w_1, \ldots, w_k\} \qquad R = \max\{r_1, \ldots, r_k\}$$
Additionally, for all numerical semigroups S in the remainder of the paper, we will impose that the generators are listed so as to satisfy the w-ordering.

Lemma 5.5. *Fix $a \in S = \langle r_1, \dots, r_k \rangle$ such that*

$$r_1 \leq_w r_2 \leq_w \cdots \leq_w r_{k-1} <_w r_k$$

Also fix $s = (s_1, \dots, s_k) \in \mathsf{Z}(a)$ and $w = (w_1, \dots, w_k) \in \mathbb{Z}^k > 0$. Then,

(a) If $\sum_{i=1}^k s_i \geq r_k$ and $s_k = 0$, then $\exists s' \in \mathsf{Z}(a)$ such that $|s'|_w < |s|_w$ and $s'_k > 0$.
(b) If $a > W r_{k-1} r_k$, then there is a factorization $s' \in \mathsf{Z}(a)$ of minimal weighted length such that $s'_k > 0$.
(c) If $a > W r_k^2$, then $|s|_w > W r_k$.

Proof of (a). This follows immediately from Corollary 4.14. $\square$

Proof of (b). To derive a contradiction, suppose that $a > W r_{k-1} r_k$ and s has minimal weighted length but $s_k = 0$. Thus,

$$W r_{k-1} r_k < a = s_1 r_1 + \cdots + s_{k-1} r_{k-1} = s_1 \frac{w_1}{\Pi_1} + \cdots + s_{k-1} \frac{w_{k-1}}{\Pi_{k-1}} \leq |s|_w \cdot \frac{1}{\Pi_{k-1}}$$
$$= |s|_w \cdot \frac{r_{k-1}}{w_{k-1}} \leq \frac{r_{k-1}}{w_{k-1}} W \sum_{i=1}^{k-1} s_i \leq r_{k-1} W \sum_{i=1}^{k-1} s_i$$

So,

$$r_k < \sum_{i=1}^{k-1} s_i = \sum_{i=1}^{k} s_i$$

By (a), we know there exists a factorization $b \in \mathsf{Z}(a)$ with $b_k > 0$ such that $|b|_w < |s|_w$. Contradiction! Therefore, we can conclude that if $a > W r_{k-1} r_k$ and s has minimal weighted length, then $s_k > 0$. $\square$

Proof of (c). Suppose $a > W r_k^2$. Then,

$$W r_k^2 < a = s_1 r_1 + \cdots + s_k r_k = s_1 \frac{w_1}{\Pi_1} + \cdots + s_k \frac{w_k}{\Pi_k} \leq |s|_w \cdot \frac{1}{\Pi_k}$$
$$= |s|_w \cdot \frac{r_k}{w_k} \leq |s|_w \cdot r_k,$$

so $W r_k < |s|_w$. $\square$

Example 5.6. Suppose we do not enforce that our weights are positive integers. Consider $S = \langle 6, 9, 20 \rangle$, $w = (1, -1, -2)$. Thus, $r_k^2 = 400$, so $(1, 0, 20) \in \mathsf{Z}(406)$ but

$$\big|(1, 0, 20)\big|_w = 1(1) + 0(-1) + 20(-2) = -39 \not> 20 = r_k$$

Hence, our claim only holds with this restriction.

Theorem 5.7. *Suppose $n > W R^2$ and let z and z' be factorizations of a Betti element $\beta \in P_n$ with weight vector $w = (w_1, w_2, \dots, w_k)$ in different connected components of ∇_β. If $|z|_w > |z'|_w$, then $z_0 > 0$ and $z'_k > 0$.*

Proof. We begin by noting that we construct P_n such that $w_0 = 1$, and we cancel all common generators in the factorizations of z and z' so that there is no overlap in non-zero factorization values. Now we can observe that

$$\beta - |z|_w n = z_0 n + \sum_{i=1}^{k} z_i(w_i n + r_i) - (z_0 + z_1 w_1 + \cdots + z_k w_k)n = \sum_{i=1}^{k} z_i r_i$$

which yields an explicit bijection between factorizations of $\beta \in P_n$ of length ℓ and the factorizations of $\beta - \ell n \in S$ of length at most ℓ. Let $s = (z_1, \ldots, z_k) \in \mathsf{Z}_s(\beta - |z|_w n)$ and $s' = (z'_1, \ldots, z'_k) \in \mathsf{Z}_s(\beta - |z'|_w n)$ denote the factorizations in S corresponding to z and z', respectively. Notice that since $|z|_w > |z'|_w$, we have

$$\beta - |z'|_w n \geq \beta - (|z|_w - 1)\, n = n + \beta - |z|_w n \geq n > WR^2 \geq Wr_k^2$$

by Lemma 5.5(c) we know $|s'|_w > Wr_k$. Next, we claim some factorization in the same connected component of ∇_β as z' has positive last component. If $z'_k > 0$, then we are done. Suppose this is not the case, i.e. $z'_k = 0$. By the same reasoning as above and our assumption on n, we have that

$$\beta - |z'|_w n > Wr_{k-1}r_k$$

By Lemma 5.5(b) we know $\exists s'' \in \mathsf{Z}_S\left(\beta - |z'|_w n\right)$ of minimal weighted length with $s''_k > 0$. Notice,

$$\begin{gathered}\beta - |z'|_w n = s''_1 r_1 + \cdots + s''_k r_k \\ = s''_1 r_1 + \cdots + s''_k r_k + (s''_1 w_1 n + s''_2 w_2 n + \cdots + s''_k w_k n) - (s''_1 w_1 n + s''_2 w_2 n + \cdots + s''_k w_k n) \\ = -|s''|_w n + s''_1(w_1 n + r_1) + \cdots + s''_k(w_k n + r_k)\end{gathered}$$

So,

$$\begin{aligned}\beta &= |z'|_w n - |s''|_w n + s''_1(w_1 n + r_1) + \cdots + s''_k(w_k n + r_k) \\ &= \left(|z'|_w - |s''|_w\right) n + s''_1(w_1 n + r_1) + \cdots + s''_k(w_k n + r_k)\end{aligned}$$

This factorization $z'' = (|z'|_w - |s''|_w, s''_1, \ldots, s''_k) \in \mathsf{Z}(\beta)$ of β under the above bijection is connected to z' in ∇_β. In particular, it is important to note that $z''_k = s''_k > 0$. Since z' is connected to z'', but z and z' are in different connected components of ∇_β, we must have that $z_k = 0$ or we could connect z and z''. This means $\sum_{i=1}^{k} s_i < r_k$ or else Lemma 5.5(a) would produce a factorization connected to z in ∇_β with positive last coordinate that we could connect to z''. Thus,

$$|z|_w > |z'|_w \geq |s'|_w > Wr_k > W \sum_{i=1}^{k} s_i \geq \sum_{i=1}^{k} s_i w_i = |s|_w$$

Hence, $z_0 = |z|_w - |s|_w > 0$. By definition of z and z' being in different connected components, we must have that $z'_0 = 0$. At this point, it is important to note that

$$\begin{array}{ll} 1) & s'_k = z'_k = 0 \\ 2) & |s'|_w > Wr_k \geq r_k \end{array}$$

Hence, by Lemma 5.5(a), we must have that $|s'|_w$ is not minimal so $|s''|_w < |s'|_w$ since we know $|s''|_w$ is minimal. So,

$$|z'|_w - |s''|_w > |z'|_w - |s'|_w \geq |z'|_w - |z'|_w = 0$$

Thus, $z''_0 > 0$ which is a contradiction since we can then connect z and z'. Therefore we must have that $z'_k > 0$, and thus, we have proven the claim. □

6. The Main Theorem

In this section, our goal is to prove properties regarding the Apéry sets of our parametrized numerical semigroups P_n. Primarily, we want to generalize Theorem 3.3, but we need several tools in order to do so. First, we need a generalization of proposition 2.9 from [3], but this, in itself, requires several new definitions.

Definition 6.1. For a numerical semigroup $S = \langle r_1, \ldots, r_k \rangle$ and $s \in S$, the *delta set of s* is

$$\Delta(s) = \{\ell_i - \ell_{i-1} : i = 2, \ldots, n\} \text{ where } L_S(s) = \{\ell_1 < \ell_2 < \cdots < \ell_n\}$$

The *selta set of S* is

$$\Delta(S) = \bigcup_{s \in S} \Delta(s)$$

We now define an analogous weighted version of the delta set.

Definition 6.2. For a numerical semigroup $S = \langle r_1, \ldots, r_k \rangle$, weight vector $w = (w_1, \ldots, r_k)$, and $s \in S$, the *weighted delta set of s* is

$$\Delta_w(s) = \{\ell_i - \ell_{i-1} : i = 2, \ldots, n\} \text{ where } \mathsf{L}_{S,w}(s) = \{\ell_1 < \ell_2 < \cdots < \ell_n\}$$

The *weighted delta set of S* is

$$\Delta_w(S) = \bigcup_{s \in S} \Delta_w(s)$$

Remark 6.3. Delta sets provide information about possible factorization lengths. In terms of this paper, though, we will be using them purely as a means for studying Apéry sets of our parametrized semigroups.

The following is a generalization of [1, Lemma 3].

Lemma 6.4. *If, for a semigroup S, $\Delta_w(S)$ is non-empty, then* $\min \Delta_w(S) = \gcd \Delta_w(S)$.

Proof. It is sufficient to show that $d = \gcd \Delta_w(S) \in \Delta_w(S)$. Let $d_1, \ldots, d_t \in \Delta_w(S)$. Thus, there exists $m_i \in \mathbb{Z}$ for $i = 1, \ldots, t$ such that $d = \sum_{i=1}^{t} m_i d_i$. We must have that there exists $z_i, z_i' \in \mathsf{Z}(s_i)$ for some $s_i \in S$ such that $|z_i|_w - |z_i'|_w = sgn\,(m_i)\,d_i$. Since z_i and z_i' are factorizations of s_i, we must have that $(z_i + z_i), (z_i' + z_i') \in \mathsf{Z}(2s_i)$. Also, note that $(z_i + z_i) = 2z_i$ which can be evaluated by adding component-wise. Continuing this pattern gives us that $\sum_{i=1}^{t} |m_i| z_i, \sum_{i=1}^{t} |m_i| z_i' \in \mathsf{Z}\left(\sum_{i=1}^{t} |m_i| s_i\right)$ Additionally, using the linearity of weighted length, we know

$$\left|\sum_{i=1}^{t} |m_i| z_i\right|_w - \left|\sum_{i=1}^{t} |m_i| z_i'\right|_w = \sum_{i=1}^{t} \Big||m_i| z_i\Big|_w - \sum_{i=1}^{t} \Big||m_i| z_i'\Big|_w$$

$$= \sum_{i=1}^{t} |m_i| \left(|z_i|_w - |z_i'|_w\right) = \sum_{i=1}^{t} |m_i| \cdot sgn(m_i) d_i = d \leq \min \Delta_w(S)$$

Hence, we can see that we must have $d \in \Delta_w(S)$, so $\min \Delta_w(S) \leq d$. Therefore, we can conclude that $\min \Delta_w(S) = \gcd \Delta_w(S)$. □

Remark 6.5. This lemma tells us that for $d = \min \Delta_w(S)$,

$$\Delta_w(S) \subseteq \{d, 2d, \ldots, pd\} \text{ for some } p \in \mathbb{N},$$

a well-known property of $\Delta(S)$.

Now we can state a generalization of [5, Corollary 3.5] which follows from Theorem 5.7.

Corollary 6.6. *Suppose $n > WR^2$, P_n is primitive, and $d = \min \Delta_w(\langle r_1, \ldots, r_k \rangle)$. Any two factorizations $z, z' \in \mathsf{Z}(\beta)$ of a Betti element $\beta \in Betti(P_n)$ lying in different connected components of ∇_β satisfy $\big||z|_w - |z'|_w\big| \in \{0, d\}$.*

Proof. By Lemma 6.4 and Remark 6.5, $|z|_w - |z'|_w \in d\mathbb{Z}$. We will show that if $|z|_w - |z'|_w \geq 2d$, then z and z' must be in the same connected component of ∇_β. Let $\ell = |z|_w - |z'|_w$. Just as in the proof of Theorem 5.7, we know

$$\beta - |z|_w n = \sum_{i=1}^{k} z_i r_i \in S$$

Because of our bound on n, we know $n \in S$. Then, by the closure of S, we have that

$$(\beta - |z|_w n) + (\ell - d)n = (\beta - |z'|_w n - \ell n) + (\ell - d)n = \beta - (|z'|_w + d)\, n \in S$$

Because of the bijection we established in Theorem 5.7, there must exist a factorization $z'' \in \mathsf{Z}(\beta)$ such that $|z''|_w = |z'|_w + d$. By Theorem 5.7, we know that $z_0'' > 0$ and $z_k'' > 0$, so both z and z' are connected to z'' in ∇_β. Hence, we must have that $\big||z|_w - |z'|_w\big| \in \{0, d\}$ for z and z' to be in different connected components. □

Lemma 6.7. *Suppose $n > WR^2$ If a and b are factorizations of some element $m \in P_n$ with $|a|_w < |b|_w$, then there is some factorization b' such that $|b|_w = |b'|_w$ where $b_0' > 0$.*

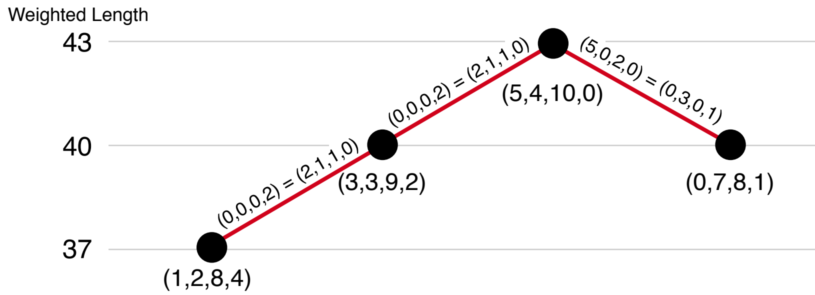

FIGURE 6. Factorization chain from Example 6.8

Proof. Suppose $a, b \in \mathsf{Z}(m)$ for some $m \in P_n$ such that $|a|_w < |b|_w$. By Corollary 6.6, we have minimum relations that either maintain the weighted length or increase it by exactly $d = \min \Delta_w(\langle r_1, \ldots, r_k \rangle)$. Since a and b are factorizations of the same element with different lengths, we are guaranteed to have a minimal relation that increases the factorization length. Moreover, by Theorem 5.7, we know that each such relation must increase the first coordinate of the next factorization. We can "chain" these relations together until we get the desired factorization b' such that $|b|_w = |b'|_w$. $\square$

Example 6.8. This example is meant to better illustrate the argument for Lemma 6.7. Let $S = \langle 3, 5, 7 \rangle$, $w = (2, 3, 2)$, and $n = 150$. Hence, $P_{150} = \langle 150, 303, 307, 455 \rangle$. Consider $m = 5032$. Thus, we have that

$$a = (1, 2, 8, 4), b = (0, 7, 8, 1) \in \mathsf{Z}(5032)$$

Notice that $|a|_w < |b|_w$ since

$$|a|_w = 1 + 2(2) + 8(3) + 4(2) = 37$$

$$|b|_w = 0 + 7(2) + 8(3) + 1(2) = 40$$

Each Betti element provides a minimal relation, so, in general, the chains of factorizations are not unique. In this example, *Betti* $(P_{150}) = \{910, 1212, 1364, 6750, 7061, 7209\}$ where

$$(0, 0, 0, 2), (2, 1, 1, 0) \in \mathsf{Z}(910)$$

$$(5, 0, 2, 0), (0, 3, 0, 1) \in \mathsf{Z}(1364)$$

Consider the factorization chain in Figure 6 at the top of the page. As we move from a "towards" b, we first reach $(3, 3, 9, 2)$ with length $40 = |b|_w$. Hence, we can let $b' = (3, 3, 9, 2)$, and we are done.

Now we have enough tools to generalize Theorem 3.4.

Theorem 6.9. *If $n \in S$ satisfies $n > WR^2$, then*

$$\mathrm{Ap}(P_n; n) = \{i + \mathsf{m}_{S,w}(i) \cdot n | i \in \mathrm{Ap}(S; dn)\}$$

where $d = gcd(S)$ and $\mathsf{m}_{S,w}$ denotes the minimum weighted factorization length in S. Moreover, we have

$$\mathsf{L}_{\mathsf{w}_{P_n}}(i + \mathsf{m}_{S,w}(i) \cdot n) = \{\mathsf{m}_{S,w}(i)\}$$

for each $i \in \mathrm{Ap}(S; dn)$.

Proof. Let $A = \{i + \mathsf{m}_{S,w}(i) \cdot n | i \in Ap(S; dn)\}$. Because each element of $\{\frac{i}{d} : i \in \mathrm{Ap}(S; dn)\}$ is distinct modulo n and $\gcd(n, d) = 1$, we must have that each element of $\mathrm{Ap}(S; dn)$ is distinct modulo n. It follows then that each element of A is distinct modulo n because for all $i \in \mathrm{Ap}(S; dn)$,

$$i + \mathsf{m}_{S,w}(i) \cdot n \equiv i \bmod n$$

Hence, $|A| = n$, so we only need to show that $A \subseteq \mathrm{Ap}(P_n)$. Fix $i \in \mathrm{Ap}(S; dn)$ and let $a = i + \mathsf{m}_{S,w}(i)n$. If $s \in \mathsf{Z}_S(i)$ has minimal weighted length, then

$$a = i + \mathsf{m}_{S,w}(i)n = i + |s|_w n = \sum_{i=1}^{k} s_i r_i + \left(\sum_{i=1}^{k} s_i w_i\right) n = \sum_{i=1}^{k} s_i(w_i n + r_i)$$

so $a \in P_n$. Hence each minimal factorization $s \in \mathsf{Z}_S(i)$ corresponds to a factorization of $a \in P_n$ where the first component is zero. In fact, for each $\ell \geq 0$, there is a natural bijection

$$\begin{array}{rcl} \{z \in \mathsf{Z}_{P_n}(a) : |z|_w = \ell\} & \to & \{s \in \mathsf{Z}_S(a - \ell n) : |s|_w \leq \ell\} \\ (z_0, z_1, \ldots, z_k) & \mapsto & (z_1, \ldots, z_k) \end{array}$$

between factorizations of $a \in P_n$ of length ℓ and factorizations of $a - \ell n \in S$ of length at most ℓ because

$$a = z_0 n + \sum_{i=1}^{k} z_i(w_i n + r_i) = |z|_w n + \sum_{i=1}^{k} z_i r_i$$

Now note that $a - \mathsf{m}_{S,w}(i)n = (i + \mathsf{m}_{S,w}(i)n) - \mathsf{m}_{S,w}(i)n = i \in \mathrm{Ap}(S; dn)$. When we have $\ell > \mathsf{m}_{S,w}(i)$ it follows that

$$a - \ell n = (i + \mathsf{m}_{S,w}(i)n) - (\mathsf{m}_{S,w}(i) + [\ell - \mathsf{m}_{S,w}(i)])n = i - [\ell - \mathsf{m}_{S,w}(i)]n \notin S$$

by definition of $i \in \mathrm{Ap}(S; dn)$. Hence, a has no factorizations of length ℓ in P_n. Furthermore, a cannot have any factorizations in P_n with length strictly less than $\mathsf{m}_{\mathsf{w}_S}$ since Lemma 6.7 would force some factorization of length $\mathsf{m}_{S,w}$ to have non-zero first coordinate. Combining these two facts, we must have that $\mathsf{L}_{S,w}(i + \mathsf{m}_{S,w}(i) \cdot n) = \{\mathsf{m}_{S,w}(i)\}$, meaning every factorization of a in P_n has first coordinate zero. As such, we conclude that $a \in \mathrm{Ap}(P_n)$. □

With these tools generalized, we can now begin to generalize several applications in [7]. First, we will need a definition.

Definition 6.10 ([7, Definition 4.1])**.** A function $f : \mathbb{Z} \to \mathbb{R}$ is an *r-quasipolynomial* of degree α if

$$f(n) = a_\alpha(n)n^\alpha + \cdots + a_1(n)n + a_0(n)$$

for periodic functions $a_0, \ldots, a_\alpha$, whose periods all divide r, with a_α not identically 0. We say f is *eventually quasipolynomial* if the above equality holds for all $n \gg 0$.

The following generalization of [7, Corollary 4.3] follows immediately.

Corollary 6.11. *For $n > WR^2$, the function $n \mapsto F(P_n)$ is r_k-quasiquadratic in n.*

Proof. Let a denote the element of $\mathrm{Ap}(S; dn)$ for which $\mathsf{m}_{S,w}(-)$ is maximal. Theorem 6.9 and Proposition 3.8 imply

$$F(P_n) = \max\{\mathrm{Ap}(P_n)\} - n = a - n + \mathsf{m}_{S,w}(a)n$$

Theorem 4.13 and Proposition 3.8 together imply that $a + r_k$ is the element of $\mathrm{Ap}(S; dn + r_k)$ for which $\mathsf{m}_{S,w}(-)$ is maximal and quasilinearity of $\mathsf{m}_{S,w}(-)$ completes the proof. □

In order to generalize [7, Corollary 4.2], we need an additional definition.

Definition 6.12. Let S be a numerical semigroup with $\gcd(S) = 1$. The *genus* of S is

$$g(S) = |\mathbb{Z}_{\geq 0} \setminus S|$$

which is the number of positive integers outside of S. For a non-primitive numerical semigroup $T = dS$ with $d \geq 1$, define $g(T) = d \cdot g(S)$.

Example 6.13. Let $S = \langle 6, 9, 20\rangle$. From Table 1, we can see that

$$\mathbb{Z}_{\geq 0} \setminus S = \{1, 2, 3, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 22, 23, 25, 28, 31, 34, 37, 43\}$$

Hence, $g(S) = |\mathbb{Z}_{\geq 0} \setminus S| = 22$.

Just as for the previous corollary, the generalization of [7, Corollary 4.2] follows immediately.

Corollary 6.14. *For $n > WR^2$, the function $n \mapsto g(P_n)$ is r_k-quasiquadratic in n.*

Proof. By counting the elements of $\mathbb{Z}_{\geq 0} \setminus S$ modulo n, we can write

$$g(P_n) = \sum_{a \in \mathrm{Ap}(P_n)} \left\lfloor \frac{a}{n} \right\rfloor$$

Theorem 6.9 and Proposition 3.8 gives us that

$$= \sum_{i \in \mathrm{Ap}(S;dn)} \left\lfloor \frac{i + \mathsf{m}_{S,w}(i)n}{n} \right\rfloor = \sum_{i \in \mathrm{Ap}(S;dn)} \left\lfloor \frac{i}{n} \right\rfloor + \sum_{i \in \mathrm{Ap}(S;dn)} \mathsf{m}_{S,w}(i)$$

$$= \sum_{t=1}^{n-1} \left\lfloor \frac{dt}{n} \right\rfloor + d \cdot g(S) + \sum_{\substack{i<n \\ di \in S}} \mathsf{m}_{S,w}(di) + \sum_{\substack{i \geq 0 \\ di \notin S}} \mathsf{m}_{S,w}(di + dn)$$

Each of the terms is eventually quasipolynomial in n. The first term is d-quasilinear in n, the second term is independent of n, and Theorem 6.9 guarantees that the last two terms are eventually r_k-quasiquadratic and r_k-quasilinear in n, respectively. Thus, we have proven our claim. □

Before generalizing [5, Corollary 5.7], we need a weighted version of [4, Theorem 2.5].

Theorem 6.15. *If S is a numerical semigroup with $0 < |\Delta_w(S)| < \infty$, then*

$$\max \Delta_w(S) = \max_{n \in Betti(S)} \max \Delta_w(n)$$

Proof. Let $M = \max_{n \in Betti(S)} \max \Delta_w(n)$. The inequality $M \leq \max \Delta_w(S)$ follows immediately. To derive a contradiction, suppose that $\max \Delta_w(S) > M$. Let $x, y \in \mathsf{Z}(s)$ for some $s \in S$ such that $|y|_w - |x|_w = \max \Delta_w(S)$ where there does not exist a factorization $z \in \mathsf{Z}(s)$ such that $|x|_w < |z|_w < |y|_w$. Since $|x|_w < |y|_w$, we know there exists a chain of factorizations between x and y. Applying [4, Lemma 2.1], let $x_1, \ldots, x_t \in \mathsf{Z}(s)$ such that $x = x_1, \ldots, x_t = y$ where $(x_i, x_{i+1}) = (a_i + c_i, b_i + c_i)$ for some factorization $c_i \in \mathsf{Z}(S)$ where a_i and b_i are disjoint factorizations of some Betti element $n \in S$ for $i = 1, 2, \ldots, t-1$. Since $|x_1|_w = |x|_w$, $|x_t|_w = |y|_w$, and no $|x_i|_w$ lies between these values, there exists some $i \in \{1, \ldots, t-1\}$ such that

$$|x_i|_w \leq |x|_w < |y|_w \leq |x_{i+1}|_w$$

Again, by [4, Lemma 2.1], we know there exists a chain of factorizations such that

$$a_i = f_{i1}, \ldots, f_{im} = b_i \text{ with } \big||f_{i(j+1)}|_w - |f_{ij}|_w\big| \leq \max \Delta_w(n) < \max \Delta_w(S)$$

for all $j \in \{1, \ldots, m-1\}$. Notice that we then have $(f_{ij} + c_i), x, b \in \mathsf{Z}(s)$ for all $j \in \{1, \ldots, m-1\}$. Also, by our assumption on x and y there is no i or j such that $|x|_w < |f_{ij}|_w < |y|_w$. By the same argument as before, we can find some j such that

$$|f_{ij} + c_i|_w \leq |x|_w < |y|_w \leq |f_{i(j+1)} + c_i|_w$$

This is a contradiction because

$$\begin{aligned} \max\{\Delta_w(S)\} &= |y|_w - |x|_w \leq |f_{i(j+1)} + c_i|_w - |f_{ij} + c_i|_w \\ &= |f_{i(j+1)} - f_{ij}|_w \leq \max_{n \in Betti(S)} \max \Delta_w(n) < \max \Delta_w(S) \end{aligned}$$

Therefore we have proven the claim. □

Corollary 6.16. *If $n > WR^2$, then*

$$\Delta_w(P_n) = \{d\},$$

for $d = \gcd \Delta_w(S)$.

Proof. By Lemma 6.4, $d = \min \Delta_w(P_n)$. Additionally, $\max \Delta_w(P_n)$ occurs in the delta set of a Betti element of P_n by Theorem 6.15, so $\max \Delta_w(P_n) = d$ by Corollary 6.6. □

7. Future Work

This paper is just the beginning for parametrized numerical semigroups and weighted length applications. Although we have been able to generalize several results from [1, 4, 5, 7], there are plenty of properties that remain.

Problem 7.1. *Generalize the remaining results from* [5, 7].

Problem 7.2. *Find and prove a formula for calculating* $\min \Delta_w(S)$ *in terms of the generators of* S *and the weight vector* w.

Solving Problem 7.2 would serve as a generalization of [3, Proposition 2.9]. Moreover, it would provide a more concrete calculation in Lemma 6.4 and Corollary 6.6.

Problem 7.3. *Prove Conjecture 4.15.*

These problems should serve as starting points for future research.

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