Anyonic Dispersion Under Weak Perturbations to the Toric Code Model

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Abstract

We propose a two-parameter Dynamical Toric Code model (DTCM), which is a small perturbation to the Toric Code model that induces dispersion in the anyons of this model, yet preserves particle number. We calculate the exact energies in the one-anyon sector, and show a $\sin(k)$ dispersion relation. We also calculate the exact energies for the two-anyon sector in the zero anyon hopping limit, and find four distinct energy bands. For the full DTCM in the two-anyon sector, we proceed for non-zero anyon hopping numerically. We show evidence for fused ribbon eigenvectors from our numerical simulation, which would make this model the first known model to exhibit fusion of anyons under the dynamics generated by the Hamiltonian.

Acknowledgements

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I would also like to thank Prof. Rajiv Singh for introducing me to research in quantum spin systems and for teaching me most of the models in this thesis. He brought me from a place where I didn't know what spin was nor what quantum meant to conducting cutting edge research in this field, and for this I will be forever grateful.

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Contents

1 Introduction

Quantum mechanics is arguably the largest jump in the history of theoretical physics. At a fundamental level, it has shattered our previous model of nature, given by Newtonian mechanics. Newtonian mechanics provides a recipe for determining the future dynamics of a particle, if you know the position and velocity at a given time, however, quantum mechanics forbids the capability of ever knowing both the position and momentum at the same time. It also claims that all particles are secretly also waves, it states that most quantities can only take discrete values rather than continuous values, and the observable quantities are described by a non-commutative algebra. Even after nearly 100 years of research in quantum mechanics, physicists and mathematicians still have much to uncover.

One major prediction from quantum mechanics is that all the matter in the universe is built out of fundamental building blocks called particles. The standard model of particle physics has been a very successful model of all the fundamental particles in the universe. However, it is known to be limited in scope. This model puts all particles into one of two catergories: fermions and bosons. One assumption made by the model is that we live in three spatial dimensions. Although this is quite a reasonable assumption, certain materials, such as graphene, are able to confine particles to live in only two spatial dimensions, where the story is different. In two dimensions, bosons and fermions still exist, but there are additional particles called anyons. These particles are sensitive to moving around each other, a process called braiding, even in the absence of any form of interaction between the anyons, a property not observed in bosons nor fermions. This process of braiding has been shown to be able to encode a quantum computation [4]. Anyons are a very exciting and not well understood phenomena, that are being studied extensively by both mathematicians and condensed matter physicists.

One of the simplest models exhibiting anyons is known as the Toric Code model, proposed by Kitaev [4]. The Toric Code was initially proposed to live on a finite square lattice with periodic boundary conditions, forming a torus, but we will discuss this model defined on $\mathbb{Z}^2$. This model has a unique frustration free ground state, and all the energy eigenvectors are completely specified by counting the number of anyon excitations in the model. The anyons in this model are in fact eigenvectors of the Toric Code Hamiltonian, and we will see that this means their dynamics are trivial. This means that if there is an anyon at a site in the lattice, there is no way for the anyon to move under the dynamics induced by the Hamiltonian. It is of great interest to introduce other terms into the Hamiltonian that preserve the anyonic structure of the Toric Code model, yet induce non-trivial dynamics in the anyons.

The ground state of the Toric Code model has a topological order, with a finite gap to the lowest excitation in the model. It was proved by Bravyi, Hastings, and Michalakis, that the ground state is stable in the sense that the gap does not close under sufficiently small perturbations [2]. In earlier work my advisor Prof. Bruno Nachtergaele, along with a Ph.D student Mathew Cha,

and postdoc Pieter Naaijkens, recently proved that the anyonic structure of the excitation spectrum in these models is stable under small perturbations [3]. The goal of this project is to develop a specific perturbation to the Toric Code model and explore the effects this perturbation has on the spectrum of the Toric Code model.

This paper will be structured in the following way. In section 2, we will discuss some necessary background information on quantum mechanics, and studying system of more then one particle. We will finish this section with a brief discussion of bosons, fermions, and anyons. In section 3, we discuss a two-spin system, and an N-spin system. We then discuss the desire to study infinite quantum spin systems rather than just finite systems, and motivate this from both mathematical and physical reasons. In section 4, we will develop the theory for studying infinite quantum spin systems of spin-1/2 particles. In section 5, we discuss two standard quantum spin systems on infinite lattices, and introduce the notion of quasi-particle excitations. In section 6, we discuss the Toric Code model, including introducing anyons, and discussing some properties of the model. In section 7, we discuss our proposed Dynamical Toric Code model, which is a perturbation to the Toric Code model that induces non-trivial dynamics of the anyons. We finish this paper in section 8 with concluding remarks, and planned future work.

2 Background

2.1 Quantum Mechanics

For simplicity, we will develop quantum mechanics initially in the case of a finite dimensional system. There are two major quantities in quantum mechanics: states, and observables. States are represented by normalized vectors in a Hilbert space.

Definition 2.1 *A Hilbert space $\mathcal{H}$ is a complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product.*

Given $v, u \in \mathcal{H}$ with coefficients v_n, u_n in some orthonormal basis, we define the inner product as

$$\langle v|u\rangle = \sum_{n=1}^{\dim(\mathcal{H})} \bar{v}_n u_n, \tag{1}$$

where $\bar{\cdot}$ denotes complex conjugation.

Observables are defined as bounded operators on this Hilbert space which form an algebra $\mathcal{B}(\mathcal{H})$. The measurement of a physical experiment will always yield an eigenvalue of an observable, and because of this the set of observables is often restricted to only include Hermitian operators. However, the set of Hermitian operators does not form an algebra, and so we will not make this

restriction, but simply say that *physical observables* are given by Hermitian operators in $\mathcal{B}(\mathcal{H})$.

Since we will always be working in an inner product space, we will use Dirac (bra-ket) notation. The idea is that an inner product between vectors ϕ and ψ takes the form

$$\langle \phi | \psi \rangle \,. \tag{2}$$

We will break this "bracket" into two parts: a bra and a ket. The ket, denoted $|\cdot\rangle$ will be an element of $\mathcal{H}$, and the bra denoted $\langle \cdot |$ will be an element of the dual space of $\mathcal{H}$. This notation makes the data-types of quantities very explicit. If we have a full bracket $\langle \cdot | \cdot \rangle$, then the data type is an element of $\mathbb{C}$, if we have just a ket $|\cdot\rangle$ then the data type is an element of $\mathcal{H}$, and if we have an outer product $|\cdot\rangle \langle \cdot|$ then the data type is an element of $\mathcal{B}(\mathcal{H})$.

Let us suppose that the dimension of $\mathcal{H}$ is D, and consider a physical observable $\mathcal{O}$. Since $\mathcal{O}$ is Hermitian, we have that it can be unitarily diagonalized. For simplicity let us assume the eigenvalues are distinct, which automatically ensures that the eigenvectors of $\mathcal{O}$ form an orthonormal basis for $\mathcal{H}$. In the case of degenerate eigenvalues, one just has an invariant subspace of $\mathcal{H}$ under $\mathcal{O}$ spanned by the degenerate eigenvectors, which could be made orthonormal by the Gram-Schmidt procedure for example, so there is no loss of generality. Let us denote the eigenpairs by $\{(\lambda_n, |\lambda_n\rangle)\}_{n=1}^{D}$. Thus if we have a state $|\psi\rangle \in \mathcal{H}$ we can expand it in a basis of eigenvectors of $\mathcal{O}$ as

$$|\psi\rangle = \sum_{n=1}^{D} a_n |\lambda_n\rangle \quad a_n \in \mathbb{C}. \tag{3}$$

Then consider the quantity

$$\langle \psi | \mathcal{O} | \psi \rangle \,. \tag{4}$$

Then, in the basis of eigenvectors of $\mathcal{O}$

$$\langle \psi | \mathcal{O} | \psi \rangle = \sum_{n,n'=1}^{D} \bar{a}_{n'} a_n \langle \lambda_{n'} | \mathcal{O} | \lambda_n \rangle \,. \tag{5}$$

Since $|\lambda_n\rangle$ is an eigenvector of $\mathcal{O}$ with eigenvalue λ_n, we can write

$$\langle \psi | \mathcal{O} | \psi \rangle = \sum_{n,n'=1}^{D} \bar{a}_{n'} a_n \lambda_n \langle \lambda_{n'} | \lambda_n \rangle \,. \tag{6}$$

Orthonormality of the eigenvectors implies $\langle \lambda_{n'} | \lambda_n \rangle = \delta_{n,n'}$, killing the sum over n' yielding

$$\langle \psi | \mathcal{O} | \psi \rangle = \sum_{n=1}^{D} |a_n|^2 \lambda_n. \tag{7}$$

We interpret this quantity as the expectation value of $\mathcal{O}$ in the state ψ. We thus interpret $|a_n|^2$ as the probability of a measurement yielding the value λ_n. Since

we assume states are normalized, this ensures $\{|a_n|^2\}_{n=1}^D$ form a well defined probability distribution, and is why we assume states are normalized.

Let us now consider the transformation

$$|\psi\rangle \longrightarrow e^{i\phi}|\psi\rangle \quad \phi \in \mathbb{R}. \tag{8}$$

Then consider how an expectation value changes under this transformation

$$\langle\psi|\mathcal{O}|\psi\rangle \longrightarrow \langle\psi|e^{-i\phi}\mathcal{O}e^{i\phi}|\psi\rangle = \langle\psi|\mathcal{O}|\psi\rangle. \tag{9}$$

Thus expectation values are invariant under such a transformation. Since expectation values are the means for calculating experimental quantities, we have that two states that differ only by an overall phase factor are physically equivalent.

2.2 Hamiltonians

One very important operator in quantum mechanics is the Hamiltonian, which for the systems studied here is equivalent to the total energy. The Hamiltonian is the generator of dynamics, as is seen from Schrödinger's equation

$$i\hbar\frac{d}{dt}|\psi\rangle = H|\psi\rangle, \tag{10}$$

where $|\psi\rangle$ is any state, H is the Hamiltonian, and $\hbar$ is the reduced Planck's constant which we will from now on set to 1. We always demand that the Hamiltonian is a Hermitian operator, as the total energy must be measured to be real.

Let us consider an eigenpair of H, call it $(E, |E\rangle)$. Then notice Shrödinger's equation for an eigenvector of H becomes

$$i\frac{d}{dt}|E\rangle = H|E\rangle \implies i\frac{d}{dt}|E\rangle = E|E\rangle \implies |E_t\rangle = e^{-iEt}|E_0\rangle. \tag{11}$$

Thus the time evolution is simply multiplication by a phase for an eigenvector of H, which is physically undetectable. Thus the dynamics of an eigenvector of H is trivial, and we call such a state a *stationary state*.

2.3 Pauli Matrices

One of the simplest quantum mechanical system is a single spin-1/2 system with Hilbert space $\mathcal{H} = \mathbb{C}^2$. All the operators on this space are given by the 2×2 matrices in $M_2(\mathbb{C})$, and so the algebra of observables is generated by the identity and the Pauli matrices $\sigma^1, \sigma^2, \sigma^3$. The Pauli matrices in the basis of eigenvectors of σ^3 are given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Pauli matrices are seen to be Hermitian, and have eigenvalues ± 1. Considering just one Pauli matrix say σ^3, we interpret the eigenvector corresponding to

an eigenvalue of $+1$ as "spin up" often denoted as $|\uparrow\rangle$, and the eigenvector with eigenvalue -1 as "spin down" denoted by $|\downarrow\rangle$. Physically, we say the particle has total spin $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$, and we interpret $\sigma^{1,2,3}$ as the projection of the spin along the x,y,z-direction. This means that the projection along a given axis can only take one of two values, in sharp contrast with a normalized classical vector that can take a projection along an axis of any value in the interval $[-1, 1]$.

We can consider the action of $\sigma^{1,2}$ on the states $|\uparrow\rangle$,$|\downarrow\rangle$. In the matrix representation we see that $|\uparrow\rangle = (1, 0)^T$ and $|\downarrow\rangle = (0, 1)^T$. Thus

$$\sigma^1 |\uparrow\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\downarrow\rangle . \tag{12}$$

Performing similar explicit calculations, we yield the following

$$\begin{aligned} \sigma^1 |\uparrow\rangle &= |\downarrow\rangle \\ \sigma^1 |\downarrow\rangle &= |\uparrow\rangle \\ \sigma^2 |\uparrow\rangle &= i |\downarrow\rangle \\ \sigma^2 |\downarrow\rangle &= -i |\uparrow\rangle . \end{aligned} \tag{13}$$

The Pauli matrices satisfy many useful properties that we will use through out this paper. First, their commutation relations are given by

$$[\sigma^i, \sigma^j] := \sigma^i \sigma^j - \sigma^j \sigma^i = 2i\epsilon_{ijk}\sigma^k, \tag{14}$$

where $i, j, k \in \{1, 2, 3\}$, ϵ_{ijk} is the Levi-Civita symbol, and repeated indices are summed over. We also have that their anti-commutation relations are

$$\{\sigma^i, \sigma^j\} := \sigma^i \sigma^j + \sigma^j \sigma^i = 2\delta_{i,j}\mathbb{I}. \tag{15}$$

From the anti-commutation relation, we also have the useful property that

$$\left(\sigma^i\right)^2 = \mathbb{I}. \tag{16}$$

We can also define spin raising and lowering operators by

$$\begin{aligned} \sigma^+ &:= \frac{\sigma^x + i\sigma^y}{2} \\ \sigma^- &:= \frac{\sigma^x - i\sigma^y}{2}. \end{aligned} \tag{17}$$

Sometimes the raising and lower operators are defined without the factor of 2, but we choose this normalization due its simple form for acting on eigenvectors of σ^3. In particular

$$\begin{aligned} \sigma^+ |\downarrow\rangle &= |\uparrow\rangle \\ \sigma^+ |\uparrow\rangle &= 0 \\ \sigma^- |\downarrow\rangle &= 0 \\ \sigma^- |\uparrow\rangle &= |\downarrow\rangle . \end{aligned} \tag{18}$$

The raising and lowering operators also have the following properties

$$\begin{aligned}\sigma^3\sigma^+ &= \sigma^+ \\ \sigma^3\sigma^- &= -\sigma^- \\ \sigma^-\sigma^+ &= \frac{(\mathbb{I}-\sigma^3)}{2}.\end{aligned} \tag{19}$$

There are many others, but these are what we need here. Also $\sigma^{+,-}$ are not Hermitian, and in particular we have that

$$\left(\sigma^+\right)^* = \sigma^- \quad ; \quad (\sigma^-)^* = \sigma^+. \tag{20}$$

2.4 Tensor Product

When studying more then one particle, the method to transfer one particle quantum mechanics into more then one particle is through the tensor product. In particular, if we have one particle living in the Hilbert space $\mathcal{H}_1$ and another particle in $\mathcal{H}_2$, then the full Hilbert space for both particles is $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, with $\dim(\mathcal{H}) = \dim(\mathcal{H}_1)\cdot\dim(\mathcal{H}_2)$. If we suppose that the first particle is in the state $|\psi\rangle \in \mathcal{H}_1$ and the second particle is in the state $|\phi\rangle \in \mathcal{H}_2$, then the state for the combined system is written $|\psi\rangle \otimes |\phi\rangle$. We have the following identification of tensor product states

$$\begin{aligned}&\lambda(\psi\otimes\phi) = (\lambda\psi)\otimes\phi = \psi\otimes(\lambda\phi) \\ &(\psi_1+\psi_2)\otimes\phi = \psi_1\otimes\phi + \psi_2\otimes\phi \\ &\psi\otimes(\phi_1+\phi_2) = \psi\otimes\phi_1 + \psi\otimes\phi_2.\end{aligned} \tag{21}$$

The inner product on the tensor product space is defined in terms of the tensor products of the one particle spaces as

$$\langle\psi_1\otimes\phi_1|\psi_2\otimes\phi_2\rangle_{\mathcal{H}} = \langle\psi_1|\psi_2\rangle_{\mathcal{H}_1}\langle\phi_1|\phi_2\rangle_{\mathcal{H}_2}. \tag{22}$$

Furthermore, if we have operator's $\mathcal{O}_1$,$\mathcal{O}_2$ acting on the first or second particle respectively, then these operators in the full Hilbert space would take the form

$$\begin{aligned}\mathcal{O}_1 &\longrightarrow \mathcal{O}_1\otimes\mathbb{I} \\ \mathcal{O}_2 &\longrightarrow \mathbb{I}\otimes\mathcal{O}_2.\end{aligned} \tag{23}$$

Suppose we wanted to have an operator act on both particles simultaneously, such as in the case of the two particles interacting, then such an operator would take the form of $\mathcal{O}_1\otimes\mathcal{O}_2$, and its action on the state $|\psi\rangle\otimes|\phi\rangle$ would be

$$(\mathcal{O}_1\otimes\mathcal{O}_2)(|\psi\rangle\otimes|\phi\rangle) = \mathcal{O}_1|\psi\rangle\otimes\mathcal{O}_2|\phi\rangle. \tag{24}$$

When studying N particles, then tensor product generalizes in the natural way, by treating the tensor product space as its own Hilbert space, and tensoring this with another Hilbert space, and so on and so forth.

2.5 Identical Particles

We believe that most particles in the universe are in fact indistinguishable from other particles of the same type. In particular, electrons are identical particles. What this means, is if we have a system containing N indistinguishable particles, then the system is invariant under the action of the group S_N. Thus if we simply take N copies of the single particle Hilbert space, this space has duplicate states that are connected by permutations of the locations of the N particles. If we consider a system of two particles described by the state

$$|\psi\rangle = |x_1\rangle \otimes |x_2\rangle \,, \tag{25}$$

then this state is physically indistinguishable from the state

$$|\psi'\rangle = |x_2\rangle \otimes |x_1\rangle \,. \tag{26}$$

Thus the relation between $|\psi\rangle$ and $|\psi'\rangle$ is at most multiplication by a phase. If we define an operator T which swaps the locations of the two particles, then we have

$$T\,|\psi\rangle = |\psi'\rangle \,. \tag{27}$$

If we then swap again, we have

$$T^2\,|\psi\rangle = |\psi\rangle \,, \tag{28}$$

which generally implies

$$T^2 = \mathbb{I}. \tag{29}$$

Since the operator T satisfies $T^2 = \mathbb{I}$, so does its eigenvalues, meaning the eigenvalues of T are ± 1. If the eigenvalue is $+1$, we call such a state a state of bosons, and if the eigenvalue is -1, we call this a state of fermions.

Unfortunately, this formalism is incomplete. It turns out to be sufficient in 3 and higher spatial dimensions, but is not the case in 2 dimensions. The major flaw is that we assume that swapping the particles is independent of the path taking to do this operation, which is not generally true in 2 dimensions. Very loosely, one way to understand this is that if we introduce 1 dimension of time, then in $2+1$ dimensions, the trajectories of the particles can move around each other and form a knot. Since knots only exits in 3 dimensions, the case of $3+1$ dimensions allows enough freedom to untie this knot, killing the path dependence.

We can understand this more formally by considering the composite Hilbert space and taking the quotient space with repsect to the action of the group S_N. I will give just a brief description of this formalism, for a more complete description I point the reader to [5]. Let us suppose that the position of the particles are represented by vectors in $\mathbb{R}^n$, and that we have N copies of identical particles. Then the configuration space for the composite system is given by the cartesian product of N copies $\mathbb{R}^n$, call this space $\mathcal{M}_N^n$. However this is an over specification of the N particles, since swapping any of the particles is physically

the same state, so we must quotient by the group action of S_N. It can be shown that

$$\mathcal{M}_N^n / S_N \cong \mathbb{R}^n \times r(n, N), \tag{30}$$

where $\mathbb{R}^n$ represents the center of mass of the particles, and $r(n, N)$ is the relative coordinate of the N particles. If we restrict to just two particles, we can write $r(n, 2)$ in terms of a radial and an angular component as

$$r(n, 2) \cong (0, \infty) \times \mathscr{P}_{n-1}, \tag{31}$$

where $\mathscr{P}_{n-1}$ is the n-dimensional real projective space. Some properties of this space is that $\mathscr{P}_0$ is a point, $\mathscr{P}_1$ is a circle which is infinitely connected, where as $\mathscr{P}_{n-1}$ for $n \geq 3$ is doubly connected. The fact that $\mathscr{P}_{n-1}$ for $n \geq 3$ is doubly connected is what ensures that there are only two cases for the phase acquired for interchanging two particles, namely ± 1 yielding bosons and fermions. Since $\mathscr{P}_1$ is infinitely connected, this implies that there are infinitely many phases that can be obtained in the process of swapping the position of two particles. If T is the operator that swaps the two particles, and $|x_1\rangle \otimes |x_2\rangle$ is the state for two particles in two dimensions, then

$$T\,|x_1\rangle \otimes |x_2\rangle = e^{i\phi}\,|x_2\rangle \otimes |x_1\rangle\,. \tag{32}$$

If the phase angle ϕ is equal to 0, then we call these particles bosons, if $\phi = \pi$, then we have fermions, and if ϕ is anything else, we call such particles anyons.

3 Quantum Spin Systems

3.1 Two Spin System

If we treat spin-1/2 particles as magnetic dipoles, then the energy of the two spins is given by the Hamiltonian

$$H = \vec{\sigma}_1 \cdot \vec{\sigma}_2. \tag{33}$$

Each spin-1/2 particle lives in $\mathbb{C}^2$, and so the full Hilbert space is $\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$. Explicitly we have

$$\vec{\sigma}_1 \cdot \vec{\sigma}_2 = \sum_{i=1}^{3} \left[(\sigma^i \otimes \mathbb{I})(\mathbb{I} \otimes \sigma^i)\right]. \tag{34}$$

The tensor product notation is quite cumbersome, and so we will denote which Hilbert space the spins act on by subscripts, and it is assumed that the operators are tensored with the identity to live in the full Hilbert space for the composite system. Explicitly

$$\begin{aligned} \sigma_1^i &:= \sigma^i \otimes \mathbb{I} \\ \sigma_2^i &:= \mathbb{I} \otimes \sigma^i. \end{aligned} \tag{35}$$

Similarly for states in this Hilbert space, we drop explicit tensor product notation and simply write the following

$$|\psi\phi\rangle := |\psi\rangle \otimes |\phi\rangle \,. \tag{36}$$

The ordering of the state label is meaningful, and matches the tensor product structure. So in particular, $|\psi\phi\rangle \neq |\phi\psi\rangle$ in general.

So let us find the eigenvalues of H. If we work in the basis of eigenvectors of σ^3, we know how the Pauli matrices act, and the basis for this space will be the tensor products of basis vectors of σ^3, namely $(|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle)$. Let us explicitly calculate the action on the first two basis vectors

$$\begin{aligned}
\vec{\sigma}_1 \cdot \vec{\sigma}_2 \, |\uparrow\uparrow\rangle &= \left[\sigma_1^1\sigma_2^1 + \sigma_1^2\sigma_2^2 + \sigma_1^3\sigma_2^3\right] |\uparrow\uparrow\rangle = [|\downarrow\downarrow\rangle - |\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle] = |\uparrow\uparrow\rangle \\
\vec{\sigma}_1 \cdot \vec{\sigma}_2 \, |\uparrow\downarrow\rangle &= \left[\sigma_1^1\sigma_2^1 + \sigma_1^2\sigma_2^2 + \sigma_1^3\sigma_2^3\right] |\uparrow\downarrow\rangle = [|\downarrow\uparrow\rangle + |\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle] = 2\,|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle \,.
\end{aligned} \tag{37}$$

Via a similar calculation, the action on all four basis vectors is given by

$$\begin{aligned}
H\,|\uparrow\uparrow\rangle &= |\uparrow\uparrow\rangle \\
H\,|\uparrow\downarrow\rangle &= 2\,|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle \\
H\,|\downarrow\uparrow\rangle &= 2\,|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \\
H\,|\downarrow\downarrow\rangle &= |\downarrow\downarrow\rangle \,.
\end{aligned} \tag{38}$$

Let us write $|\uparrow\uparrow\rangle = 2\,|\uparrow\uparrow\rangle - |\uparrow\uparrow\rangle$, and similarly for $|\downarrow\downarrow\rangle$. Define a flip operator $F_{1,2}$ which exchanges the spins of the two particles. Then notice we can write H compactly as

$$H = 2F_{1,2} - 1^1. \tag{39}$$

Since every eigenvector is an eigenvector of 1, diagonalizing $F_{1,2}$ diagonalizes H. Since $F_{1,2}$ simply swaps the two spins, we have immediately that $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ are eigenvectors of $F_{1,2}$ with eigenvalue 1. Also, if we take a symmetric and anti-symmetric combination of $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$, since $F_{1,2}$ maps these states into each other, we can find the other two eigenvectors. In particular

$$\begin{aligned}
F_{1,2}\frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right] &= \frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right] \\
F_{1,2}\frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right] &= -\frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right] .
\end{aligned}$$

Thus, $\frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle\right]$ are eigenvectors of $F_{1,2}$ with eigenvalues ± 1. Since $\dim(\mathcal{H}) = 4$, these are all the eigenvectors of $F_{1,2}$. The same vectors are eigenvectors of H, now with eigenvalues of $(-3, 1, 1, 1)$.

[1] Whenever a number c is added to an operator, we will interpret this as adding $c\mathbb{I}$ to the operator.

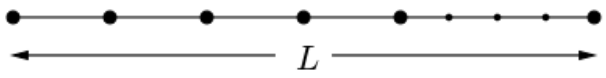

Figure 1: A finite spin chain of length L.

3.2 Spin Chain

Let us now consider a collection of N interacting spin-1/2 particles. Let us suppose that the $N = L$ particles live on the vertices of the chain shown in Figure 1. We will allow the particles to interact with each other only if the vertices that the spin's live on are connected by an edge. We will say that at each vertex, the spin lives in the Hilbert space $\mathbb{C}^2$, and so the Hilbert space for the full chain is

$$\mathcal{H} = \bigotimes_{i=1}^{N} \mathbb{C}^2 \cong \mathbb{C}^{2^N}. \tag{40}$$

A real material has $N \sim 10^{23}$, and so nearly every particle in the material has the exact same surroundings. However, when we examine a finite chain, there are boundaries on both sides of the chain, at $x = 1, L$. It is nearly the case that the system is translationally invariant, and so we wish to build this into our model. In addition, a result from Statistical Mechanics states that phase transitions only exist in the *thermodynamic limit*, where we take $N \to \infty$. Thus if we study an infinite spin chain, then we would have both the thermodynamic limit, and translational invariance. Another method is to work with finite N, but we connect the spin at site $x = L$ back to the spin at site $x = 1$, known as applying *periodic boundary conditions*. This builds in translational symmetry, and then taking the limit as $N \to \infty$ yields the thermodynamic limit. However, doing this induces a global topology different from that of an infinite chain, which can lead to differences between the two models. On the other hand, if done with care, this provides a finite system with translational symmetry which in many cases is ideal for a computer simulation, as it captures the translational symmetry of the infinite system. We will discuss both methods in this paper, but first we will discuss working directly in the thermodynamical limit. This makes our Hilbert space infinite dimensional, and so we will need to proceed with some care.

One major issue is an ill-defined inner product. To see this, suppose we have infinitely many copies of the Hilbert space $\mathcal{H}_1$. Then the Hilbert space would be

$$\mathcal{H} = \bigotimes_{i=1}^{\infty} \mathcal{H}_i. \tag{41}$$

Then if we have two normalized states $|\psi\rangle, |\phi\rangle \in \mathcal{H}$, the natural definition of inner product would be

$$\langle \psi | \phi \rangle = \prod_{i=1}^{\infty} \langle \psi_i | \phi_i \rangle_{\mathcal{H}_i}. \tag{42}$$

This need not converge as is seen if $|\psi_i\rangle = -|\phi_i\rangle$ for each i. This is potentially catastrophic, as all physical measurements are calculated via inner products.

4 Quantum Spin Systems on Infinite Lattices

The majority of this section is based upon [6]. Let us assume that we have infinitely many copies of spin-1/2 particles located in a periodic manner at *sites*, such as at the vertices of a lattice like $\mathbb{Z}^2$, with Hilbert space $\mathbb{C}^2$ at each site. We saw before that there is a possible issue of an ill-defined inner product. Notice though that if we were able to ensure that every state in our full Hilbert space differed from any other state at only finitely many sites, then the inner product would reduce to a finite product, and would be well defined. Doing this will be the goal of this section. The big picture idea is that we are going to construct our Hilbert space $\mathcal{H}$ out of a single reference state $|\Omega\rangle$ by acting on this state with all possible observables that act only on a finite number of sites, called *local observables*. This will make any other state differ from $|\Omega\rangle$ at only finitely many sites, moreover this would mean any two states in $\mathcal{H}$ would differ at only finitely many sites. To achieve this, we first want to generalize away from a Hilbert space, to define observables in a C*-algebra, rather than as bounded operators on a Hilbert space. We then will develop the notion of a state directly on observables. We will then discuss the GNS Hilbert space construction, which generates a Hilbert space from our C*-algebra. Lastly, we will discuss how the algebra of local observables form a C*-algebra, from which we can construct a well-defined local Hilbert space, which will allow us to work in the thermodynamic limit.

4.1 C*-algebras

For operators acting on a Hilbert space, there is a well-defined adjoint operation, due to a Hilbert space being an inner product space. This operation plays an important role in quantum mechanics, as all physical observables are self-adjoint, meaning

$$H^* = H, \tag{43}$$

where H^* is the adjoint of H. The adjoint operation is only defined through the inner product through the defining relation

$$\langle \psi | H\phi \rangle = \langle H^*\psi | \phi \rangle \,. \tag{44}$$

We want to define an adjoint like operation directly on our algebra of observables, and such an algebra is called a *$*$-algebra*.

Definition 4.1 *A $*$-algebra $\mathfrak{A}$ is an algebra on which an (anti-linear) involution $*$ is defined. That is, there is a map $* : \mathfrak{A} \to \mathfrak{A}$ with the following properties:*

1. *$(A^*)^* = A$ for all $A \in \mathfrak{A}$.*
2. *$(AB)^* = B^*A^*$ for $A, B \in \mathfrak{A}$.*

3. $(\lambda A + B)^* = \bar{\lambda} A^* + B^*$ *for* $A, B \in \mathfrak{A}$ *and* $\lambda \in \mathbb{C}$.

An example of a $*$-algebra is the set of $n \times n$ matrices, where the $*$-operation is given by the adjoint. We want our final algebra of observables to also include limits of sums and sequences of operators, and so we need a space that is complete with respect to a norm. This leads us to a more general form of $*$-algebra, known as a *C*-algebra*.

Definition 4.2 *A C*-algebra* $\mathfrak{A}$ *is a* $*$*-algebra which is complete with respect to a norm* $\|\cdot\|$*. Moreover, the norm satisfies the following properties:*

1. $\|AB\| \leq \|A\|\|B\|$ *for* $A, B \in \mathfrak{A}$.
2. $\|A\| = \|A^*\|$ *for all* $A \in \mathfrak{A}$.
3. $\|A^*A\| = \|A\|^2$ *for all* $A \in \mathfrak{A}$.

Also, a *unital C*-algebra* is a C*-algebra that has an identity element. We will always work with a unital C*-algebra.

4.2 States

Recall that the expectation value in a Hilbert space of the operator $\mathcal{O}$ in the state $|\psi\rangle$ is given by

$$\langle \mathcal{O} \rangle := \langle \psi | \, \mathcal{O} \, | \psi \rangle \, . \tag{45}$$

We want a definition of a state without making reference to a vector in a Hilbert space, so we can address expectations of operators in an algebra. We observe that since expectation values are elements of $\mathbb{C}$, we can define a state as a linear functional from our C*-algebra to $\mathbb{C}$.

Definition 4.3 *A linear functional on a C*-algebra* $\mathfrak{A}$ *is a linear map* $\omega : \mathfrak{A} \to \mathbb{C}$*. It is called positive if* $\omega(A^*A) \geq 0$ *for all* $A \in \mathfrak{A}$.

Then a *state* is defined as a linear functional with norm 1, i.e. $\omega(\mathbb{I}) = 1$. This is the abstraction of demanding our states are normalized.

4.3 GNS Hilbert Space Construction

Ultimately, quantum mechanics is performed in a Hilbert space, and so we need to construct a Hilbert space from our C*-algebra to perform calculations. To do this we will need a representation of our C*-algebra on a Hilbert space.

Definition 4.4 *Let* $\mathfrak{A}$ *be a C*-algebra and let* $\mathcal{H}$ *be a Hilbert space. A representation of* $\mathfrak{A}$ *on* $\mathcal{H}$ *is a* $*$*-homomorphism* $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$*. That is, a linear map such that* $\pi(AB) = \pi(A)\pi(B)$ *for all* $A, B \in \mathfrak{A}$*. For a* $*$*-representation one has in addition that* $\pi(A)^* = \pi(A^*)$.

Now, if we have a state, a C*-algebra, and a representation, the GNS Hilbert space construction guarantees that we can construct a Hilbert space from these quantities.

Theorem 4.1 (GNS Hilbert Space Construction) *Let $\mathfrak{A}$ be a unital C^*-algebra and ω a state on $\mathfrak{A}$. Then there is a triple $(\pi_\omega,\mathcal{H}_\omega,\Omega)$, where $\mathcal{H}_\omega$ is a Hilbert space, π_ω a representation of $\mathfrak{A}$ on $\mathcal{H}_\omega$ and $\Omega \in \mathcal{H}_\omega$, such that Ω is cyclic for π_ω and in addition we have*

$$\omega(A) = \langle\Omega| \pi_\omega(A) |\Omega\rangle, \quad A \in \mathfrak{A}. \tag{46}$$

This triple is unique in the sense that if $(\pi, \mathcal{H}, \Psi)$ is another such triple, there is a unitary $U : \mathcal{H}_\omega \to \mathcal{H}$ such that $U\Omega = \Psi$ and $\pi(A) = U\pi(A)U^*$ for all $A \in \mathfrak{A}$.

4.4 Algebra of Local Observables

In the case of a finite-dimensional Hilbert space, we defined observables as the set of bounded operators acting on that Hilbert space $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$. Using the adjoint operation from the Hilbert space, $\mathcal{B}(\mathcal{H})$ forms a $*$-algebra. In the case of an infinite system, one approach to define our observable algebra is to again just use the set of bounded operators, but then locality is lost. By locality we mean that an operator only has a finite range, and cannot have an effect infinitely far away. The importance of locality is two-fold: first it ensures that the inner product is well defined in the thermodynamic limit, and secondly it is physically meaningful because we expect that a measurement made here on Earth will have no effect on a distant galaxy. Note that addition and composition of local operators is still local, and the set of local operators form an algebra. In the case of an finite system, then the set of local operators on a Hilbert space is the same as the set of bounded operators. In the case of the infinite system, the set of local operators is a subset of bounded operators. We will construct the set of local operators on an infinite system as follows.

First, we will always work with the lattice $\mathbb{Z}^2$. Define a set of sites $\mathcal{S}$ on $\mathbb{Z}^2$, this set could be the set of vertices, or the set of edges connecting vertices, or something else, but we will always assume the set of sites is countable. We will also always study spin-1/2 systems, so say that at each site is a Hilbert space $\mathcal{H}_s = \mathbb{C}^2$. Consider a finite set of sites $\Lambda \in \mathbb{Z}^2$, then

$$\mathcal{H}_\Lambda := \bigotimes_{s\in\Lambda} \mathcal{H}_s \tag{47}$$

is a finite dimensional Hilbert space. We then define

$$\mathfrak{A}_\Lambda := \mathcal{B}(\mathcal{H}_\Lambda) = \bigotimes_{x\in\Lambda} M_2(\mathbb{C}) \tag{48}$$

to be the algebra of observables on this space. We want to generalize this notion to the case of infinitely many sites, while preservering locality. To do this, we need a way to compare the algebra of observables from one set of sites to the other.

Let us suppose $\Lambda_1 \subset \Lambda_2 \subset \mathbb{Z}^2$, and let $A \in \mathfrak{A}_{\Lambda_1}$. We can interpret $A \in \mathfrak{A}_{\Lambda_2}$ by acting as A on the sites in $\Lambda_1 \cup \Lambda_2$, but trivially everywhere else. Suppose that

there are n sites in Λ_1 and $n+m$ sites in Λ_2, then we say $A_1 \otimes A_2 \otimes \cdots \otimes A_n \in \mathfrak{A}_{\Lambda_1}$ then $A_1 \otimes A_2 \otimes \cdots \otimes A_n \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} \in \mathfrak{A}_{\Lambda_2}$, where we have inserted copies of the identity operator at the m sites in $\Lambda_2 \backslash \Lambda_1$. For the case that $\Lambda_1 \subset \Lambda_2$ we can see via this extension that $\mathfrak{A}_{\Lambda_1} \subseteq \mathfrak{A}_{\Lambda_2}$

Let us denote the set of all finite subsets of $\mathbb{Z}^2$ as $\mathcal{P}_f$. Since we can make sense of algebras being embedded in a larger space, we can define the $*$-algebra called the *strictly local algebra* of observables for the infinite systems as

$$\mathfrak{A}_{loc} := \bigcup_{\Lambda \in \mathcal{P}_f} \mathfrak{A}_\Lambda. \tag{49}$$

To yield a C*-algebra, we take the closure with respect to the operator norm

$$\mathfrak{A}_{\mathbb{Z}^2} := \overline{\mathfrak{A}_{loc}}^{\|\cdot\|}. \tag{50}$$

This space is called the *quasi-local algebra* of observables. This is a C*-algebra for the infinite system $\mathbb{Z}^2$. Thus, if we have any state $\omega : \mathfrak{A}_{\mathbb{Z}^2} \to \mathbb{C}$, then from the GNS Hilbert space construction Theorem, we will have a Hilbert space $\mathcal{H}_\omega$, a representation π_ω of $\mathfrak{A}_{\mathbb{Z}^2}$ on $\mathcal{H}_\omega$, and a reference state $|\Omega\rangle$, such that $|\Omega\rangle$ is cyclic for π_ω. Furthermore, since $|\Omega\rangle$ is cyclic for π_ω and π_ω is a representation of local operators, this means that every vector in $\mathcal{H}_\omega$ differs from $|\Omega\rangle$ at finitely many sites, and so our inner product is well defined.

Picking such a state ω is somewhat of an art. Different choices of ω can in fact yield inequivalent representations of our observables $\mathfrak{A}_{\mathbb{Z}^2}$, and it is not always clear which representation is preferred. Ultimately we are interested in studying the low energy excitations of our system, and this makes taking the ground state of our Hamiltonian an obvious candidate for our state. It is sometimes the case that the ground state is degenerate, and we will discuss such models. In this case, one can simply pick one of the ground states as their state for input into the GNS construction. In the cases studied here, the energies will not depend on the choice of ground state, but expectation values of certain observables can. The different inequivalent representations correspond to different physical configuration with a global conserved quantity, loosely called *charge*.

4.5 Unbounded Operators and the Hamiltonian

In general the Hamiltonian H is an unbounded operator in the thermodynamic limit. To remedy this, we will always work in an invariant subspace under H, in which the action of H is well defined. Finding the ground state of the Hamiltonian will take some care though. We will first work on a finite system in which the Hamiltonian is bounded, and find the ground state there. We then will extend this ground state to the thermodynamic limit, giving us our state for the GNS construction.

5 Quantum Spin Models on $\mathbb{Z}^2$

Many real materials have planes of magnetically active atoms seperated by planes of mangetically inactive atoms, for which their bulk properties are well modelled by two-dimensional planes of spin-1/2 particles. There are also single atom thick materials, such as graphene, for which a two-dimensional model is the most accurate. One of the simplest two-dimenonsional lattices is a square lattice, given mathematically by $\mathbb{Z}^2$. We will restrict ourselves to studying the energy levels of quantum spin-1/2 systems on $\mathbb{Z}^2$.

5.1 Ising Model

Define Λ as a finite set of connected vertices in $\mathbb{Z}^2$. We then define the Ising model on Λ by its Hamiltonian

$$H_\Lambda = \frac{1}{2}\sum_{v\in\Lambda}\sum_{\delta\sim x}(1-\sigma_v^3\sigma_\delta^3), \tag{51}$$

where the second sum is over all vertices $\delta \in \Lambda$ that are connected to $v \in \Lambda$.

Define the total spin operator along the x_3 direction

$$\Sigma^3 = \sum_{v\in\Lambda}\sigma_v^3. \tag{52}$$

Let us note that $[H_\Lambda, \Sigma^3] = 0$, and so we can find simultaneous eigenvectors of H and Σ^3. The eigenvectors of Σ^3 are simply the tensor product of σ^3 eigenvectors at each vertex, labelled by whether the spin is up or down at each vetex $x \in \mathbb{Z}^2$. We also note that for each v, δ that

$$[H, (1-\sigma_v^3\sigma_\delta^3)] = 0, \tag{53}$$

and so we can diagonalize H by diagnonalizing each term. We also note that

$$(1-\sigma_v^3\sigma_\delta^3) \geq 0, \tag{54}$$

and in particular, the four eigenvalues of $(1-\sigma_x^3\sigma_\delta^3)$ are $(0, 0, 2, 2)$. Thus $H \geq 0$, and there are exactly two states $|\psi_{1,2}\rangle$ satisfying

$$H\,|\psi_{1,2}\rangle = 0. \tag{55}$$

These states are written in the spin basis as

$$|\psi_1\rangle = |\uparrow\uparrow\uparrow\cdots\uparrow\rangle \quad ; \quad |\psi_2\rangle = |\downarrow\downarrow\downarrow\cdots\downarrow\rangle\,. \tag{56}$$

Let us use $|\psi_2\rangle$ to define a state on our algebra. In particluar, define

$$|\Omega_\Lambda\rangle := \bigotimes_{x\in\Lambda}|\downarrow\rangle\,. \tag{57}$$

Then we will define a state ω_Λ on our local algebra $\mathfrak{A}_\Lambda$ by

$$\omega_\Lambda(A) := \langle \Omega_\Lambda | A | \Omega_\Lambda \rangle, \quad \forall A \in \mathfrak{A}_\Lambda. \tag{58}$$

We want to generalize this to the infinite system, to be a state on $\mathfrak{A}_{\mathbb{Z}^2}$. Notice that if we increase the size of Λ, we can just increase the tensor product of $|\Omega_\Lambda\rangle$ with more copies of $|\downarrow\rangle$, and so this naturally extends to a state on $\mathfrak{A}_{loc}$. To yield a state ω on $\mathfrak{A}_{\mathbb{Z}^2}$, we will take the limit that Λ goes to $\mathbb{Z}^2$

$$\omega(A) := \lim_{\Lambda \uparrow \mathbb{Z}^2} \omega_\Lambda(A). \tag{59}$$

Since A must be local and bounded, A will only effect finitely many sites of $|\Omega\rangle_\Lambda$, making the expectation value a finite product of finite terms. This argument is independent on how we extend Λ to $\mathbb{Z}^2$, and thus the sequence converges.

We then use ω along with $\mathfrak{A}_{\mathbb{Z}^2}$ to construct a GNS triple $(\pi_\omega, \mathcal{H}_\omega, |\Omega\rangle)$. We can now work directly in the thermodynamic limit to study this model.

5.1.1 Thermodynamic Limit

We wish to construct an invariant subspace for which we can discuss the action of the Hamiltonian in the infinite system. Let $|S_1\rangle, |S_2\rangle$ be eigenvectors of Σ^3 with eigenvalues s_1, s_2, and notice that since $[H_\Lambda, \Sigma^3] = 0$

$$\begin{aligned} 0 = \langle S_1 | [H_\Lambda, \Sigma^3] | S_2 \rangle &\implies 0 = \langle S_1 | H_\Lambda \Sigma^3 - \Sigma^3 H_\Lambda | S_2 \rangle \\ &\implies 0 = (s_2 - s_1) \langle S_1 | H_\Lambda | S_2 \rangle . \end{aligned} \tag{60}$$

Thus, either $s_2 = s_1$ or $\langle S_1 | H_\Lambda | S_2 \rangle = 0$. Thus, if we work in a space of degenerate eigenvectors of Σ^3, then H_Λ leaves this space invariant, for every $\Lambda \in \mathbb{Z}^2$. In particular, define the following subspaces of $\mathcal{H}_\omega$

$$\begin{aligned} \mathcal{H}^{(1)} &:= \overline{span(\sigma_x^+ |\Omega\rangle | x \in \mathbb{Z}^2)}^{\|\cdot\|} \\ \mathcal{H}^{(2)} &:= \overline{span(\sigma_x^+ \sigma_y^+ |\Omega\rangle | x, y \in \mathbb{Z}^2, x \neq y)}^{\|\cdot\|}, \end{aligned} \tag{61}$$

where the over line denotes the closure with respect to the norm $\|\cdot\|$. These subspaces are invariant under H_Λ, and restricted to this subspace, the infinite system Hamiltonian H will be well defined, defined by

$$H = \frac{1}{2} \sum_{v \in \mathbb{Z}^2} \sum_{\delta \sim v} (1 - \sigma_v^3 \sigma_\delta^3), \tag{62}$$

where we restrict the domain of H to be in $\mathcal{H}^{(1,2)}$.

If we adopt the particle physics nomenclature, we interpret $|\Omega\rangle$ as the vacuum and $\sigma_x^+ |\Omega\rangle$ as creation of a particle at the position x. However, this kind of particle is not part of the standard model of physics, and so we call such an excitation a *quasi-particle*. Since the Pauli matrices at different sites in a lattice commute, we have that

$$\sigma_x^+ \sigma_y^+ |\Omega\rangle = \sigma_y^+ \sigma_x^+ |\Omega\rangle , \tag{63}$$

and so these quasi-particles are bosons, and called *magnons*. We will define magnon states as follows

$$\begin{aligned} |m_x\rangle &:= \sigma_x^+ |\Omega\rangle \\ |m_x m_y\rangle &:= \sigma_x^+ \sigma_y^+ |\Omega\rangle . \end{aligned} \tag{64}$$

Let us consider the action of H on a one-magnon state

$$H |m_x\rangle = \frac{1}{2} \sum_{v \in \mathbb{Z}^2} \sum_{\delta \sim v} (1 - \sigma_v^3 \sigma_\delta^3) \sigma_x^+ |\Omega\rangle . \tag{65}$$

Since Pauli matrices of different vertices commute, each term in the sum passes through σ_x^+ and returns 0 on $|\Omega\rangle$ except for the case of $v, \delta = x$. Each pair will happen exactly twice, so we can remove the extra factor of two out front and consider only the case $v = x$,

$$H |m_x\rangle = \sum_{\delta \sim x} (1 - \sigma_x^3 \sigma_\delta^3) \sigma_x^+ |\Omega\rangle . \tag{66}$$

Since σ_x^+ is on a different vertex than σ_δ^3, they commute. Also $\sigma_x^3 |\Omega\rangle = - |\Omega\rangle$ for all $x \in \mathbb{Z}^2$, thus

$$H |m_x\rangle = \sum_{\delta \sim x} (\sigma_x^+ + \sigma_x^3 \sigma_x^+) |\Omega\rangle . \tag{67}$$

Also, by the commutation relations of the Pauli matrices, we have that $\sigma^3 \sigma^+ = \sigma^+$, so we have

$$H \sigma_x^+ |\Omega\rangle = 2 \sum_{\delta \sim x} \sigma_x^+ |\Omega\rangle . \tag{68}$$

Since there are exactly four term in the sum, and the summand is independent of δ we conclude

$$H |m_x\rangle = 8 |m_x\rangle , \tag{69}$$

and therefore $|m_x\rangle$ is an eigenvector of H with eigenvalue 8 for each $x \in \mathbb{Z}^2$. Since $|m_x\rangle$ is an eigenvector, its dynamics is simply given by multiplication by a phase, and is trivial. In other words, magnon's in the Ising model are stationary, and do not move under their own dynamics. A nearly identical calculation shows that for the two-magnon states $\sigma_x^+ \sigma_y^+ |\Omega\rangle$ yields

$$H |m_x m_y\rangle = \begin{cases} 12 |m_x m_y\rangle & if \quad x \sim y \\ 16 |m_x m_y\rangle & otherwise \end{cases} \tag{70}$$

In general, any N magnon state defined by

$$|m_{x_n} \cdots m_{x_2} m_{x_1}\rangle := \sigma_{x_N}^+ \cdots \sigma_{x_2}^+ \sigma_{x_1}^+ |\Omega\rangle \tag{71}$$

is an eigenvector of H, and thus the dynamics of any collection of magnons in this model is trivial. To yield non-trivial dynamics in the magnons, we need other terms in the Hamiltonian. A common model that does this is called the Heisenberg model.

5.2 Heisenberg Model

Let Λ be a finite set of connected vertices in $\mathbb{Z}^2$. Then the Heisenberg model is defined by its Hamiltonian

$$H_\Lambda = \frac{1}{2}\sum_{v\in\Lambda}\sum_{\delta\sim v}(1-\vec{\sigma}_v\cdot\vec{\sigma}_\delta). \tag{72}$$

Again we have the property that

$$[H_\Lambda, \Sigma^3] = 0. \tag{73}$$

Therefore we have that H_Λ and Σ^3 can be simultaneously diagonalized. From the two-spin Hamiltonian problem, we know that $\vec{\sigma}_1\cdot\vec{\sigma}_2$ has eigenvalues of $-3, 1$, and so $1-\vec{\sigma}_1\cdot\vec{\sigma}_2$ has eigenvalues of $0, 4$. Thus H is a sum of positive operators, and so $H \geq 0$. In this case there are in fact infinitely many states $|\psi\rangle$ satisfying $H|\psi\rangle = 0$. In particular, $|\Omega_\Lambda\rangle$ from the Ising model satisfies $H_\Lambda|\Omega_\Lambda\rangle = 0$. We thus can use the same GNS Hilbert space that we constructed for the Ising model here.

Since $[H_\Lambda, \Sigma^3] = 0$ for each Λ, we have that the subspaces $\mathcal{H}^{(1,2)}$ are invariant under H, and so we define the infinite system Hamiltonian which is well defined on these subspaces as

$$H_\Lambda = \frac{1}{2}\sum_{v\in\mathbb{Z}^2}\sum_{\delta\sim v}(1-\vec{\sigma}_v\cdot\vec{\sigma}_\delta). \tag{74}$$

In this case, however, the state $|m_x\rangle$ is not an eigenvector of H. To see this, let us determine the action of H on $|m_x\rangle$

$$H|m_x\rangle = \frac{1}{2}\sum_{v\in\mathbb{Z}^2}\sum_{\delta\sim v}(1-\vec{\sigma}_v\cdot\vec{\sigma}_\delta)\sigma_x^+|\Omega\rangle. \tag{75}$$

By definition

$$\vec{\sigma}_v\cdot\vec{\sigma}_\delta = \sigma_v^3\sigma_\delta^3 + \sigma_v^1\sigma_\delta^1 + \sigma_v^2\sigma_\delta^2. \tag{76}$$

We can rewrite this dot product in terms of σ^+, σ^- instead of σ^1, σ^2 yielding

$$\vec{\sigma}_v\cdot\vec{\sigma}_\delta = \sigma_v^3\sigma_\delta^3 + 2\left[\sigma_\delta^+\sigma_v^- + \sigma_\delta^+\sigma_v^-\right]. \tag{77}$$

With this we can write

$$H|m_x\rangle = \frac{1}{2}\sum_{v\in\mathbb{Z}^2}\sum_{\delta\sim v}\left(1-\sigma_v^3\sigma_\delta^3 - 2\left[\sigma_\delta^+\sigma_v^- + \sigma_\delta^+\sigma_v^-\right]\right)\sigma_x^+|\Omega\rangle. \tag{78}$$

We have that each term in the sum is 0 unless $v, \delta = x$. Combining both these instances into one eliminates the extra factor of $1/2$.

$$H|m_x\rangle = \sum_{\delta\sim x}\left(1-\sigma_v^3\sigma_\delta^3 - 2\left[\sigma_\delta^+\sigma_x^- + \sigma_\delta^+\sigma_x^-\right]\right)\sigma_x^+|\Omega\rangle. \tag{79}$$

The first two terms is identical to the Ising model, and yields $8\,|m_x\rangle$. For the other terms, we use the fact that $\sigma^+\sigma^+ = 0$, and $\sigma^-\sigma^+ = (1-\sigma^3)/2$, thus

$$H\,|m_x\rangle = 8\,|m_x\rangle - \sum_{\delta\sim x} \sigma^+_\delta (1-\sigma^3_x)\,|\Omega\rangle\,. \tag{80}$$

Lastly, $\sigma^3_x\,|\Omega\rangle = -\,|\Omega\rangle$, and so we have

$$H\,|m_x\rangle = 8\,|m_x\rangle - 2\sum_{\delta\sim x} |m_\delta\rangle\,. \tag{81}$$

Thus we see that $|m_x\rangle$ is not an eigenvector of H in the Heisenberg model. We do know, however, that since $\mathcal{H}^{(1)}$ is an invariant subspace under H, in general we can write an eigenvector as a super-position over all $|m_x\rangle$ for $x \in \mathbb{Z}^2$. Thus define a general state in this subspace by

$$|\psi\rangle = \sum_{x\in\mathbb{Z}^2} A_x\,|m_x\rangle\,. \tag{82}$$

Then finding the eigenvectors is equivalent to solving for the coefficients $\{A_x\}$ in the equation

$$H\,|\psi\rangle = E\,|\psi\rangle \implies \sum_{x\in\mathbb{Z}^2} A_x H\,|m_x\rangle = \sum_{x\in\mathbb{Z}^2} A_x E\,|m_x\rangle \tag{83}$$

using Equation (81)

$$\sum_{x\in\mathbb{Z}^2}\left[8\,|m_x\rangle + 2\sum_{\delta\sim x} A_x\,|m_\delta\rangle\right] = \sum_{x\in\mathbb{Z}^2} A_x E\,|m_x\rangle\,. \tag{84}$$

From translational invariance in $\mathbb{Z}^2$, we can convert the $|m_\delta\rangle$ to $|m_x\rangle$ by a translation of $-\delta$. Then define u as the set of unit vectors in the $\pm x_{1,2}$ directions, then

$$\sum_{x\in\mathbb{Z}^2}\left[8A_x + 2\sum_{y\in u} A_{x-y}\right]|m_x\rangle = \sum_{x\in\mathbb{Z}^2} A_x E\,|m_x\rangle\,. \tag{85}$$

Note that $\langle m'_x|m_x\rangle = \delta_{x,x'}$, since $\langle\downarrow|\uparrow\rangle = 0$, and so we can equate the coefficents in front of $|m_x\rangle$ on each side of the equals sign. This yields the following recursion relation

$$8A_x + 2\sum_{y\in u} A_{x-y} = EA_x. \tag{86}$$

To solve this recursion relation, consider the ansatz $A_x = e^{i\vec{k}\cdot\vec{x}}$

$$8e^{i\vec{k}\cdot\vec{x}} + 2e^{i\vec{k}\cdot\vec{x}}\sum_{y\in u} e^{-i\vec{k}\cdot\vec{y}} = Ee^{i\vec{k}\cdot\vec{x}}. \tag{87}$$

Using the fact that $e^{i\phi} + e^{-i\phi} = 2\cos(\phi)$ we conclude

$$E = 4(1-\cos(k_1)) + 4(1-\cos(k_2)) = 4\left[\sin^2\left(\frac{k_1}{2}\right) + \sin^2\left(\frac{k_2}{2}\right)\right], \tag{88}$$

where $(k_1, k_2) = \vec{k}$.

5.3 A Few Remarks

We saw that the Ising model had quasi-particle excitations called magnons, but the magnons were eigenvectors of the Hamiltonian and so they had trivial dynamics. The Heisenberg model however also has magnon excitations, but they were not eigenvectors of the Hamiltonian. We found that the eigenvectors were a superposition of magnon states, with coefficients given by $e^{i\vec{k}\cdot\vec{x}}$. Although we algebraically found the coefficients in the superposition by solving a recursion relation, the coefficients have a simple physical meaning.

Since $\mathbb{Z}^2$ is translationally invariant, this induces a group action of the translation group on our model, which has a unitary representation. Since the Hamiltonian acts the same, it is invariant under this transformation, and so the eigenvectors of H must also respect this transformation. This means that translating the entire system by an element $\xi \in \mathbb{Z}^2$ would be physically equivalent before and after. Thus, the difference before and after the transformation would at most pick up a phase that depends on ξ. In particular, if $T(\xi)$ is the representation of a translation by ξ on our Hilbert space and $|\psi\rangle$ is an eigenvector of H

$$T(\xi)\,|\psi\rangle = e^{i\phi(\xi)}\,|\psi\rangle\,. \tag{89}$$

Since translation by ξ_1 and then translation by ξ_2 is equivalent to a translation by $\xi_1 + \xi_2$ we need

$$T(\xi_2)T(\xi_1)\,|\psi\rangle = T(\xi_1+\xi_2)\,|\psi\rangle \implies e^{i\phi(\xi_1)+\phi(\xi_2)} = e^{i\phi(\xi_1+\xi_2)}. \tag{90}$$

Also translation by $\xi = 0$ must return the original state. These conditions constrain $\phi(\xi) = k\xi$ for some constant k. Thus we say

$$T(\xi)\,|\psi\rangle = e^{i\vec{k}\cdot\vec{\xi}}\,|\psi\rangle\,. \tag{91}$$

What we find from this is that the eigenvectors of the Heisenberg Hamiltonian are in fact eigenvectors of the translation operator, which we could have predicted by exploiting translation symmetry rather than simply solving for the coefficients generally. Translational symmetry is not always enough to solve a problem. The fact that all the Hamiltonians we will study are translational invariant only means that we can simultaneoulsy diagonalize both operators, but this does not always guarantee that the eigenvectors of the translation operator will be eigenvectors of the Hamiltonian. This is analogous to the fact that when we found eigenvectors of Σ^3, these eigenvectors were the eigenvectors of the Ising Hamiltonian, but not the eigenvectors of the Heisenberg Hamiltonian.

Also, the vector $\vec{k}$ in the eigenvector of the translation operator we interpret loosely as the momentum. In the case of the continuum, such as in a Quantum Field Theory, the Translation operator is generated by the momentum operator, and so we carry this over into the case of a discrete system such as the ones defined on a lattice here. This momentum is sometimes called the crystalline momentum, quasi-momentum, or simply the wave-vector. One major difference is the distinct values that $\vec{k}$ can take. We usually restrict $\vec{k} \in [-\pi,\pi] \times [-\pi,\pi]$.

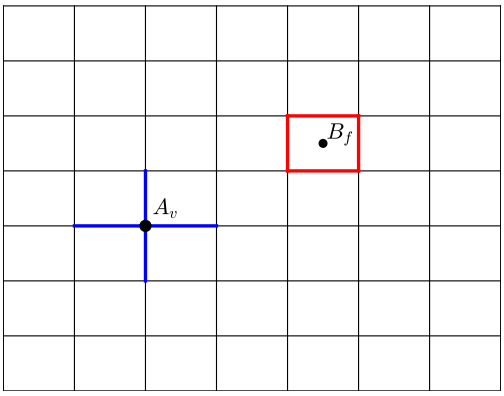

Figure 2: Example of a vertex operator (blue) and a face operator (red) of the Toric Code model.

We see that under the transformation $\vec{k} \to \vec{k} + 2\pi\vec{n}, \vec{n} \in \mathbb{Z}^2$, then the eigenvalues of the translation operator transform

$$e^{i\vec{k}\cdot\vec{x}} \longrightarrow e^{i(\vec{k}+2\pi\vec{n})\cdot\vec{x}} = e^{i\vec{k}\cdot\vec{x}}. \tag{92}$$

Thus the eigenvalues are invariant under this transformation. Thus, all the physically significant information from $\vec{k}$ is extracted in any square of length 2π, we choose $\vec{k} \in [-\pi, \pi] \times [-\pi, \pi]$ by convention.

Lastly, we found that the energies in the Heisenberg model were written as a function of $\vec{k}$. The energy as a function of $\vec{k}$ is known as a *dispersion relation*. This relation yields a direct measurement of the speed of propagation of information v_g in the material by

$$v_g := \frac{dE}{dk}. \tag{93}$$

This relation also makes calculating thermodynamic quantities explicitly clear, since thermodynamic quantities can be expressed as integrals of the energy over all physically distinct values of k weighted by some kernel. In many cases, once the dispersion relation is known for the system, essentially any other quantity can, at least in principle, be calculated from this. Because of this, we focus the majority of our attention to determining the dispersion relation for these models.

6 Toric Code Model

Let Λ be a finite subset of $\mathbb{Z}^2$. In this model, the sites are given by edges in Λ. A face is defined as the center point of a square, and an edge is a line connecting

two vertices. We will denote by $e \sim f$,$e \sim v$ if the edge e is adjacent to f, v, and we will denote $v \sim f$ if there exists an edge e such that $e \sim v$ and $e \sim f$. define the following operators, illustrated in Figure 2

$$A_v = \prod_{e \sim v} \sigma_e^1 \quad ; \quad B_f = \prod_{e \sim f} \sigma_e^3. \tag{94}$$

We then define the Hamiltonian on Λ by

$$H_\Lambda = \sum_{\{v | A_v \in \mathfrak{A}_\Lambda\}} (1 - A_v) + \sum_{\{f | B_f \in \mathfrak{A}_\Lambda\}} (1 - B_f). \tag{95}$$

Since both A_v and B_f are products of Pauli matrices, each with eigenvalues ± 1, we have that the eigenvalues of $A_v, B_f = \pm 1$. Thus each term in the Hamiltonian is non-negative, and so we have that $H \geq 0$. Note that every A_v, B_f shares either 0 or 2 edges. If they share 0 edges, then it is clear that $[A_v, B_f] = 0$. When they share two edges, call them e_3, e_4 then we have

$$A_v B_f = \sigma_{e_1}^1 \sigma_{e_2}^1 \sigma_{e_3}^1 \sigma_{e_4}^1 \sigma_{e_3}^3 \sigma_{e_4}^3 \sigma_{e_5}^3 \sigma_{e_6}^3. \tag{96}$$

Since the Pauli matrices acting on different edges commute, we have

$$A_v B_f = \sigma_{e_5}^3 \sigma_{e_6}^3 (\sigma_{e_3}^1 \sigma_{e_3}^3 \sigma_{e_4}^1 \sigma_{e_4}^3) \sigma_{e_1}^1 \sigma_{e_2}^1. \tag{97}$$

Since the Pauli matrices anti-commute, we can reverse the order of the Pauli matrices at e_3 at the cost of a minus sign. Then doing this again for the Pauli matrices at e_4 produces an additional minus sign cancelling the first. Thus

$$A_v B_f = \sigma_{e_3}^3 \sigma_{e_4}^3 \sigma_{e_5}^3 \sigma_{e_6}^3 \sigma_{e_1}^1 \sigma_{e_2}^1 \sigma_{e_3}^1 \sigma_{e_4}^1 = B_f A_v. \tag{98}$$

Thus

$$[A_v, B_f] = 0 \quad \forall v, f \in \Lambda. \tag{99}$$

What this also means is that since H_0 is a sum of terms including the identity and A_v or B_f, then

$$[H, A_v] = [H, B_f] = 0 \quad \forall v, f \in \Lambda. \tag{100}$$

Thus we can diagonalize H by diagonalizing each A_v, B_f.

The ground state of this model, $|\Omega_\Lambda\rangle$ is given by the unique state satisfying $A_v |\Omega_\Lambda\rangle = B_f |\Omega_\Lambda\rangle = |\Omega_\Lambda\rangle$ for all $v, f \subset \Lambda$. We then define a state on our local algebra by

$$\omega_\Lambda := \langle \Omega_\Lambda | A | \Omega_\Lambda \rangle, \quad \forall A \in \mathfrak{A}_\Lambda. \tag{101}$$

This state is less clear to show that this extends to the infinite system then in the case of the Ising model. It was shown in [1] that there is a unique frustration free ground state $\omega : \mathfrak{A}_{\mathbb{Z}^2} \to \mathbb{C}^2$ in the thermodynamic limit of this model, satisfying $\omega(A_v) = \omega(B_f) = 1$ for all $v, f \subset \mathbb{Z}^2$. This will be our state that we use in the GNS Hilbert space construction.

Let us label the set of faces in $\mathbb{Z}^2$ by $\mathcal{F}$, the set of vertices in $\mathbb{Z}^2$ by $\mathcal{V}$, and the set of edges in $\mathbb{Z}^2$ by $\mathcal{E}$. Then the infinite system Hamiltonian is defined by

$$H_0 = \sum_{v \in \mathcal{V}} (1 - A_v) + \sum_{f \in \mathcal{F}} (1 - B_f). \tag{102}$$

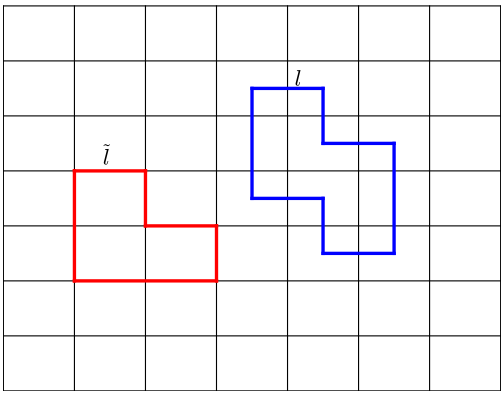

Figure 3: Examples of loop operators for the Toric Code model.

6.1 Understanding The Ground State of the Toric Code

Let us first discuss the state $|\Omega\rangle$ satisfying $H_0 |\Omega\rangle = 0$. Such a state is realized when each Pauli matrix at each edge satisfies

$$\sigma_e^1 |\Omega\rangle = \sigma_e^3 |\Omega\rangle = |\Omega\rangle . \tag{103}$$

Such a state is interpreted as a vacuum state, where all the operators act trivially. However, if at any vertice or at any face, exactly two of the Pauli matrices in A_v, B_f return -1, then still the energy of such a state is 0. We interpret such states as closed loops of Pauli matrices. Explicitly, define a loop in $\mathbb{Z}$ by $\widetilde{l}$ and a loop l in $dual(\mathbb{Z}^2)$ (the dual of $\mathbb{Z}^2$ is the square lattice formed by connecting all the faces), and define loop operators by

$$L^3(\widetilde{l}) = \prod_{e \in \widetilde{l}} \sigma_e^3 \quad ; \quad L^1(l) = \prod_{e \in l} \sigma_e^1. \tag{104}$$

The claim is that $L^1(l) |\Omega\rangle$ and $L^3(\widetilde{l}) |\Omega\rangle$ are eigenvectors H_0 with eigenvalue 0. An example of such loops are shown in Figure 3. Let us consider the quantity

$$H_0 L^3(\widetilde{l}) = \left[\sum_{v \in \mathcal{V}} (1 - A_v) + \sum_{f \in \mathcal{F}} (1 - B_f) \right] \prod_{e \in \widetilde{l}} \sigma_e^3. \tag{105}$$

Since B_f is comprised of all σ^3 operators, the second term commutes with $L^3(\widetilde{l})$. The first term is a little more subtle. Not that since L^3 is a closed loop of σ^3s each vertex that it crosses, it acts on two edges $e \sim v$. This is very analogous to the case of a B_f interacting with A_v, and since there is an even number of

overlaps between $L^3(\widetilde{l})$ and A_v for each $v \in \mathbb{Z}^2$, $L^3(\widetilde{l})$ also commutes with the first term, and so

$$H_0 L^3(\widetilde{l}) = L^3(\widetilde{l}) H_0. \tag{106}$$

Then consider the state $L^3 |\Omega\rangle$. We have

$$H_0 L^3(\widetilde{l}) |\Omega\rangle = L^3(\widetilde{l}) H_0 |\Omega\rangle = L^3(\widetilde{l}) 0 = 0. \tag{107}$$

Thus we conclude that $L^3(\widetilde{l}) |\Omega\rangle$ is an eigenvector of H_0 with eigenvalue 0. A very similar argument shows that $L^1(l) |\Omega\rangle$ is also an eigenvector of H_0 with eigenvalue 0. Since $\mathbb{Z}^2$ is homotopic to a point (i.e. the topology of $\mathbb{Z}^2$ has genus 0) every closed loop can be contracted away, yielding a unique ground state $|\Omega\rangle$ for H_0 satisfying

$$A_v |\Omega\rangle = B_f |\Omega\rangle = |\Omega\rangle . \tag{108}$$

Thus we interpret the ground state $|\Omega\rangle$ as an equal superposition of all possible closed loops in $\mathbb{Z}^2$.

6.2 Excited States of the Toric Code

We saw that when we have a closed loop of Pauli matrices acting on $|\Omega\rangle$ that this returned $|\Omega\rangle$ again. This was because every vertex and face operator saw an even number of Pauli matrices from the loop. What if we have a non-closed path of Pauli matrices instead? Consider a path $\gamma \subset dual(\mathbb{Z}^2)$, and define a string operator by

$$S^1(\gamma) = \prod_{e \in \gamma} \sigma_e^1. \tag{109}$$

Then let us consider the action of H_0 on $S^1(\gamma) |\Omega\rangle$

$$H_0 S^1(\gamma) |\Omega\rangle = \left[\sum_{v \in \mathcal{V}} (1 - A_v) + \sum_{f \in \mathcal{F}} (1 - B_f) \right] S^1(\gamma) |\Omega\rangle . \tag{110}$$

Since A_v is comprised of σ^1s, it passes through $S^1(\gamma)$ and returns 0 on $|\Omega\rangle$. Thus

$$H_0 S^1(\gamma) |\Omega\rangle = \sum_{f \in \mathcal{F}} (1 - B_f) S^1(\gamma) |\Omega\rangle . \tag{111}$$

Only the endpoints of the string don't commute with any B_f, since any interior points will always share exactly two edges with B_f. Then let us denote the endpoints of γ by a, b. Thus only if $f \sim a$ or $f \sim b$ do we get a non-zero contribution from H_0

$$H_0 S^1(\gamma) |\Omega\rangle = \left[(1 - B_{f \sim a}) + (1 - B_{f \sim b}) \right] S^1(\gamma) |\Omega\rangle . \tag{112}$$

In both cases, $B_f S^1(\gamma) = -S^1(\gamma) B_f$. Then using the fact that $B_f |\Omega\rangle = |\Omega\rangle$ we have

$$H_0 S^1(\gamma) |\Omega\rangle = 4 S^1(\gamma) |\Omega\rangle . \tag{113}$$

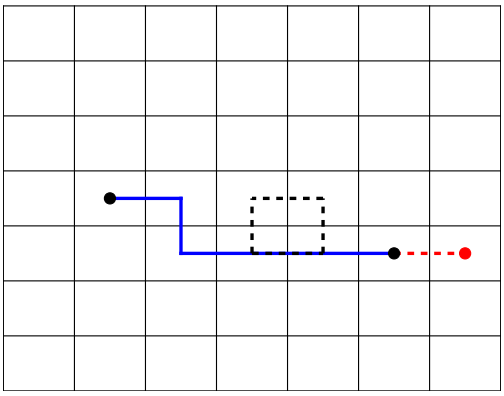

Figure 4: Example's of adjusting the string between two anyons in the Toric Code model. The black dotted line represents a loop which moves the string, and the red dotted line extends the string by multiplcation with a Pauli matrix.

Thus, $S_x^1 |\Omega\rangle$ is an eigenvector of H with eigenvalue 4. If we define a path $\widetilde{\gamma} \subset \mathbb{Z}^2$, and string operator

$$S^3(\widetilde{\gamma}) = \prod_{e \in \widetilde{\gamma}} \sigma_e^3. \tag{114}$$

We would find by an identical analysis that

$$H_0 S^3(\widetilde{\gamma}) |\Omega\rangle = 4 S^3(\widetilde{\gamma}) |\Omega\rangle . \tag{115}$$

These states are another example of quasi-particle excitations, except instead of living at a particular edge, they must live at the endpoints of strings. Also we note that suppose we have a string as shown by the blue line in Figure 4, and we act with the loop shown in the same figure by a dotted black line, then the state is left invariant. This is seen because the string and loop commute, and then the loop returns 1 on $|\Omega\rangle$. Therefore, we can use closed loops to move the string around arbitrarily. Since the Pauli matrices square to 1, we can also interpret such a state as a new state with string given by the union of both strings, but subtracting away their overlap. Furthermore, we can hop the particle to a neighboring site by extending the string with another σ^1, as depicted in red in Figure 4. The story is analogous for strings operators $S^3(\widetilde{\gamma})$.

Define the following subspaces of the full Hilbert space

$$\begin{aligned} \mathcal{H}^{(\epsilon)} &= \overline{\{S^3(\widetilde{\gamma}) |\Omega\rangle : \widetilde{\gamma} \subset \mathbb{Z}^2\}}^{\|\cdot\|} \\ \mathcal{H}^{(\mu)} &= \overline{\{S^1(\gamma) |\Omega\rangle : \gamma \subset dual(\mathbb{Z}^2)\}}^{\|\cdot\|}. \end{aligned} \tag{116}$$

We see that these spaces are invariant subspaces under the action of H_0.

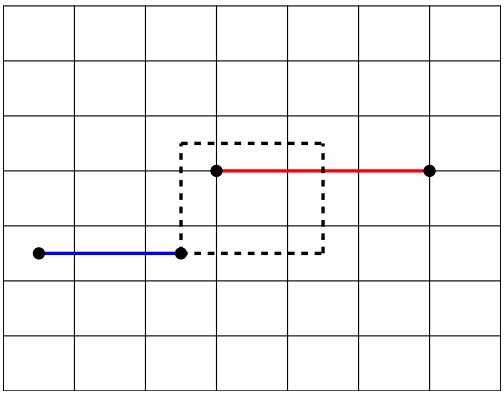

Figure 5: An example of the anyon in blue braiding around the anyon in red, in the Toric Code model.

6.3 Anyons

Next let us consider a state of the form form

$$S^3(\widetilde{\gamma})S^1(\gamma)\left|\Omega\right\rangle. \tag{117}$$

Nearly identical analysis as for the one particle case shows that

$$H_0 S^3(\gamma)S^1(\gamma)\left|\Omega\right\rangle = 8S^3(\gamma)S^1(\gamma)\left|\Omega\right\rangle. \tag{118}$$

We saw before that we can hop a one-particle state to a neighboring face by the action of a σ^3 at a neighboring edge to the particle, effectively extending the path. Let us consider hopping one particle around the other. We show one particle in blue, the other particle in red, and the path the blue particle takes hopping around the red particle, in a process called *braiding*, illustrated by a black dotted line in Figure 5. This process returns the blue particle back to its original place, and so we would naturally believe that the final state after this process is the same as the initial state. To determine the relation between the two, define the following states

$$\left|\psi_{initial}\right\rangle := S^3(\widetilde{\gamma})S^1(\gamma)\left|\Omega\right\rangle \quad ; \quad \left|\psi_{final}\right\rangle := L^1(l)S^3(\widetilde{\gamma})S^1(\gamma)\left|\Omega\right\rangle, \tag{119}$$

where $\widetilde{\gamma}$ is the red string in the Figure 5, γ is the blue string, and l is the dotted black loop. Note that $L^1(l)$ and $S^3(\widetilde{\gamma})$ share exactly one edge, and so they anti-commute. Thus

$$\left|\psi_{final}\right\rangle = -S^3(\widetilde{\gamma})L^1(l)S^1(\gamma)\left|\Omega\right\rangle. \tag{120}$$

Since $L^1(l)$ and $S^1(\gamma)$ are both products of σ^1s, they commute. Then $L^1(l)\,|\Omega\rangle = |\Omega\rangle$, and so we have

$$|\psi_{final}\rangle = -S^3(\widetilde{\gamma})S^1(\gamma)\,|\Omega\rangle = -\,|\psi_{initial}\rangle\,. \tag{121}$$

These particles are quite unlike usual particles, fermions and bosons, which do not change their phase in such a process. Particles with this unusual property are known as *anyons*. The anyons that live in $\mathbb{Z}^2$ are named *electric anyons*, and the anyons in $dual(\mathbb{Z}^2)$ are named *magnetic anyons*. We then usually denote the anyon states as

$$|\epsilon\rangle := S^3(\widetilde{\gamma})\,|\Omega\rangle \quad ; \quad |\mu\rangle := S^1(\gamma)\,|\Omega\rangle \quad ; \quad |\epsilon\mu\rangle := S^3(\widetilde{\gamma})S^1(\gamma)\,|\Omega\rangle\,. \tag{122}$$

These particles also exist in real solid state materials, namely, they are intrinsicly related to the fractional quantum Hall effect.

If we define the following subspace of our Hilbert space

$$\mathcal{H}^{(\epsilon\mu)} = \overline{\{S^3(\widetilde{\gamma})S^1(\gamma)\,|\Omega\rangle : \widetilde{\gamma} \subset \mathbb{Z}^2, \gamma \subset dual(\mathbb{Z}^2)\}}^{\|\cdot\|}, \tag{123}$$

we see also that this subspace is left invariant under H_0.

We see that the Toric Code model harbors exotic quasi-particle excitations that develop non-trivial phases when the anyons braid around each other. However, since the anyons are eigenvectors of H_0, such a process would not occur under its own dynamics. This is very analogous to magnons in the Ising model, in which the magnons were eigenvectors of the Ising Hamiltonian. The goal of this work is to add additional terms to the Hamiltonian (much like the Heisenberg model to the Ising model), that induce non-trivial dynamics in the anyons, yet at the same time leave $\mathcal{H}^{(\epsilon)}$, $\mathcal{H}^{(\mu)}$, and $\mathcal{H}^{(\epsilon\mu)}$ invariant and preserve $|\Omega\rangle$.

7 Dynamical Toric Code Model

The goal we are trying to produce with our Dynamical Toric Code Model is to induce dispersion into the anyons of the Toric Code model. This will be analogous to the Heisenberg model inducing disersion into the magnons of the Ising model. From the Heisenberg model, we can learn a few lessons of how to do this. First, the Heisenberg model respects the total spin symmetry, meaning that it does not couple a one-magnon state to a state with a different number of magnons. Second, a single magnon state was not an eigenvector of the Heisenberg Hamiltonian, but rather this Hamiltonian hopped a single magnon to a neighboring vertex. Lastly, if a magnon exists at vertex v, the Heisenberg model acts trivially on all vertices that are not v or neighbors of v. We want add a term to the Toric Code model that has these same properties.

7.1 Hamiltonian Construction

First, one major difference in the Toric Code model is that the excitations come in pairs connected by a string. We saw, however, that we can extend this string

by multiplying by $\sigma^{1,3}$ at the endpoint of a string. To study the disersion in just one anyon, we consider a semi-infinite path of $\sigma^{1,3}$s that effectively pushes one of the anyons out of the model. Formally, we let $\gamma \subset dual(\mathbb{Z}^2)$ be a semi-infinite path terminated at the face x, and let $\widetilde{\gamma} \subset \mathbb{Z}^2$ be a semi-infinite path terminated at the vertex $\widetilde{x}$, then we define the one particle states by

$$\begin{aligned} |\epsilon_{\widetilde{x}}\rangle &:= S^3_{\widetilde{x}}(\widetilde{\gamma})\,|\Omega\rangle \quad ; \quad S^3_{\widetilde{x}}(\widetilde{\gamma}) = \prod_{j\in\widetilde{\gamma}} \sigma^3_j \\ |\mu_x\rangle &:= S^1_x(\gamma)\,|\Omega\rangle \quad ; \quad S^1_x(\gamma) = \prod_{j\in\gamma} \sigma^1_j. \end{aligned} \tag{124}$$

This process leaves behind a semi-infinite string attached to each anyon state that cannot be removed. We want our perturbation to not interact with the strings in anyway. These states are somewhat different from the Toric Code model, and so let us define the one-particle Hilbert space by

$$\begin{aligned} \mathcal{H}^{(\epsilon)} &= \overline{\{S^3_{\widetilde{x}}(\widetilde{\gamma})\,|\Omega\rangle : \widetilde{\gamma} \subset \mathbb{Z}^2\}}^{\|\cdot\|} \\ \mathcal{H}^{(\mu)} &= \overline{\{S^1_x(\gamma)\,|\Omega\rangle : \gamma \subset dual(\mathbb{Z}^2)\}}^{\|\cdot\|}. \end{aligned} \tag{125}$$

We will see that the ground state of the Toric Code model is annihilated by our perturbation, and that it leaves $\mathcal{H}^{(\epsilon)}, \mathcal{H}^{(\mu)}$ invariant. Thus we will work directly in the thermodynamic limit in the Hilbert space constructed for the Toric Code model.

Let us just consider the case of a one-anyon state at the face x given by $|\mu_x\rangle$. We saw from the Toric Code model that we can hop the anyon by acting with a σ^1 at the end of the string, effectively extending the string. This will produce the hopping that we desire. Thus we want something of the form

$$\sum_{f\in\mathcal{F}} \sum_{e\sim f} \sigma^1_e. \tag{126}$$

However, note that such an operator could create anyon pairs out of $|\Omega\rangle$, as there is nothing restricting it to only act at the end point of a string. The characteristic of an anyon at the face x is that

$$B_x\,|\mu_x\rangle = -\,|\mu_x\rangle \quad ; \quad B_f\,|\mu_x\rangle = |\mu_x\rangle\,, f \neq x. \tag{127}$$

Thus the operator $(1 - A_v)$ would be 0 on $|\mu_x\rangle$ unless $v = x$. Thus the operator

$$\sum_{f\in\mathcal{F}} \sum_{e\sim f} \sigma^1_e (1 - B_f) \tag{128}$$

would certainly hop an anyon, and also conserve anyon number. The issue with this operator is that it is not Hermitian. We need the Hamiltonian to be Hermitian as we demand that time evolution is unitary, as well as physically the Hamiltonian represents the total energy which is always measured to be real. To see this, define

$$b_f := \sum_{e\sim f} \sigma^1_e. \tag{129}$$

Observe that $b_f B_f = -B_f b_f$, since each term in the sum of b_f anti-commutes with B_f. Thus

$$[b_f(1-B_f)]^* = (1-B_f)^* b_f^* = (1-B_f)b_f = b_f(1+B_f). \tag{130}$$

Thus each term in the sum over f is not Hermitian. To remedy this, let us observe that given any operator $\mathcal{O}$, the quantity

$$i[\mathcal{O}, \mathcal{O}^*] = i\left[\mathcal{O}\mathcal{O}^* - \mathcal{O}^*\mathcal{O}\right] \tag{131}$$

is Hermitian. We also note that the sum of Hermitian operators is also Hermitian. Thus if we modify our perturbation to

$$\frac{i}{2}\sum_{f\in\mathcal{F}}\left[(1-B_f)b_f - b_f(1-B_f)\right], \tag{132}$$

this operator is Hermitian. Note that this term simplifies to

$$i\sum_{f\in\mathcal{F}} b_f B_f. \tag{133}$$

Unfortunately, this term now has the problem that it creates anyons out of $|\Omega\rangle$, which is not what we want. The final correction is to redefine b_f with alternating signs as follows

$$b_f \to \sum_{e\sim f}\Theta(e)\sigma_e^1, \tag{134}$$

where $\Theta(e) = +1$ if e is above or to the right of f, and $\Theta(e) = -1$ if e is below or to the left of f. We still have $b_f B_f = -B_f b_f$, and thus

$$H_\mu = i\sum_{f\in\mathcal{F}} b_f B_f \tag{135}$$

is still Hermitian. Since both B_f and b_f are comprised of operators that live on edges, we can rewrite this sum as a sum over edges instead of faces as

$$H_\mu = i\sum_{e\in\mathcal{E}}\left[\sigma_e^1\sum_{f\sim e}\Theta(f)B_f\right], \tag{136}$$

where $\Theta(f) = +1$ if f is above or to the right e, and $\Theta(f) = -1$ if f is below or to the left of e. This will be our final form for the perturbation to induce dispersion. We can create an analogous perturbation that acts on electron anyons as follows

$$H_\epsilon = i\sum_{e\in\mathcal{E}}\left[\sigma_e^3\sum_{v\sim e}\Theta(v)A_v\right]. \tag{137}$$

7.2 Action on the Ground State

Let us first just verify that H_μ and H_ϵ do not couple $|\Omega\rangle$ to any other states. We will check this is true for H_μ, and the case for H_ϵ is nearly identical.

$$H_\mu |\Omega\rangle = i \sum_{e\in\mathcal{E}} \left[\sigma_e^1 \sum_{f\sim e} \Theta(f) B_f \right] |\Omega\rangle . \tag{138}$$

Note that by definition of $|\Omega\rangle$, $B_f |\Omega\rangle = |\Omega\rangle$ for all f, thus

$$H_\mu |\Omega\rangle = i \sum_{e\in\mathcal{E}} \left[\sigma_e^1 \sum_{f\sim e} \Theta(f) \right] |\Omega\rangle . \tag{139}$$

Now, $\Theta(f)$ is just a sign, that is ± 1. There are exactly one $+1$ and one -1 in the sum, and so

$$H_\mu |\Omega\rangle = i \sum_{e\in\mathcal{E}} \left[\sigma_e^1 (+1-1) \right] |\Omega\rangle = 0. \tag{140}$$

Therefore $H_\mu |\Omega\rangle = 0$. Similarly we have that $H_\epsilon |\Omega\rangle = 0$. This means that if we define our full Hamitonian as

$$H = H_0 + \lambda \left[H_\mu + H_\epsilon \right], \tag{141}$$

then we have $H |\Omega\rangle = 0$.

7.3 Action on one-anyon States

Let us now consider

$$H |\mu_x\rangle = \left[H_0 + \lambda \left(H_\mu + H_\epsilon \right) \right] |\mu_x\rangle . \tag{142}$$

Let us discuss each term independently. First, we know that $H_0 |\mu_x\rangle = 2 |\mu_x\rangle$ from the calcuation done in the Toric Code model. The reason it is not 4 is because one of the anyons was sent to infinity, and does not contribute. Next consider

$$H_\epsilon |\mu_x\rangle = i \sum_{e\in\mathcal{E}} \left[\sigma_e^3 \sum_{v\sim e} \Theta(v) A_v \right] S^1 |\Omega\rangle . \tag{143}$$

Note that since A_v and $S_x^1(\gamma)$ are both comprised of σ^1s, they commute. Then $A_v |\Omega\rangle = |\Omega\rangle$, thus

$$H_\epsilon |\mu_x\rangle = i \sum_{e\in\mathcal{E}} \left[\sigma_e^3 \sum_{v\sim e} \Theta(v) \right] S^1 |\Omega\rangle . \tag{144}$$

Similar to the case of the ground state, the $\Theta(v)$ term will have one $+1$ and one -1 terms in the sum, and will thus yield zero, implying $H_\epsilon |\mu_x\rangle = 0$. Let us

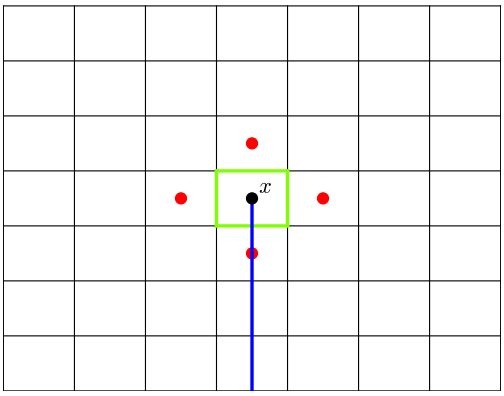

Figure 6: Illustration for what results from the calculation of $H_\mu |\mu_x\rangle$, which leads to Equation (147).

consider the action of H_μ

$$H_\mu |\mu_x\rangle = i \sum_{e \in \mathcal{E}} \left[\sigma_e^1 \sum_{f \sim e} \Theta(f) B_f \right] S_x^1(\gamma) |\Omega\rangle . \tag{145}$$

There are three cases: f is a face that does not contain the string nor the anyon, f contains the string, or $f = x$. If f contains neither a string nor an anyon, then it commutes with S_x^1, and the the $\Theta(f)$ term will kill the sum. If f contains the string, but not the anyon, then there will be exactly two edges in common between B_f and $S_x^1(\gamma)$, and the two edges will anti-commute meaning that $[B_f, S_x^1(\gamma)] = 0$, and so again the $\Theta(f)$ terms will yield 0 killing the sum. Thus all terms in the sum are 0 unless $f = x$, forcing $e \sim x$. Thus

$$H_\mu |\mu_x\rangle = i \sum_{e \sim x} \left[\sigma_e^1 \sum_{f \sim e} \Theta(f) B_f \right] S_x^1(\gamma) |\Omega\rangle . \tag{146}$$

What remains is shown in figure 6. The edges in green are the edges e that remain in the sum. each edge has exactly two faces f that satisfy $f \sim e$, one is the face x, and the other are shown in red. If f is one of the red faces, then B_f commutes with $S_x^1(\gamma)$ and returns 1 on $|\Omega\rangle$. However, if $f = x$, then B_f anti-commutes with $S_x^1(\gamma)$ picking up a minus sign. If we denote the edges that are in green by subscripts l, u, r, d if the edge is to the left,up,right, or down compared to x we have that

$$H_\mu |\mu_x\rangle = 2i \left[-\sigma_l^1 - \sigma_d^1 + \sigma_u^1 + \sigma_r^1 \right] S_x^1(\gamma) |\Omega\rangle . \tag{147}$$

The additional σ^1 simply extends the string (or reduces the string in the case of σ_d^1) yielding a new one-anyon state. This can be written compactly by introducing $\Theta(e;x)$ and $\Theta(e;\widetilde{x})$ by returning $+1$ if the edge is above or to the right of $x, \widetilde{x}$ and -1 otherwise, we conclude

$$H_\mu |\mu_x\rangle = 2i \sum_{e\sim x} \Theta(e;x) |\mu_e\rangle . \tag{148}$$

Thus, the full action of H on $|\mu_x\rangle$ is given by

$$H |\mu_x\rangle = 2 |\mu_x\rangle + 2i\lambda \sum_{e\sim x} \Theta(e;x) |\mu_e\rangle . \tag{149}$$

An essentially identical calculation shows

$$H |\epsilon_{\widetilde{x}}\rangle = 2 |\epsilon_{\widetilde{x}}\rangle + 2i\lambda \sum_{e\sim \widetilde{x}} \Theta(e;\widetilde{x}) |\epsilon_e\rangle . \tag{150}$$

We see that H takes one magnetic anyon states to other magnetic anyon states, and same for electric anyons. Thus we have that the spaces $\mathcal{H}^{(\epsilon)}$ and $\mathcal{H}^{(\mu)}$ are invarant subspaces under the action of H.

To calculate the dispersion relation, we do the same as for the Heisenberg model. We will, however, exploit translational symmetry immediately and define the state

$$|\psi\rangle = \sum_{x\in\mathcal{F}} e^{i\vec{k}\cdot\vec{x}} |m_x\rangle . \tag{151}$$

Then we will solve for E in the equation

$$H |\psi\rangle = E |\psi\rangle . \tag{152}$$

This yields

$$\sum_{x\in\mathcal{F}} e^{i\vec{k}\cdot\vec{x}} H |m_x\rangle = \sum_{x\in\mathcal{F}} E e^{i\vec{k}\cdot\vec{x}} |m_x\rangle . \tag{153}$$

We know how H acts on $|\mu_x\rangle$ from before, so

$$\sum_{x\in\mathcal{F}} 2e^{i\vec{k}\cdot\vec{x}} \left[|m_x\rangle + i\lambda \sum_{e\sim x} \Theta(e;x) |\mu_e\rangle \right] = \sum_{x\in\mathcal{F}} E e^{i\vec{k}\cdot\vec{x}} |\mu_x\rangle . \tag{154}$$

We rewrite the terms $|\mu_e\rangle$ as $|\mu_x\rangle$ by translating by unit vectors δ, yielding

$$\sum_{x\in\mathcal{F}} 2e^{i\vec{k}\cdot\vec{x}} \left[1 + i\lambda \sum_{\delta} \Theta(\delta) e^{i\vec{k}\cdot\vec{\delta}} \right] |\mu_x\rangle = \sum_{x\in\mathcal{F}} E e^{i\vec{k}\cdot\vec{x}} |\mu_x\rangle . \tag{155}$$

Since $\langle m_x | m_{x'} \rangle = \delta_{x,x'}$ we have that we can compare coefficients yielding

$$E = 2 \left[1 + i\lambda \sum_{\delta} e^{-i\vec{k}\cdot\vec{\delta}} \Theta(\delta) \right] . \tag{156}$$

Recalling that $e^{ix} - e^{-ix} = 2i \sin(x)$ we conclude

$$E = 2 + 4\lambda \left[\sin(k_1) + \sin(k_2) \right] . \tag{157}$$

7.4 Ribbons

A third anyon excitation exists in this model known as a ribbon. A ribbon is formed when a magnetic and an electric anyon fuse together, and thus travel together as a new particle. More mathematically, the dynamics of a ribbon would satisfy that if the position of the electric anyon is v and the position of the magnetic anyon is f, then $v \sim f$ for all time. We want to develop a perturbation that hops a ribbon, yet always satisfies $v \sim f$ after each hop. Let us define the two-anyon Hilbert space by

$$\mathcal{H}^{(\epsilon\mu)} = \overline{\{S^3_{\widetilde{x}}(\widetilde{\gamma})S^1_x(\gamma)\left|\Omega\right\rangle : \widetilde{\gamma} \subset \mathbb{Z}^2, \gamma \subset dual(\mathbb{Z}^2)\}}^{\|\cdot\|}. \tag{158}$$

We will see that our perturbation on ribbon states annihilates the ground state of the Toric Code model, and leaves $\mathcal{H}^{(\epsilon\mu)}$ so we work directly in the thermodynamic limit.

We induce dispersion in the ribbons by adding an additional term to our old perturbation, that enforces the condition $v \sim f$. This is achieved with the perturbation

$$H_{\epsilon\mu} = i\sum_{e\in\mathcal{E}}\left[\sigma^1_e\sum_{f\sim e}\Theta(f)B_f\sum_{v\sim e}(1-A_v) + \sigma^3_e\sum_{v\sim e}\Theta(v)A_v\sum_{f\sim e}(1-B_f)\right]. \tag{159}$$

Note that since $[B_f, A_v] = 0$ for all $f, v \in \mathbb{Z}^2$, we have that the sum's over f and v commute, and so they act independently. We also note that $\left[\sigma^1_e, \sum_{v\sim e}(1-A_v)\right] = \left[\sigma^3_e, \sum_{f\sim e}(1-B_f)\right] = 0$, and so the final sum in each term commutes with everything. We saw before that with out the final sum of each term, we have either H_ϵ or H_μ, which we saw in the previous section that these are Hermitian. Since $1-A_v$ and $1-B_f$ are Hermitian, and the product of two commuting Hermitian operators is Hermitian, we conclude the $H_{\epsilon\mu}$ is Hermitian. Explicitly for the first term

$$\begin{aligned}\left[i\sum_{e\in\mathcal{E}}\sigma^1_e\sum_{f\sim e}\Theta(f)B_f\sum_{v\sim e}(1-A_v)\right]^* &= (-i)\sum_{e\in\mathcal{E}}\sum_{v\sim e}(1-A_v)\sum_{f\sim e}\Theta(f)B_f\sigma^1_e \\ &= (-i)\sum_{e\in\mathcal{E}}\sum_{f\sim e}\Theta(f)B_f\sigma^1_e\sum_{v\sim e}(1-A_v) \\ &= i\sum_{e\in\mathcal{E}}\sigma^1_e\sum_{f\sim e}\Theta(f)B_f\sum_{v\sim e}(1-A_v), \end{aligned}\tag{160}$$

similarly for the other term, and thus $H_{\epsilon\mu}$ is Hermitian. Note that the final term in $H_{\epsilon\mu}$ kills $\left|\Omega\right\rangle$, since $A_v\left|\Omega\right\rangle = B_f\left|\Omega\right\rangle = \left|\Omega\right\rangle$ for all $v, f \in \mathbb{Z}^2$, so $H_{\epsilon\mu}\left|\Omega\right\rangle = 0$.

Let us define a two particle state by

$$\left|\epsilon_{\widetilde{x}}\mu_x\right\rangle = S^3_{\widetilde{x}}(\widetilde{\gamma})S^1_x(\gamma)\left|\Omega\right\rangle, \tag{161}$$

where $\widetilde{\gamma}$,γ are semi-infinite paths in $\mathbb{Z}^2$,$dual(\mathbb{Z}^2)$ terminated at $\widetilde{x}, x$ respectively. Let us then find the action of $H_{\epsilon\mu}$ on $|\epsilon_{\widetilde{x}}\mu_x\rangle$

$$H_{\epsilon\mu}\,|\epsilon_{\widetilde{x}}\mu_x\rangle = i\sum_{e\in\mathcal{E}}\left[\sigma_e^1\sum_{f\sim e}\Theta(f)B_f\sum_{v\sim e}(1-A_v)\right.$$
$$\left.+\,\sigma_e^3\sum_{v\sim e}\Theta(v)A_v\sum_{f\sim e}(1-B_f)\right]|\epsilon_{\widetilde{x}}\mu_x\rangle\,. \tag{162}$$

We have that $(1-B_f)\,|\epsilon_{\widetilde{x}}\mu_x\rangle = 0$ unless $e\sim x$, and $(1-A_v)\,|\epsilon_{\widetilde{x}}\mu_x\rangle = 0$ unless $e\sim\widetilde{x}$, and thus

$$H_{\epsilon\mu}\,|\epsilon_{\widetilde{x}}\mu_x\rangle = i\left[\sum_{e\sim\widetilde{x}}\sigma_e^1\sum_{f\sim e}\Theta(f)B_f\sum_{v\sim e}(1-A_v)\right.$$
$$\left.+\sum_{e\sim x}\sigma_e^3\sum_{v\sim e}\Theta(v)A_v\sum_{f\sim e}(1-B_f)\right]|\epsilon_{\widetilde{x}}\mu_x\rangle\,. \tag{163}$$

Since $e\sim\widetilde{x}$, exactly one of $v\sim e$ satisfies $v=\widetilde{x}$, and thus the last sum over v returns 2 on $|\epsilon_{\widetilde{x}}\mu_x\rangle$, similarly for the sum over f in the second sum. Thus

$$H_{\epsilon\mu}\,|\epsilon_{\widetilde{x}}\mu_x\rangle = i\left[\sum_{e\sim\widetilde{x}}\sigma_e^1\sum_{f\sim e}\Theta(f)B_f 2+\sum_{e\sim x}\sigma_e^3\sum_{v\sim e}\Theta(v)A_v 2\right]|\epsilon_{\widetilde{x}}\mu_x\rangle\,. \tag{164}$$

The alternating signs induced by Θ force that $f=x$ in the first sum, and $v=\widetilde{x}$ in the second sum, forcing $e\sim x$ and $e\sim\widetilde{x}$ for both sums. If it is the case that $x\not\sim\widetilde{x}$, then $H_{\epsilon\mu}\,|\epsilon_{\widetilde{x}}\mu_x\rangle = 0$[2], otherwise

$$H_{\epsilon\mu}\,|\epsilon_{\widetilde{x}}\mu_x\rangle = 2i\sum_{\substack{e\sim x\\ e\sim\widetilde{x}}}\left[\sigma_e^1\sum_{f\sim e}\Theta(f)B_f+\sigma_e^3\sum_{v\sim e}\Theta(v)A_v\right]|\epsilon_{\widetilde{x}}\mu_x\rangle\,. \tag{165}$$

When $f=x$ or $v=\widetilde{x}$, the sum over f and the sum over v yield ± 2. In particluar, if the edge is to the left or bottom of x we get -2 and if the edge is above or to the right of x we get $+2$, similarly for $\widetilde{x}$. Thus,

$$H_{\epsilon\mu}\,|\epsilon_{\widetilde{x}}\mu_x\rangle = 4i\sum_{\substack{e\sim x\\ e\sim\widetilde{x}}}\left[\Theta(e;\widetilde{x})\sigma_e^1+\Theta(e;x)\sigma_e^3\right]|\epsilon_{\widetilde{x}}\mu_x\rangle\,. \tag{166}$$

As an example, if we have the state shown in Figure 7, we would have

$$H_{\epsilon\mu}\,|\epsilon_{\widetilde{x}}\mu_x\rangle = 4i[-\sigma_{e_l}^1-\sigma_{e_d}^1+\sigma_{e_l}^3+\sigma_{e_d}^3]\,|\epsilon_{\widetilde{x}}\mu_x\rangle\,, \tag{167}$$

[2]This is the crucial property that forces ribbons to stay together.

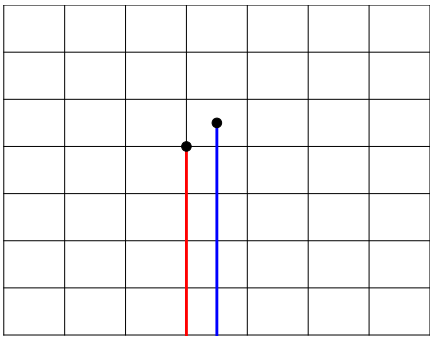

Figure 7: An example of a ribbon for the state $|\epsilon_{\widetilde{x}}\mu_x\rangle$ used in Equation (167).

where e_l is the edge to the left of the red anyon, and e_d is the edge below the red anyon.

To combine the $\sigma^{1,3}$'s into $|\epsilon_{\widetilde{x}}\mu_x\rangle$ to produce a new state depends on the orientation of the string. In order to proceed we introduce a convention for the string, which is equivalent to defining a basis to work in. We choose for all strings to go to infinitely straight downward, towards infinity in the minus x_2-direction. If the anyon hops horizontally, we must return this state to one of the basis states, by constructing an infinite loop. Such a loop is not a local operator, and thus is not in our algebra of observables. However, we can construct a sequence of finite loops, and take the limite as the loop size goes to infinity. Doing this will determine the sign that the state picks up, as creating such a loop may cross the string from the other anyon, yielding a minus sign.

7.5 Ribbon Dispersion

In this section, we wish to find the dispersion relation for $H_{\epsilon\mu}$. If we define

$$H = H_0 + \rho H_{\epsilon\mu}, \tag{168}$$

then the dispersion relation will be the dispersion relation for $H_{\epsilon\mu}$, just multiplied by ρ, and shifted by 4 from H_0. Thus anything interesting is due to $H_{\epsilon\mu}$ alone.

A general two particle state (one electric and one magnetic) is given by

$$S^3_{\widetilde{x}}(\widetilde{\gamma})S^1_x(\gamma)\,|\Omega\rangle\,. \tag{169}$$

We say such a state is a ribbon state if it is the case that $x \sim \widetilde{x}$. Such a state can be thought of as a particle living at a *site* defined by the arithmetic mean of x and $\widetilde{x}$. In Figure 8 we show the location of sites denoted by vectors δ_j stemming from a face denoted by x. Each site can be uniquely located by the vector $x + \delta_j$ where $x \in dual(\mathbb{Z}^2)$.

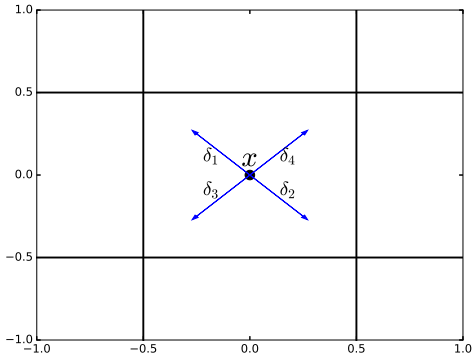

Figure 8: Location of ribbon sites given by δ_j in reference to a face x. By translational invariance, the same δ_js apply for all $x \in \mathcal{F}$.

Note that $H_{\epsilon\mu}$ is translationally invariant under translations in $\mathbb{Z}^2$, but since sites have a lattice spacing of $1/2$, all ribbon states are not equivalent under translations. We must consider a lattice with a basis, given by $x \in dual(\mathbb{Z}^2)$, with basis vectors δ_j. We thus denote a ribbon state by $|x;\delta_j\rangle$, $j \in \{1,2,3,4\}$.

Let us choose a basis in which $\gamma, \widetilde{\gamma}$ are semi-infinite vertical paths coming from $-\infty$ in the x_2 direction. From the last section, we have that the action on a generic two-particle state is given by

$$H_{\epsilon\mu}\,|\epsilon_{\widetilde{x}}\mu_x\rangle = 4i \sum_{\substack{e\sim x \\ e\sim\widetilde{x}}} \left[\Theta(e;\widetilde{x})\sigma_e^1 + \Theta(e;x)\sigma_e^3\right] |\epsilon_{\widetilde{x}}\mu_x\rangle\,. \tag{170}$$

This translates to the four following equations in terms of the action on $|x;\delta_j\rangle$

$$\begin{aligned}
H_{\epsilon\mu}\,|x,\delta_1\rangle &= 4i[-|x-\hat{e}_1;\delta_4\rangle - |x;\delta_4\rangle + |x+\hat{e}_2;\delta_3\rangle - |x;\delta_3\rangle] \\
H_{\epsilon\mu}\,|x,\delta_2\rangle &= 4i[|x;\delta_3\rangle - |x+\hat{e}_1;\delta_3\rangle + |x;\delta_4\rangle - |x-\hat{e}_2;\delta_3\rangle] \\
H_{\epsilon\mu}\,|x,\delta_3\rangle &= 4i[|x-\hat{e}_1;\delta_2\rangle - |x;\delta_2\rangle + |x;\delta_1\rangle - |x-\hat{e}_2;\delta_1\rangle] \\
H_{\epsilon\mu}\,|x,\delta_4\rangle &= 4i[|x;\delta_1\rangle + |x+\hat{e}_1;\delta_1\rangle + |x+\hat{e}_2;\delta_2\rangle - |x;\delta_2\rangle].
\end{aligned} \tag{171}$$

We note that when an anyon hops down or left a minus sign is picked up, but up or right no sign is picked up. The sign for vertical hopping comes just from the $\Theta(B_f, A_v)$ terms. For horizontal hopping, there is a minus sign from these terms as well, but there is an additional minus sign that comes from returning the state to the basis state defined by a semi-infinite straight path from $-\infty$ in the vertical direction. This extra minus sign is always seen when the electric anyon hops, since we defined the electric anyons to be above the magnetic anyons (the S^3 operator is applied after the S^1 operator on $|\Omega\rangle$). As for the horizontal hopping of the magnetic anyon, it only induces an additional minus sign when the magnetic anyon is below the electric anyon in the vertical direction. This is

because to place the state back in this position after hopping, we need to create a loop, which will pick up an additional minus sign in this case, and in the limit the loop height goes to ∞, we yield a net minus sign.

To find the dispersion relation, we consider the ansatz

$$|\psi\rangle = \sum_{x \in dual(\mathbb{Z}^2)} \sum_{j=1}^{4} A_j(\vec{k}) e^{i\vec{k}\cdot\vec{x}} |x, \delta_j\rangle \,. \tag{172}$$

We then want to solve for the coefficients A_j from the equation

$$H|\psi\rangle = E|\psi\rangle \tag{173}$$

analogously to the case of the Heisenberg model. Using Equations (171) we find that

$$\begin{aligned} \sum_{x \in dual(\mathbb{Z}^2)} 4ie^{i\vec{k}\cdot\vec{x}} \big\{ &\big[(1-e^{ik_2})A_3 + (1-e^{-ik_1})A_4\big] |x, \delta_1\rangle \\ &+ \big[-(1+e^{ik_1})A_3 - (1-e^{-ik_2})A_4\big] |x, \delta_2\rangle \\ &+ \big[-(1-e^{-ik_2})A_1 + (1+e^{-ik_1})A_2\big] |x, \delta_1\rangle \\ &+ \big[-(1-e^{ik_1})A_1 + (1-e^{ik_2})A_2\big] |x, \delta_1\rangle \big\} \\ &= \sum_{x \in dual(\mathbb{Z}^2)} \sum_{j=1}^{4} EA_j(\vec{k}) e^{i\vec{k}\cdot\vec{x}} |x, \delta_j\rangle \,. \end{aligned} \tag{174}$$

Since $\langle x', \delta_{j'}|x, \delta_j\rangle = \delta_{x,x'}\delta_{j,j'}$ we have the following matrix equation for the A_j's

$$4i \begin{pmatrix} 0 & 0 & (1-e^{ik_2}) & (1-e^{-ik_1}) \\ 0 & 0 & -(1+e^{ik_1}) & -(1-e^{-ik_2}) \\ -(1-e^{-ik_2}) & (1+e^{-ik_1}) & 0 & 0 \\ -(1-e^{ik_1}) & (1-e^{ik_2}) & 0 & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} = E \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}. \tag{175}$$

Diagonalizing this 4×4 matrix yields the dispersion relation

$$E = \pm\sqrt{4 - 2\cos(k_2) \pm \sqrt{10 + 2\cos(2k_1) - 8\cos(k_2)}}, \tag{176}$$

where all four combinations of the $\pm$ are the four eigenvalues. We illustrate these dispersion relations for a few paths of k in the region $[-\pi, \pi] \times [-\pi, \pi]$ in Figure 9.

7.6 Full Hamiltonian

We have shown two seperate perturbations to the Toric Code model. One that induced dispersion in the one anyon sector, and one that induced dispersion in the ribbons. We now want to study the case where we have both perturbations

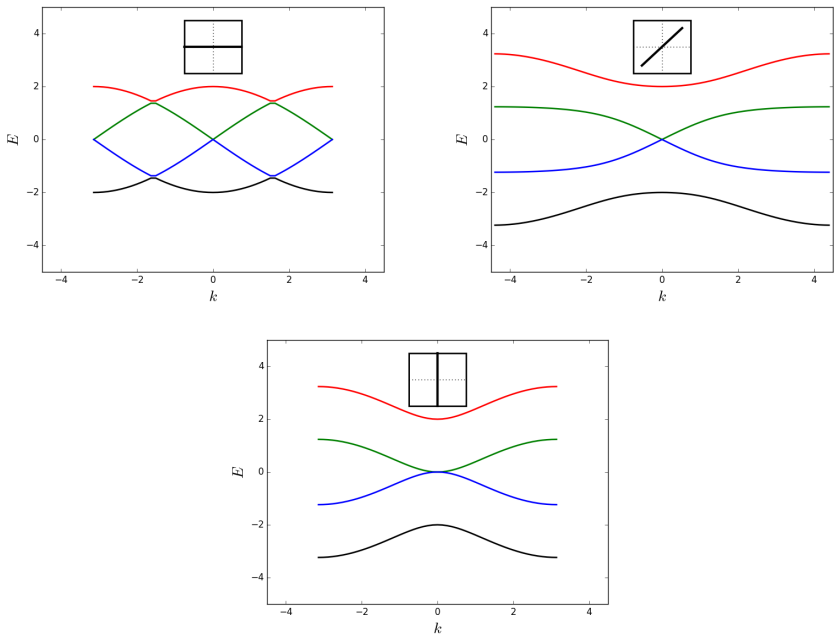

Figure 9: Plots of the dispersion relation of $H_{\epsilon\mu}$. The black box at the top of each figure represents the region $k \in [-\pi, \pi] \times [-\pi, \pi]$ and the black line represents the path taken in this region.

to the Toric Code model, and the full Dynamical Toric Code model Hamiltonian will be given by

$$H = H_0 + \lambda[H_\epsilon + H_\mu] + \rho H_{\epsilon\mu}, \tag{177}$$

where λ, ρ are real numbers which can tune the strength of the two perturbations. We call λ the coupling constant for free anyon hopping, and ρ the coupling constant for ribbon hopping. Let us first observe that on a one-particle state, $H_{\epsilon\mu}$ acts trivially. Let us consider $H_{\epsilon\mu}|\mu_x\rangle$. Note that $(1 - A_v)|\mu_x\rangle = 0$ annihilating the first term, and thus

$$H_{\epsilon\mu}|\mu_x\rangle = \sum_{e\in\mathcal{E}} \sigma_e^3 \sum_{v\sim e} \Theta(v) A_v \sum_{f\sim e} (1 - B_f)|\mu_x\rangle . \tag{178}$$

Unless $f = x$, $(1 - B_f)|\mu_x\rangle = 0$, forcing $e \sim x$ for a (potentially) non-zero term. If $f = x$ then $(1 - B_f)|\mu_x\rangle = 2|\mu_x\rangle$, thus

$$H_{\epsilon\mu}|\mu_x\rangle = \sum_{e\sim x} \sigma_e^3 \sum_{v\sim e} \Theta(v) A_v 2|\mu_x\rangle . \tag{179}$$

However, note that $A_v|\mu_x\rangle = 1$ for all $v \in \mathbb{Z}^2$. Since every edge has a vertex to the left and right, or above and below, and so the term

$$\sum_{v\sim e} \Theta(v) A_v \tag{180}$$

always takes the form of $(A_v - A'_v)$, which will annihilate $|\mu_x\rangle$. Thus $H_{\epsilon\mu}|\mu_x\rangle = 0$. A similar argument implies $H_{\epsilon\mu}|\epsilon_{\widetilde{x}}\rangle = 0$. Thus, we have that

$$H|\mu_x\rangle = 2|\mu_x\rangle + 2\lambda i \sum_{e\sim x} \Theta(e)|\mu_e\rangle , \tag{181}$$

and so adding $H_{\epsilon\mu}$ to the Hamiltonian does nothing to the one-anyon sector.

The story is different for the case of two-anyon excitations. Although $H_{\epsilon\mu}$ forces the anyons to move together as a ribbon, the terms H_ϵ and H_μ have no such restriction. If $\lambda = 0$, then we recover the ribbon Hamiltonian discussed previously, in which the anyons move together as a fused ribbon. The question is, if $\lambda > 0$ do fused ribbon states exist as eigenvectors of the Hamiltonian.

The question is quite tricky to answer explicitly, as it is not clear how to find the exact eigenpairs of the Hamiltonian in the two-anyon invariant subspace. Since the Hamiltonian permits interactions between the anyons, exploiting translational invariance alone is not sufficient to diagonalize H. To address this question, we turn to a numerical calculation.

7.7 Numerical Calculation

To encapsulate the thermodynamic limit, we first calculate the matrix elements in the infinite system, and then simulate these matrix elements on a finite lattice. For simplicity, we rescale the coupling constants in our Hamiltonian, so that all

matrix elements are ± 1, defining $\lambda' = \lambda/2$ and $\rho' = \rho/4$, and then we rescale our Hamiltonian to be

$$\begin{aligned} H &= H_0 + H' \\ H' &= \lambda' \left[H_\epsilon + H_\mu \right] + \rho' H_{\epsilon\mu}. \end{aligned} \tag{182}$$

We only simulate H', as H_0 has every vector in the subspace $\mathcal{H}^{(\epsilon\mu)}$ is an eigenvector with energy 4, and thus only shifts the spectrum by a constant.

Since the infinite system is translationally invariant, we want to build this into our numerical calculation. To to this, we consider a finite square lattice, where we identify the points on the left boundary with those on the right, and the points on the top boundary with those on the bottom, forming geometrically a torus. There is one caveat, we have that when the x_1 coordinates of x and $\widetilde{x}$ are neighbors, then matrix elements are different depending on whether the x_2 coordinate of x is greater than or less than $\widetilde{x}$. To remedy this, we consider both paths going up and going down from x to $\widetilde{x}$, including looping around the periodic boundary. If going up is faster, then we say $\widetilde{x}$ is above x. We note that this scheme also makes it equally likely for $\widetilde{x}$ to be above x as it is for it to be below, which is a property of the infinite system as well.

Now to address the question of if there are fused ribbon eigenvectors of the Hamiltonian. To do this, we construct the Hamiltonian for the finite system as described, and then numerically diagonalize H'. We then project the eigenvectors into the subspace of all ribbon states, for which the anyons are right next to each other. Then the norm of the projected eigenvector is a reflection of how much weight the eigenvector has in this subspace. In particular, if the norm is 1, then all the weight is in this subspace, and so we say the eigenvector is thus a ribbon state. We know that when $\lambda = 0$, that there is clearly ribbon states as seen from our analytical calculation.

We show in Figure 10 our results for the norms $\mathcal{N}_p$ in this projected subspace. For this calculation, we fixed $\rho' = 1$ and we varied λ' from 0 to 1 in increments of 0.01, and we performed our calculation on a 10×10 square lattice with periodic boundary conditions. We see when $\lambda' = 0$, we have many states with $\mathcal{N}_p = 1$ which corresponds to the ribbon states discussed previously. As we turn on λ', we see that these degenerate states fan out, and many of them have $\mathcal{N}_p$ drop dramatically, signifying these fused anyons are splitting. However, we do see that there exists states with $\mathcal{N}_p \cong 1$ in this entire range of λ'. We expect fused anyons can produce some non-zero propability of coupling to non-fused states, and so slight deviations from $\mathcal{N}_p$ are still in agreement with the existence of ribbon states. Due to quantum mechanical uncertainty, there will always be a propability of a ribbon state quantum tunneling into two free anyons, and we expect that this probability will grow proportionally with λ'. This is even the case with the electron and the proton in the Hydrogen atom. There is a finite, non-zero probability that the electron would be found arbitrarily far from the proton, yet we generally say they form a bound composite system. From our numerics, it is unclear where the cutoff between a fused ribbon, and two free anyons would be from $\mathcal{N}_p$ alone, but we believe this calculation strongly

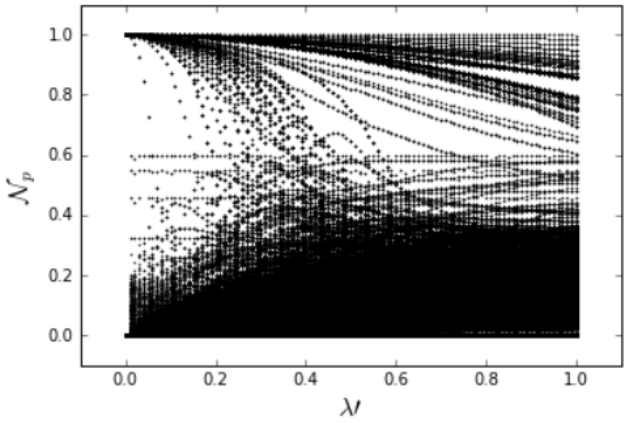

Figure 10: A plot of the norm of the eigenvectors projected into the ribbon subspace as a function of λ' in the Dynamical Toric Code model. This calculation is performed on a 10×10 square lattice with periodic boundary conditions, and we set $\rho' = 1$.

suggests that there exists fused eigenvectors of the Hamiltonian.

8 Conclusions and Future Work

In this paper, we show the development of studying infinite quantum spin systems, leading up to the GNS Hilbert space construction. We then apply the GNS Hilbert space construction to the Ising and Heisenberg model. We find that both the Ising and Heisenberg model harbor magnon quasi-particle excitations, but the Heisenberg model has a non-trivial dispersion relation for the magnons. We then discuss the anyons in the Toric Code model, and make analogy with magnons in the Ising model.

Inspired greatly by the Heisenberg model, we propose a perturbation $H_\epsilon + H_\mu$ to the Toric Code model that leaves the anyonic structure of the Toric Code invariant, and induces a non-trivial $\sin(k)$ dispersion relation in the anyons. We also propose a perturbation $H_{\epsilon\mu}$ that acts invariantly on the ribbons of the Toric Code model, and found an analytical expression for the dispersion relation of the ribbons. We then define the Dynamical Toric Code model as

$$\begin{aligned} H &= H_0 + H' \\ H' &= \lambda[H_\epsilon + H_\mu] + \rho H_{\epsilon\mu}. \end{aligned} \tag{183}$$

We show $H_{\epsilon\mu}$ acts trivially on single anyon states, and so the dispersion relation for single anyons is completely specifed by just $H_\epsilon + H_\mu$. For $\lambda = 0$, we showed that the anyons form fused ribbon states for the energy eigenvectors. When $\lambda > 0$, $H_\epsilon + H_\mu$ want to separate the ribbon into two free anyons, but it is not obvious if this transition is abrupt, or if fused ribbon states exits for non-zero

λ. To answer this we simulated H' numerically. We computed the norm of the eigenvectors projected onto the ribbon subspace, $\mathcal{N}_p$. We found eigenvectors with $\mathcal{N}_p$ very near one, with slight decay with increasing λ. This result provides evidence for the existence of fused ribbon eigenvectors for non-zero λ.

In the future, we hope to prove in a rigorous manner that the Dynamical Toric Code model harbors fused ribbon eigenvectors for non-zero λ. We hope that our numerical simulation can provide insight into the spectrum that will allow us to proceed analytically. We also hope to study scattering of anyons under this model. We believe that the existence of fused ribbon eigenvectors would lead to scattering of two free anyons producing an output state of the anyons fusing. If this were true, this would be the first known model that exhibits fusion of anyons after scattering.

References

[1] R. Alicki, M. Fannes, and M. Horodecki. A statistical mechanics view on kitaev's proposal for quantum memories. *Journal of Physics A: Mathematical and Theoretical*, 40(24):6451, 2007.

[2] S. Bravyi, M. B. Hastings, and S. Michalakis. Topological quantum order: Stability under local perturbations. *Journal of Mathematical Physics*, 51(9):093512, 2010.

[3] M. Cha, P. Naaijkens, and B. Nachtergaele. The complete set of infinite volume ground states for kitaev's abelian quantum double models. *Communications in Mathematical Physics*, 357(1):125–157, Jan 2018.

[4] A. Kitaev. Fault-tolerant quantum computation by anyons. *Annals of Physics*, 303(1):2 – 30, 2003.

[5] J. M. Leinaas and J. Myrheim. On the theory of identical particles. *Il Nuovo Cimento B (1971-1996)*, 37(1):1–23, Jan 1977.

[6] P. Naaijkens. Quantum spin systems on infinite lattices. *Lecture Notes in Physics 933, Springer International Publishing*, 2017.