RECURSIVE RELATIONS FOR THE HILBERT SERIES FOR CERTAIN QUADRATIC IDEALS

AUTHOR: YUZHE BAI
SUPERVISOR: DR. EUGENE GORSKY

Abstract. In this thesis, we will provide a basic recursive formula of the Hilbert Series of a quotient ring R_n/I_n where $R_n = \mathbb{C}[x_0, ..., x_{n-1}]$ and I_n is the quadratic defining ideal of the n^{th} jet scheme of a double point. Knowing the Hilbert Series of the quotient will give us a way to derive the Hilbert Series of the ideal. To achieve our goal, we will study the first syzygy module of the quotient ring and describe its generators. In fact, by learning the structure of the first syzygy module, we can derive a recursive formula for the minimal free resolution of the quotient ring as R_n module which will provide an explicit form for the Hilbert Series.

1. Introduction

Consider the ring

$$R_n = \mathbb{C}[x_0, x_1, x_2..., x_{n-1}]$$

and the ideal $I_n = (f_1, f_2..., f_n) \subset R_n$, where

$$f_n = \sum_{k=0}^{n-1} x_k x_{n-1-k}.$$

For example, when $n - 1$, we have $I_1 = (x_0^2)$, and when $n = 2$, we have $I_2 = (x_0^2, 2x_0x_1)$. It is easy to see that R_n is an infinite dimensional complex vector space where the basis consists of all monomials of $x_0, x_1..., x_{n-1}$.

Now, we define the bi-degree $\deg(m) = (a_1, a_2)$ to be the degree of a monomial m, where $\deg(x_i) = (i, 1)$ and $\deg(m_1m_2) = \deg(m_1) + \deg(m_2)$. Then obviously, R_n is a commutative bi-graded $\mathbb{C}$ algebra, and f_i are homogeneous with respect to this bi-grading.

Definition 1.1. For any field k and any bi-graded k vector space V, the Hilbert Series $H_V(q,t)$ of V over k is defined as

$$H_V(q,t) = \sum_{a_1,a_2} q^{a_1}t^{a_2} \dim_V(a_1, a_2),$$

where $\dim_V(a_1, a_2)$ is the dimension of the subspace of V that has bi-degree (a_1, a_2).

Now, let's consider the Hilbert series of R_n over $\mathbb{C}$

$$H(q,t)_n = \sum_{a_1,a_2}^{\infty} q^{a_1}t^{a_2} d_{R_n}(a_1, a_2)$$

where $d_{R_n}(a_1, a_2) = \dim_{R_n}(a_1, a_2)$ is the dimension of the subspace with bi-degree (a_1, a_2) in R_n; in other word, it is the number of monomials in R_n having bi-degree (a_1, a_2). For example, when $n = 1$, $R_1 = \mathbb{C}[x_0]$, the standard basis for R_1 as a complex vector space is $B_1 = \{1, x_0, x_0^2...\}$. Then $a_1(x_0^k) = 0$, $a_2(x_0^k) = k$, and that implies when $a_1 = 0$ and $a_2 \in \mathbb{Z}_{\geq 0}$, $d_{R_1}(a_1, a_2) = 1$, and when $a_1 \neq 0$, $d_{R_1}(a_1, a_2) = 0$. Thus, the Hilbert series of R_1 is

$$H(q,t)_1 = \sum_{a_1,a_2}^{\infty} q^{a_1}t^{a_2} d_{R_1}(a_1, a_2) = \sum_{a_2=0}^{\infty} t^{a_2} = \frac{1}{1-t}$$

.

Proposition 1.2. *The Hilbert series $H(q,t)_n$ of R_n is given by the equation*

$$H(q,t)_n = \prod_{k=0}^{n-1} \frac{1}{1-q^k t}.$$

Proof. We are going to prove this proposition by using mathematical induction. For the base case, as we stated earlier, when $n = 1$,

$$H(q,t)_1 = \frac{1}{1-t} = \prod_{k=0}^{0} \frac{1}{1-q^k t}.$$

Suppose for some $m \in \mathbb{Z}_{>0}$,

$$H(q,t)_m = \prod_{k=0}^{m-1} \frac{1}{1-q^k t},$$

Let's consider $n = m+1$. Let B_m be the standard basis of R_m, then the standard basis of R_{m+1} is $B_{m+1} = \bigcup_{k=0}^{\infty} x_m^k B_m$. Moreover, we know $\deg(x_m^k) = (km, k)$. Since $\deg(m_1 m_2) = \deg(m_1) + \deg(m_2)$, we can convert the Hilbert series $H(q,t)_{m+1}$ of R_{m+1} as following

$$\begin{aligned}
H(q,t)_{m+1} &= \sum_{a_1,a_2}^{\infty} q^{a_1} t^{a_2} d_{R_{m+1}}(a_1,a_2) \\
&= \sum_{j=0}^{\infty} q^{jm} t^j \sum_{a_1,a_2}^{\infty} q^{a_1} t^{a_2} d_{R_m}(a_1,a_2) \\
&= \sum_{j=0}^{\infty} q^{jm} t^j H(q,t)_m \\
&= \frac{1}{1-q^m t} H(q,t)_m \\
&= \prod_{k=0}^{m} \frac{1}{1-q^k t}.
\end{aligned}$$

Therefore, by mathematical induction, for $\forall n \in \mathbb{Z}_{>0}$, the Hilbert series $H(q,t)_n$ of R_n is

$$H(q,t)_n = \prod_{k=0}^{n-1} \frac{1}{1-q^k t}.$$

□

Remark 1.3. For any commutative bi-graded $\mathbb{C}$ algebra R and any bi-graded R module M, denote the Hilbert series of M as $H(M)$. We know that for any bi-graded submodule $S \subseteq M$, the Hilbert Series over $\mathbb{C}$ satisfy

$$H(M) = H(S) + H(M/S).$$

Thus, instead of calculate the Hilbert Series of I_n directly, we can first calculate $H(R_n/I_n)$ and subtract it from $H(R_n)$ which is known. Because of the relation, we can calculate the Hilbert Series of I_n over $\mathbb{C}$ by subtracting the Hilbert Series of R_n/I_n from the Hilbert Series of R_n. Since we already have the formula for $H(R_n)$ for $\forall n \in \mathbb{N}$, once we know $H(R_n/I_n)$, we can easily deduce $H(I_n)$.

In this sense, the main goal of this thesis is to prove the following theorem.

Main Theorem. Let $H_n(q,t) = H(R_n/I_n)$, then when $n \geq 4$, it can be derived by the following recursion:

$$H_n(q,t) = \frac{tH_{n-3}(q,q^2t) + H_{n-2}(q,qt)}{1-q^{n-1}t}.$$

When n is approching to ∞, an explicit formula for the Hilbert Series of R_n/I_n is proven by Bruschek, Mourtada and Schepers[1], which relates the Hilbert series of the arc space for the double point to the Rogers-Ramanujan identity. A similar result for $n = \infty$ was obtained by Feigin-Stoyanovsky [2, 3], Lepowsky et al. [4, 5], and Gorsky, Oblomkov and Rasmussen in [6].

Moreover, the result of this thesis, together with the explicit formula for the Hilbert Series of R_n/I_n can be found in [7].

2. Build up for recursive relation

Basis of R_n/I_n for small n

In order to calculate the Hilbert Series of R_n/I_n, we need to know the basis of R_n/I_n over $\mathbb{C}$. Now, let's consider the basis B_n of R_n/I_n for $n = 1, 2, 3$.

When $n = 1$, $R_1 = \mathbb{C}[x_0]$ and $I_1 = (x_0^2)$, then $B_1 = \{1, x_0\}$;

When $n = 2$, $R_2 = \mathbb{C}[x_0, x_1]$ and $I_2 = (x_0^2, 2x_0x_1)$, then $B_2 = \{1, x_0, x_1^k | k \in \mathbb{N}\}$;

When $n = 3$, $R_3 = \mathbb{C}[x_0, x_1, x_2]$ and $I_3 = (x_0^2, 2x_0x_1, 2x_0x_2 + x_1^2)$, we can see that by choosing the reversed degree lexicographic order as the monomial order, $\{x_0^2, 2x_0x_1, 2x_0x_2 + x_1^2\}$ satisfies Buchberger's Criterion and is indeed a reduced Groebner basis for I_3.

Therefore, $B_3 = \{x_2^i, x_0x_2^j, x_1x_2^k | i, j, k \in \mathbb{Z}_{\geq 0}\}$ which consists of all monomials which are not divisible by any of the leading terms of the elements in the Groebner basis.

For bigger n, we will get the basis from a recursive relation, but before we do that, we need some tools.

Definition 2.1. A **shift** of a polynomial $f(x_1, ..., x_n) \in R_n$ is $f(x_2, ..., x_{n+1})$, which is in R_m for all $m > n$, we denote it as $S(f)$. Moreover, for any subset $M \subseteq R_n$ or quotient of R_n, we denote the **shift** of M as $S(M) = \{S(f) : f \in M\}$. If we consider S as a linear function from R_n to R_m where $m \geq n$, then S is an injection as its kernel is trivial.

Lemma 2.2. *For any $\alpha_1, ..., \alpha_n \in R_n$, $\sum_{i=1}^n S(\alpha_i)S(f_i) = 0$ if and only if $\sum_{i=1}^n \alpha_i f_i = 0$.*

Proof. Since $\sum_{i=1}^n S(\alpha_i)S(f_i) = S(\sum_{i=1}^n \alpha_i f_i)$. Because S is an injection, then $S(g) = 0$ if and only if $g = 0$. Thus, $\sum_{i=1}^n S(\alpha_i)S(f_i) = 0$ if and only if $\sum_{i=1}^n \alpha_i f_i = 0$. ☐

Recursive relation between the bases of R_n/I_n

Now, we claim that for $n > 3$, $B_n = \bigcup_{k=0}^{\infty} x_{n-1}^k [x_0 S^2(B_{n-3}) \bigcup S(B_{n-2})]$.

The way we consider this is to construct two subsets of R_n/I_n, which we name as Q_n and P_n, where Q_n consists of all elements in R_n/I_n that are divisible by x_0 and P_n consists of all elements in R_n/I_n which contain no term divisible by x_0.

We can see that Q_n is an ideal in R_n/I_n and P_n is a subring of R_n/I_n which is isomorphic to $[R_n/I_n]/Q_n$. Consequently, we have $H(R_n/I_n) = H(Q_n) + H(P_n)$. In fact, Q_n can be considered as $x_0R_n/[I_n \bigcap (x_0)]$ and P_n can be considered as $R_n/[I_n + (x_0)]$. Now, we need to study their structure respectively. First, let's consider the quotient ring $R_n/[I_n + (x_0)]$.

Lemma 2.3. *For the subring P_n of R_n/I_n which consists of elements having no term divisible by x_0, we have*

$$P_n = \frac{R_n}{I_n + (x_0)} = S\left(\frac{R_{n-2}}{I_{n-2}}\right)[x_{n-1}].$$

Proof. By the Third Isomorphism Theorem, it's isomorphic to

$$\frac{R_n/(x_0)}{[I_n + (x_0)]/(x_0)},$$

where $R_n/(x_0) = S(R_{n-2})[x_{n-1}]$. Since $I_n + (x_0)$ is generated by $f_1, f_2, ..., f_n, x_0$, by the natural map, the quotient $[I_n + (x_0)]/(x_0)$ is generated by the residue of the generator of $I_n + (x_0)$, where the residue of f_1, f_2 is $\overline{0}$ since they contain x_0, and residue of f_n is $\overline{S(f_{n-2})}$ for $n \geq 3$ as $f_n = S(f_{n-2}) + 2x_0x_{n-1}$. As a result,

$$\frac{R_n}{I_n + (x_0)} = S\left(\frac{R_{n-2}}{I_{n-2}}\right)[x_{n-1}].$$

Thus, the basis of P_n is $\bigcup_{k=0}^{\infty} x_{n-1}^k S(B_{n-2})$. ☐

In order to study Q_n, we first need to know the structure of $I_n \bigcap (x_0)$, which requires us to study the module of first syzygy of $f_1, ..., f_n$.

3. Study of the first syzygy

Definition 3.1. Let R be a commutative ring and M be an R-module generated by $m_1, m_2, ...m_n$, where $n < \infty$. Let F be an rank n free R-module with basis $B = \{e_1, ..., e_n\}$ and there is a R-module homomorphism $\phi : F \to M$ such that $\phi(e_i) = m_i$ for $i = 1, ..., n$. The **first syzygy module** of $m_1, ...m_n$ is $\ker \phi$ and an element of $\ker \phi$ is a **syzygy** of $m_1, ..., m_n$. Moreover, since F is a free module, every element in F can be written as a unique R combination of $e_1, ..., e_n$. As a result, we can write the element in $\ker \phi$ in n-tuple form.

In our case, I_n is an R_n-module and $\{f_1, ..., f_n\}$ is a generating set of I_n. It is important for us to learn the first syzygy module of $f_1, ..., f_n$ as it will help us understand the structure of $I_n \bigcap (x_0)$.

Lemma 3.2. *For $f_1, f_2, ...f_n$, the following relation holds*

$$\sum_{i=0}^{n-1} (n-1-3i)x_i f_{n-i} = 0.$$

Proof. It is easy to see that all terms in $\sum_{i=0}^{n-1}(n-1-3i)x_i f_{n-i} = 0$ are scalar multiples of monomials $x_a x_b x_c$ such that $a+b+c = n-1$ since f_n is the summation of monomials $x_a x_b$ such that $a+b = n-1$.

If $a \neq b \neq c$, then the coefficient of $x_a x_b x_c$ is

$$\begin{aligned} 2(n-1-3a) + 2(n-1-3b) + (n-1-3c) &= 6(n-1) - 6a - 6b - 6c \\ &= 6(n-1) - 6(a+b+c) \\ &= 6(n-1) - 6(n-1) \\ &= 0; \end{aligned}$$

If $a = b \neq c$, then the coefficient of $x_a x_b x_c$ is

$$\begin{aligned} 2(n-1-3a) + (n-1-3c) &= 3(n-1) - (6a+3c) \\ &= 3(n-1) - 3(a+b+c) \\ &= 3(n-1) - 3(n-1) \\ &= 0; \end{aligned}$$

If $a = b = c$, then $3a = n-1$ and $a = b = c = (n-1-a)/2$. Thus, the coefficient of $x_a x_b x_c$ is

$$\begin{aligned} n-1-3a &= n-1-\frac{3(n-1-a)}{2} \\ &= \frac{3a-(n-1)}{2} \\ &= 0. \end{aligned}$$

As a result, $\sum_{i=0}^{n-1}(n-1-3i)x_i f_{n-i} = 0$. ∎

From Lemma 1, we can recognize some syzygies of $f_1, f_2, ..., f_n$.

Proposition 3.3. *Let*

$$\mu_i = (-2ix_i, (-2i+3)x_{i-1}, ..., ix_0, 0, ..., 0),$$
$$\nu_{j,k} = -f_k e_j + f_j e_k.$$

Then μ_i and $\nu_{j,k}$ are syzygies of $f_1, ..., f_n$ for $i \neq n-1$ and $j, k \neq n$ such that $j < k$.

Proof. For μ_i, it directly follows from Lemma 3.2. For $\nu_{j,k}$, we realize that $f_j f_k - f_k f_j = 0$ when $j > k$, the preimage of this relation is $\nu_{j,k}$. As a result, μ_i and $\nu_{j,k}$ are in the first syzygy module of $f_1, ..., f_n$. ∎

In fact, they are the generators of the first syzygy module of $f_1, .., f_n$ and we will prove this later. If we have already had the conclusion in hand, we will have enough information to discover the structure of $I_n \bigcap (x_0)$, which gives us an explicit form of Q_n.

Proposition 3.4. *When $n \geq 3$,*

$$\sum_{i=1}^{n-1}(n-3i)x_i f_{n+1-i} = -nx_0 S(f_{n-1}) = -n[x_0 x_1 x_{n-1} + x_0 S^2(f_{n-3})],$$

where the second equation holds when $n \geq 4$.

Proof. Similar to Lemma 1, we can see that each term of the sum is a scalar multiple of monomial $x_a x_b x_c$ where $a+b+c = n$. When $a, b, c \neq 0$, if $a \neq b \neq c$, then the coefficient of $x_a x_b x_c$ is

$$\begin{aligned} 2(n-3a) + 2(n-3b) + 2(n-3c) &= 6n - 6(a+b+c) \\ &= 0; \end{aligned}$$

if $a = b \neq c$, then the coefficient of $x_a x_b x_c$ is

$$\begin{aligned} 2(n-3a)+(n-3c) &= 3n-6a-3c \\ &= 3n-3(a+b+c) \\ &= 0; \end{aligned}$$

if $a = b = c$, then $a = b = c = n/3$. The coefficient of $x_a x_b x_c$ is

$$n - 3a = 0.$$

Thus, if $a, b, c \neq 0$, then the coefficient of $x_a x_b x_c$ is 0. As a result, $x_0 | \sum_{i=1}^{n-1} (n-3i) x_i f_{n+1-i}$.

Now, let's consider the coefficient of $x_0 x_a x_b$. From the observation above, we have $a + b = n$, if $a \neq b$, then the coefficient of $x_0 x_a x_b$ is

$$\begin{aligned} 2(n-3a)+(2n-3b) &= 4n-6(a+b) \\ &= 4n-6n \\ &= -2n; \end{aligned}$$

if $a = b$, then the coefficient of $x_0 x_a x_b$ is

$$\begin{aligned} 2(n-3a) &= 2n-6a \\ &= 2n-3(a+b) \\ &= 2n-3n \\ &= -n. \end{aligned}$$

Therefore, we can rewrite $\sum_{i=1}^{n-1} (n-3i) x_i f_{n+1-i}$ as

$$-nx_0 \sum_{i=1}^{n-1} x_i x_{n-i} = -nx_0 \sum_{i=0}^{n-2} x_{i+1} x_{(n-2-i)+1} = -nx_0 S(f_{n-1}).$$

Moreover, because $f_i = 2x_0 x_{i-1} + S(f_{i-2})$ when $i \geq 3$, then $S(f_i) = 2x_1 x_i + S^2(f_{i-2})$ when $i \geq 3$. Hence, when $n \geq 4$,

$$-nx_0 S(f_{n-1}) = -nx_0(2x_1 x_{n-1} + S^2(f_{n-3})) = -n[x_0 x_1 x_{n-1} + x_0 S^2(f_{n-3})].$$

□

By Proposition 3.4, we notice that $x_0 S^2(f_{n-3}) \in I_n$ when $n \geq 4$, which implies that

$$I_n \bigcap (x_0) \supseteq (x_0 S^2(I_{n-3}) + (f_1, f_2))[x_{n-1}]$$

.

4. Main theorems

Theorem 4.1. *If the first syzygy module of $f_1, ..., f_n$ is generated by μ_i and $\nu_{j,k}$ where $i \leq n-1$ and $j, k \leq n$ such that $j \neq k$,then for $n \geq 4$,*

$$\frac{x_0 R_n}{I_n \bigcap (x_0)} = x_0 S^2 (\frac{R_{n-3}}{I_{n-3}})[x_{n-1}].$$

Proof. In order to prove Theorem 4.1, we first of all need to show that

$$I_n \bigcap (x_0) = (x_0 S^2(I_{n-3}) + (f_1, f_2))[x_{n-1}]$$

when the first syzygy module is generated by μ and ν. One direction is already derived by Proposition 2, we just need to show the other direction.

For all $g \in I_n$, there exists $\alpha_i \in R_n$ such that $g = \sum_{i=1}^{n} \alpha_i f_i$. For each α_i, we can separate it into two parts, one is divisible by x_0 and the other contains no term divisible by x_0:

$$\alpha_i = x_0 \alpha_i' + S(\beta_i).$$

Do the similar thing to f_i, we get

$$\begin{aligned} f_1 &= x_0^2 \\ f_2 &= 2x_0 x_1 \\ f_i &= 2x_0 x_{i-1} + S(f_{i-2}) \qquad \text{for} \quad i \geq 3 \end{aligned}$$

Thus, we can rewrite g into the following form

$$\begin{aligned}g =& x_0^2(x_0\alpha_1' + S(\beta_1)) + 2x_0x_1(x_0\alpha_2' + S(\beta_2)) + \sum_{i=3}^{n} 2x_0^2x_{i-1}\alpha_i' + \sum_{i=3}^{n} x_0\alpha_i' S(f_{i-2}) \\ &+ \sum_{i=3}^{n} 2x_0x_{i-1}S(\beta_i) + \sum_{i=3}^{n} S(\beta_i)S(f_{i-2}) \\ =& f_1(x_0\alpha_1' + S(\beta_1)) + f_2(x_0\alpha_2' + S(\beta_2)) + \sum_{i=3}^{n} 2f_1x_{i-1}\alpha_i' + \frac{1}{2}\alpha_3' x_1 f_2 \\ &+ \alpha_4' x_2 f_2 + \sum_{i=5}^{n} x_0\alpha_i'(S^2(f_{i-4}) + 2x_1x_{i-2}) + \sum_{i=3}^{n} 2x_0x_{i-1}S(\beta_i) + \sum_{i=3}^{n} S(\beta_i)S(f_{i-2}) \\ =& f_1(x_0\alpha_1' + S(\beta_1)) + f_2(x_0\alpha_2' + S(\beta_2)) + \sum_{i=3}^{n} 2f_1x_{i-1}\alpha_i' + \frac{1}{2}\alpha_3' x_1 f_2 \\ &+ \alpha_4' x_2 f_2 + \sum_{i=5}^{n} [x_{i-2}\alpha_i' f_2 + \alpha_i' x_0 S^2(f_{i-4})] + \sum_{i=3}^{n} 2x_0x_{i-1}S(\beta_i) + \sum_{i=3}^{n} S(\beta_i)S(f_{i-2})\end{aligned}$$

The only thing that may not in $(x_0S^2(I_{n-3})+(f_1,f_2))[x_{n-1}]$ is $\sum_{i=3}^{n} 2x_0x_{i-1}S(\beta_i)+\sum_{i=3}^{n} S(\beta_i)S(f_{i-2})$ and it's totally depends on β_i.

If $g \in I_n \bigcap x_0R_n$, we know $\sum_{i=3}^{n} S(\beta_i)S(f_{i-2}) = \sum_{i=1}^{n-2} S(\beta_{i+2})S(f_i) = 0$, which implies $\sum_{i=1}^{n-2} \beta_{i+2}f_i = 0$. By assumptions, $(\beta_3, ..., \beta_n)$ is a combination of μ_i and $\nu_{j,k}$ where $i \le n-3$, and $j < k \le n-2$.

For $\mu_i = (-2ix_i, (-2i+3)x_{i-1}, ..., ix_0, 0, ..., 0)$, we have $S(\mu_i) = (0, 0, -2ix_{i+1}, (-2i+3)x_i, ..., ix_1, 0, ..., 0)$, and the image of $S(\mu_i)$ onto R_n is

$$\begin{aligned}\sum_{j=0}^{i}(i-3j)x_{j+1}f_{i-j+3} &= \sum_{j=0}^{i}(i-3j)x_{j+1}(S(f_{i-j+1}) - 2x_0x_{i-j+2}) \\ &= \sum_{j=0}^{i}(i-3j)S(x_j)S(f_{i-j+1}) - \sum_{j=0}^{i}(i-3j)2x_0x_{j+1}x_{i-j+2} \\ &= -2x_0\sum_{j=0}^{i}(i-3j)x_{j+1}x_{i-j+2} \\ &= (3-i)x_0S(f_{i+1}) - 6x_0x_1x_{i+2} \\ &= 2(3-i)x_0x_1x_{i+2} + (3-i)x_0S^2(f_{i-1}) - 6x_0x_1x_{i+2} \\ &= (3-i)x_0S^2(f_{i-1}) - 2if_2x_{i+2}\end{aligned}$$

Since $i \le n-3$, then the image is inside $(x_0S^2(I_{n-3}) + (f_1, f_2))[x_{n-1}]$.

For $\nu_{j,k} = -f_ke_j + f_je_k$, we have $S(\nu_{j,k}) = -S(f_k)e_{j+2} + S(f_j)e_{k+2}$, and the image of $S(\nu_{j,l})$ onto R_n is

$$\begin{aligned}-S(f_k)f_{j+2} + S(f_j)f_{k+2} &= -S(f_k)(S(f_j) + 2x_0x_{j-1}) + S(f_j)(S(f_k) + 2x_0x_{k-1}) \\ &= -S(f_k)S(f_j) + S(f_j)S(f_k) - 2S(f_k)x_0x_{j-1} + 2S(f_j)x_0x_{k-1} \\ &= 2(S^2(f_{j-2}) + 2x_1x_j)x_0x_{k-1} - 2(S^2(f_{k-2}) + 2x_1x_k)x_0x_{j-1} \\ &= 2x_0(S^2(f_{j-2})x_{k-1} - S^2(f_{k-2})x_{j-1}) + 4f_2(x_jx_{k-1} - x_kx_{j-1})\end{aligned}$$

Since $j < k \le n-2$, the image is in $(x_0S^2(I_{n-3}) + (f_1, f_2))[x_{n-1}]$.

Therefore, by our assumption on the generators of the first syzygy module, we have for all $g \in I_n \bigcap x_0R_n$, $g \in (x_0S^2(I_{n-3}) + (f_1, f_2))[x_{n-1}]$.

As a result, we have

$$\frac{x_0R_n}{I_n \bigcap (x_0)} = \frac{x_0R_n}{(x_0S^2(I_{n-3}) + (f_1, f_2))[x_{n-1}]}.$$

Since $R_n = \mathbb{C}[x_0, x_1, ..., x_{n-1}]$, we can consider the ideal x_0R_n as $\mathbb{C}[x_0^2, x_0x_1, ..., x_0x_{n-1}]$. Then

$$\begin{aligned}\frac{x_0R_n}{(x_0S^2(I_{n-3}) + (f_1, f_2))[x_{n-1}]} &= \frac{\mathbb{C}[x_0^2, x_0x_1, ..., x_0x_{n-1}]}{(x_0S^2(I_{n-3}) + (x_0^2, 2x_0x_1))[x_{n-1}]} \\ &= \frac{\mathbb{C}[x_0x_2, x_0x_{,3}, ..., x_0x_{n-1}]}{x_0S^2(I_{n-3})[x_{n-1}]} \\ &= \frac{x_0\mathbb{C}[S^2(x_0), S^2(x_1), ..., S^2(x_{n-4})][x_{n-1}]}{x_0S^2(I_{n-3})[x_{n-1}]} \\ &= \frac{x_0S^2(R_{n-3})[x_{n-1}]}{x_0S^2(I_{n-3})[x_{n-1}]} \\ &= x_0S^2\left(\frac{R_{n-3}}{I_{n-3}}\right)[x_{n-1}]\end{aligned}$$

□

Now, since we already have the recursive form for the ideal Q_n of R_n/I_n which consists of all elements divisible by x_0 and the quotient $P_n = (R_n/I_n)/Q_n$, we are ready to calculate the recursive relation of the Hilbert Series of R_n/I_n. However, before we start the calculation, we still need to know how the shift map affects the Hilbert Series.

Lemma 4.2. *For any bi-graded M where M consists of polynomials of $x_1, x_2, ..., x_n$, denote the Hilbert Series of M as $H_M(q,t)$. Then the Hilbert Series of $S(M)$, which is denoted as $H_{S(M)}(q,t)$, satisfies $H_{S(M)}(q,t) = H_M(q,qt)$.*

Proof. By definition, we know that

$$H_M(q,t) = \sum_{a_1,a_2} q^{a_1}t^{a_2}\dim_M(a_1,a_2),$$

then we have

$$H_{S(M)}(q,t) = \sum_{a_1,a_2} q^{a_1}t^{a_2}\dim_{S(M)}(a_1,a_2).$$

Since $S(x_i) = x_{i+1}$, which shift the first degree for all x_i. Then because $\deg(m_1m_2) = \deg(m_1) + \deg(m_2)$ for all monomials m_1 and m_2, $a_1[S(m)] = a_1(m) + a_2(m)$ and $a_2[S(m)] = a_2(m)$ for any monomial m as a_2 denotes the total degree of a monomial and that won't be changed by shifting.

Hence, $\dim_M(a_1,a_2) = \dim_{S(M)}(a_1+a_2,a_2)$ for all $a_1, a_2 \in \mathbb{Z}_{\geq 0}$, and we have

$$\begin{aligned}H_{S(M)}(q,t) &= \sum_{a_1,a_2} q^{a_1}t^{a_2}\dim_{S(M)}(a_1,a_2) \\ &= \sum_{a_1,a_2} q^{a_1+a_2}t^{a_2}\dim_{S(M)}(a_1+a_2,a_2) \\ &= \sum_{a_1,a_2} q^{a_1}(qt)^{a_2}\dim_M(a_1,a_2) \\ &= H_M(q,qt)\end{aligned}$$

□

Proposition 4.3. *For all $n \in \mathbb{N}$, $H_{S^n(M)}(q,t) = H_M(q,q^nt)$.*

We can derive this result by repeating the process n times. Notice that $H_{S^2(M)}(q,t) = H_M(q,q^2t)$.

Now, we have enough tools to prove our main theorem.

Theorem 4.4. *Denote $H_n(q,t)$ to be the Hilbert Series of R_n/I_n, then for $n \geq 4$, it follows the following recursive relation:*

$$H_n(q,t) = \frac{tH_{n-3}(q,q^2t) + H_{n-2}(q,qt)}{1 - q^{n-1}t}.$$

Proof. Notice that $x_0R_n/[I_n \bigcap (x_0)]$ is an ideal of R_n/I_n and their quotient yields

$$\frac{R_n}{I_n}\Big/\frac{x_0R_n}{I_n \bigcap (x_0)} = \frac{R_n}{I_n}/(x_0) = \frac{R_n}{I_n + (x_0)}.$$

Then, because $H(M) = H(M/S) + H(S)$ for all bi-graded $\mathbb{C}$ space M and bi-graded subspace $S \subseteq M$. we have

$$H_n(q,t) = H\bigg(\frac{x_0R_n}{I_n \bigcap (x_0)}\bigg) + H\bigg(\frac{R_n}{I_n + (x_0)}\bigg).$$

Because we know that

$$\frac{x_0R_n}{I_n \bigcap (x_0)} = x_0S^2\bigg(\frac{R_{n-3}}{I_{n-3}}\bigg)[x_{n-1}] \qquad \text{and} \qquad \frac{R_n}{I_n + (x_0)} = S\bigg(\frac{R_{n-2}}{I_{n-2}}\bigg)[x_{n-1}],$$

we can derive

$$H\bigg(\frac{x_0R_n}{I_n \bigcap (x_0)}\bigg) = t\sum_{k=0}^{\infty} H_{n-3}(q,q^2t)(q^{n-1}t)^k = \frac{tH_{n-3}(q,q^2t)}{1-q^{n-1}t},$$

as well as

$$H\bigg(\frac{R_n}{I_n + (x_0)}\bigg) = \sum_{k=0}^{\infty} H_{n-2}(q,qt)(q^{n-1}t)^k = \frac{H_{n-2}(q,qt)}{1-q^{n-1}t}.$$

As a result, we will have the recursive formula for $H_n(q,t)$ as

$$H_n(q,t) = \frac{tH_{n-3}(q,q^2t) + H_{n-2}(q,qt)}{1-q^{n-1}t}.$$

□

By Theorem 4.4, we will have

$$H(I_n) = \prod_{k=0}^{m} \frac{1}{1-q^kt} - \frac{tH_{n-3}(q,q^2t) + H_{n-2}(q,qt)}{1-q^{n-1}t}.$$

For the base cases of $H_n(q,t)$, we have:
$H_0(q,t) = 0$;
$H_1(q,t) = 1 + t$ as R_1/I_1 has basis $\{1, x_0\}$;
$H_2(q,t) = t + 1/(1-qt)$ as R_2/I_2 has basis $\{x_0, x_1^k | k \in \mathbb{Z}_{\geq 0}\}$.

Although from Theorem 4.1, we can derive the recursive formula of Hilbert Series in Theorem 4.4, the proof of Theorem 4.1 is based on the assumption that the first syzygy module of $f_1, ..., f_n$ is generated by certain relations. Now, we are going to prove that it is true.

Theorem 4.5. *For $\forall n \in \mathbb{N}$, the first syzygy module of $f_1, f_2, ...f_n$, namely M_n, is generated by μ_i and $\nu_{j,k}$ where $i \leq n-1$, $j \neq k$ where $j, k \leq n$.*

Before we start the proof of the theorem, let's first see some examples of cases with small n.

Example 4.6. (when $n = 3$)

In I_3, consider $\sum_{i=1}^{3} \alpha_i f_i = 0$, write $\alpha_i = x_0\alpha_i' + \alpha_i''$ where $x_0 \nmid \alpha_i'$ and any term of α_j'' for $i = 2,3$ and $j = 1,2,3$. Then,

$$\sum_{i=1}^{3} \alpha_i f_i = (x_0\alpha_1' + \alpha_1'')x_0^2 + (x_0\alpha_2' + \alpha_2'')2x_0x_1 + (x_0\alpha_3' + \alpha_3'')(2x_0x_2 + x_1^2).$$

Since $\alpha_3''x_1^2$ is the only term not divisible by x_0, we must have $\alpha_3'' = 0$, what remains is

$$(x_0\alpha_1' + \alpha_1'')x_0^2 + (x_0\alpha_2' + \alpha_2'')2x_0x_1 + x_0\alpha_3'(2x_0x_2 + x_1^2).$$

Similarly, $2\alpha_2''x_0x_1$ and $x_0\alpha_3'x_1^2$ are the only two terms not divisible by x_0^2, so $2\alpha_2''x_0x_1 + x_0\alpha_3'x_1^2 = 0$, which implies $\alpha_2'' = x_1\alpha_3'/2$. Moreover, since $2x_0(2x_0x_2 + x_1^2) - x_1(2x_0x_1) - 4x_2(x_0^2) = 0$ is a standard syzygy for f_1, f_2, f_3, which is μ_2, then

$$\begin{aligned}\alpha_2''(2x_0x_1) + x_0\alpha_3'(2x_0x_2 + x_1^2) &= -\frac{x_1\alpha_3'f_2}{2} + x_0\alpha_3'f_3 \\ &= -2x_2\alpha_3'f_1 - \frac{x_1\alpha_3'f_2}{2} + x_0\alpha_3'f_3 + 2x_2\alpha_3'f_1 \\ &= \frac{\alpha_3'\mu_2}{2} + 2x_2\alpha_3'f_1\end{aligned}$$

Therefore, we can rewrite the relation as

$$(x_0\alpha_1' + \alpha_1'' + 2x_2\alpha_3')f_1 + x_0\alpha_2' f_2 + \frac{\alpha_3'\mu_2}{2}.$$

Notice that $\mu_2 = 0$, then $(x_0\alpha_1' + \alpha_1'' + 2x_2\alpha_3')f_1 + x_0\alpha_2' f_2 = 0$ as well which is a relation between f_1 and f_2 and we know it must be a multiple of μ_1. As a result, any arbitrary syzygy of f_1, f_2, f_3 is generated by μ_1 and μ_2.

Example 4.7. (when $n = 4$)

Similar to the case in I_3, write the arbitrary syzygy as $\sum_{i=1}^{4} \alpha_i f_i$ as

$$(x_0\alpha_1' + \alpha_1'')x_0^2 + (x_0\alpha_2' + \alpha_2'')2x_0x_1 + (x_0\alpha_3' + \alpha_3'')(2x_0x_2 + x_1^2) + (x_0\alpha_4' + \alpha_4'')(2x_0x_3 + 2x_1x_2).$$

Since the only terms not divisible by x_0 are $\alpha_3''x_1^2$ and $\alpha_4''(2x_1x_2)$ where $x_1^2 = S(f_1)$ and $2x_1x_2 = S(f_2)$, then there $\exists\beta \in \mathbb{C}[x_1, x_2, x_3]$ such that $\alpha_3'' = -2x_2\beta$ and $\alpha_4'' = x_1\beta$.

Rewrite the relation as

$$(x_0\alpha_1' + \alpha_1'')x_0^2 + (x_0\alpha_2' + \widetilde{\alpha_2''})2x_0x_1 + x_0\alpha_3'(2x_0x_2 + x_1^2) + x_0\alpha_4'(2x_0x_3 + 2x_1x_2) - 4\beta x_0x_2^2,$$

where $\widetilde{\alpha_2''} = \alpha_2'' + x_3\beta$. Then the terms not divisible by x_0^2 are $\widetilde{\alpha_2''}(2x_0x_1)$, $\alpha_3'x_0x_1^2$, $2\alpha_4'x_0x_1x_2$, and $-4\beta x_0x_2^2$. If $\widetilde{\alpha_2''}, \alpha_3'$, and α_4' are all 0, then $\beta = 0$ and we can precede to the next step. Otherwise, we have first three terms are divisible by x_1, then the last one must be divisible by x_1 as well. Therefore, there $\exists\gamma \in \mathbb{C}[x_1, x_2, x_3]$ such that $\beta = x_1\gamma$.

As a result, $\alpha_3'' = -2x_1x_2\gamma = -\gamma f_4 + 2x_0x_3\gamma$, and $\alpha_4'' = x_1^2\gamma = \gamma f_3 - 2x_0x_2\gamma$. Then, the relation becomes

$$(x_0\alpha_1' + \alpha_1'')x_0^2 + (x_0\alpha_2' + \widetilde{\alpha_2''} + x_1x_3\gamma - 2x_2^2\gamma)2x_0x_1 + x_0\alpha_3'(2x_0x_2 + x_1^2) + x_0\alpha_4'(2x_0x_3 + 2x_1x_2) - \gamma\nu_{3,4}.$$

Then, by taking out a $\alpha_4'\mu_3$, we get

$$(x_0\alpha_1' + \alpha_1'' - x_3\alpha_4')f_1 + (x_0\alpha_2' + \widetilde{\alpha_2''} + x_1x_3\gamma - 2x_2^2\gamma - x_2\alpha_4')f_2 + (x_0\alpha_3' - x_1\alpha_4')f_3 + \alpha_4'\mu_3 - \gamma\nu_{3,4}.$$

Because $\mu_3 = 0$ and $\nu_{3,4} = 0$, we have

$$(x_0\alpha_1' + \alpha_1'' - x_3\alpha_4')f_1 + (x_0\alpha_2' + \widetilde{\alpha_2''} + x_1x_3\gamma - 2x_2^2\gamma - x_2\alpha_4')f_2 + (x_0\alpha_3' - x_1\alpha_4')f_3 = 0,$$

which is a syzygy of f_1, f_2, f_3, by the last part, it is generated by μ_1 and μ_2.

As a result, the module of syzygies of f_1, f_2, f_3, f_4 is generated by μ_1, μ_2, μ_3, and $\nu_{3,4}$.

Now, let's start the proof of the general case, which follows the similar idea. Moreover, to make the proof more accessible, we will precede in the n-tuple language.

Proof. Let F_n be a rank n free module with basis $B = \{b_1, ..., b_n\}$, and define R_n map $\phi : F_n \to I_n$ such that $\phi(b_i) = f_i$. The first syzygy module of $f_1, ..., f_n$ is $\ker\phi$ and since we have a basis for F_n, we can write element of F_n as n-tuples. For any arbitrary element $\alpha \in M_n \subset F_n$, write α as

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n),$$

then for each α_i, we can rewrite it as $x_0\alpha_i' + \alpha_i''$ where x_0 doesn't divide any term of α_i'' for all i, and by taking out proper number of $\nu_{1,i}$, we can have $x_0 \nmid \alpha_i'$ for all $i \geq 2$.

By mapping the syzygy into I_n, we get

$$\sum_{i=1}^{n}(x_0\alpha_i' + \alpha_i'')f_i = 0$$

$$(x_0\alpha_1' + \alpha_1'')f_1 + (x_0\alpha_2' + \alpha_2'')f_2 + \sum_{i=3}^{n}(x_0\alpha_i' + \alpha_i'')(2x_0x_{i-1} + S(f_{i-2})) = 0$$

and that implies $\sum_{i=3}^{n} \alpha_i'' S(f_{i-2}) = 0$ because any other terms are divisible by x_0. Then let

$$\alpha' = (0, 0, \alpha_3'', ..., \alpha_n''),$$

because of Lemma 1, we have

$$\alpha' = \sum_{i=3}^{n-1} \beta_{i+1}S(\mu_{i-2}) + \sum_{3\leq j,k\leq n, j\neq k} \beta_{j,k}S(\nu_{j-2,k-2}).$$

For any $j, k \geq 3$ such that $j \neq k$. Because

$$\begin{aligned} S(\nu_{j-2,k-2}) &= -S(f_{k-2})e_j + S(f_{j-2})e_k \\ &= -(f_k - 2x_0x_{k-1})e_j + (f_j - 2x_0x_{j-1})e_k \\ &= \nu_{j,k} + 2x_0x_{k-1}e_j - 2x_0x_{j-1}e_k, \end{aligned}$$

we can take out all the $S(\nu_{j-2,k-2})$ as $\nu_{j,k}$ and what remains will go to the $x_0\alpha_i'$ parts. Let

$$\tilde{\alpha} = \alpha - \sum_{3 \leq j,k \leq n, j \neq k} \beta_{j,k}\nu_{j,k},$$

then $\tilde{\alpha} \in M_n$. Write $\tilde{\alpha}$ as

$$\tilde{\alpha} = (x_0\tilde{\alpha_1'} + \tilde{\alpha_1''}, x_0\tilde{\alpha_2'} + \tilde{\alpha_2''}, ..., x_0\tilde{\alpha_n'} + \tilde{\alpha_n''}).$$

Let

$$\alpha'' = (0, 0, \tilde{\alpha_3''}, ..., \tilde{\alpha_n''}),$$

notice that $\tilde{\alpha_n''} = (n-3)x_1\beta_n$ and we have

$$\alpha'' = \sum_{i=3}^{n-1} \beta_{i+1}S(\mu_{i-2}).$$

Now, we consider the image of $S(\mu_{i-2})$, for $i = 3, .., n-1$.

$$\begin{aligned} \phi(S(\mu_{i-2})) &= \sum_{k=1}^{i}(i+1-3k)x_kf_{i+2-k} + (2i-1)x_if_2 \\ &= -(i+1)x_0S(f_i) + (2i-1)x_{i-1}f_2 \end{aligned}$$

Therefore, we have

$$\begin{aligned} \phi(\tilde{\alpha}) = (x_0\tilde{\alpha_1'} + \tilde{\alpha_1''})f_1 + (x_0\tilde{\alpha_2'} + \tilde{\alpha_2''} + \sum_{i=3}^{n-1}(2i-1)\beta_{i+1}x_{i-1})f_2 + \sum_{i=3}^{n} x_0\tilde{\alpha_i'}f_i - \sum_{i=3}^{n-1}(i+1)\beta_{i+1}x_0S(f_i) \\ = 0. \end{aligned}$$

By removing the terms divisible by x_0^2, we get

$$(\tilde{\alpha_2''} + \sum_{i=3}^{n-1}(2i-1)\beta_{i+1}x_{i-1})f_2 + \sum_{i=3}^{n} x_0\tilde{\alpha_i'}S(f_{i-2}) - \sum_{i=3}^{n-1}(i+1)\beta_{i+1}x_0S(f_i) = 0.$$

Since $f_2 = 2x_0x_1$, we can cancel a x_0 in each of the terms and simplify the equation as

$$2(\tilde{\alpha_2''} + \sum_{i=5}^{n-1}(2i-1)\beta_{i+1}x_{i-1})x_1 + \sum_{i=3}^{n} \tilde{\alpha_i'}S(f_{i-2}) - \sum_{i=3}^{n-1}(i+1)\beta_{i+1}S(f_i) = 0.$$

Because $S(f_{i-2}) = 2x_1x_{i-2} + S^2(f_{i-4})$ for $i \geq 5$, we can write $\tilde{\alpha_i'} = x_1\delta_i' + \delta_i''$ and $\beta_i = x_1\gamma_i' + \gamma_i''$ where δ_i'' and γ_i'' contain no term divisible by x_1. Therefore, we notice the terms that are not divisible by x_1 will sum up to be 0. That is

$$\begin{aligned} \sum_{i=5}^{n} \delta_i''S^2(f_{i-4}) - \sum_{i=3}^{n-1}(i+1)\gamma_{i+1}''S^2(f_{i-2}) = 0 \\ \sum_{i=5}^{n} \delta_i''S^2(f_{i-4}) - \sum_{i=5}^{n+1}(i-1)\gamma_{i-1}''S^2(f_{i-4}) = 0 \\ \sum_{i=5}^{n}(\delta_i'' - (i-1)\gamma_{i-1}'')S^2(f_{i-4}) + n\gamma_n''S^2(f_{n-3}) = 0. \end{aligned}$$

Because the equation is a relation of $S^2(f_i)$, where $i = 1, ..., n-3$, by Lemma **??** and induction hypothesis, we know

$$\gamma_n'' = x_2\eta' + \sum_{i=5}^{n} \eta_iS^2(f_{i-4}).$$

Then,

$$\begin{aligned}\beta_n &= x_1\gamma_n' + x_2\eta' + \sum_{i=5}^{n}\eta_i(f_i - 2x_0x_{i-1} - 2x_1x_{i-2})\\ &= x_1\tilde{\gamma_n'} + x_2\eta' + \sum_{i=5}^{n-1}\eta_i(f_i - 2x_0x_{i-1}) + S(f_{n-2}),\end{aligned}$$

so that

$$\begin{aligned}\tilde{\alpha_n''} &= (n-3)[x_1^2\tilde{\gamma_n'} + x_1x_2\eta' + \sum_{i=5}^{n-1}\eta_i x_1(f_i - 2x_0x_{i-1}) + x_1S(f_{n-2})]\\ &= (n-3)[(f_3 - 2x_0x_2)\tilde{\gamma_n'} + \frac{1}{2}(f_4 - 2x_0x_3)\eta' + \sum_{i=5}^{n-1}\eta_i x_1(f_i - 2x_0x_{i-1})\\ &\quad - \frac{1}{n-3}\sum_{i=1}^{n-3}(n-3-3i)x_{i+1}S(f_{n-2-i})\\ &= (n-3)[(f_3 - 2x_0x_2)\tilde{\gamma_n'} + \frac{1}{2}(f_4 - 2x_0x_3)\eta' + \sum_{i=5}^{n-1}\eta_i x_1(f_i - 2x_0x_{i-1})\\ &\quad - \frac{1}{n-3}\sum_{i=1}^{n-3}(n-3-3i)x_{i+1}(f_{n-i} - 2x_0x_{n-i-1})\\ &= \sum_{i=3}^{n-1}\theta_i f_i - x_0\theta\end{aligned}$$

As a result, let

$$\tilde{\alpha}' = \tilde{\alpha} - \sum_{i=3}^{n-1}\theta_i\nu_{i,n} - \frac{1}{n-1}(\tilde{\alpha_n'} - \theta)\mu_{n-1},$$

then $\tilde{\alpha}' \in M_n$ and the last entry is 0. Then by induction hypothesis, it is a combination of μ_i and $\nu_{j,k}$ for $i = 1, 2, ..., n-2$ and $j, k = 1, 2, ..., n-1$ such that $j \neq k$.

In conclusion, M_n is generated by μ_i and $\nu_{j,k}$ for $i = 1, 2, ..., n-1$ and $j, k = 1, 2, ..., n$ such that $j \neq k$. □

In fact, this result of the structure of the first syzygy module will induce a recursive formula for the minimal free resolution of the quotient ring, and it can further imply an explicit formula of the Hilbert Series, which, as we stated in the introduction, can be found in [7].

References

[1] C. Bruschek, H. Mourtada, J. Schepers. Arc spaces and the Rogers–Ramanujan identities. Ramanujan J. 30 (2013), no. 1, 9–38.

[2] B. Feigin. Abelianization of the BGG Resolution of Representations of the Virasoro Algebra. Funct. Anal. Appl. 45:4 (2011), 297–304.

[3] B. Feigin, A. Stoyanovsky. Functional models of the representations of current algebras, and semi-infinite Schubert cells. Funct. Anal. Appl. 28:1 (1994), 55–72

[4] C. Calinescu, J. Lepowsky, A. Milas. Vertex-algebraic structure of the principal subspaces of certain $A_1^{(1)}$-modules. I. Level one case. Internat. J. Math. 19 (2008), no. 1, 71–92.

[5] S. Capparelli, J. Lepowsky, A. Milas. The Rogers-Ramanujan recursion and intertwining operators. Commun. Contemp. Math. 5 (2003), no. 6, 947–966.

[6] E. Gorsky, A. Oblomkov, J. Rasmussen. On stable Khovanov homology of torus knots. Exp. Math. 22 (2013), no. 3, 265–281.

[7] Y. Bai, E. Gorsky, O. Kivinen. Quadratic ideals and Rogers-Ramanujan recursions. arXiv:1805.01593v1 [math.AG].