Undergraduate Thesis

Global Solution to a Non-linear Wave Equation of Liquid Crystal in the Constant Electric Field

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GLOBAL SOLUTION TO A NON-LINEAR WAVE EQUATION OF LIQUID CRYSTAL IN THE CONSTANT ELECTRIC FIELD

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ABSTRACT. We construct a global conservative weak solution to the Cauchy problem for the non-linear variational wave equation $v_{tt} - c(v)(c(v)v_x)_x + \frac{1}{2}g(v) = 0$ where $g(v)$ is defined in (2.5) and $c(\cdot)$ is any smooth function with uniformly positive bounded value. This wave equation is derived from a wave system modelling nematic liquid crystals in a constant electric field.

1. INTRODUCTION

1.1. Physical background.

Liquid crystal is an intermediate phase between solid and liquid. Liquid crystalline is a phase that the molecules can flow with some liquid property as well as have some optical properties of solids. The crystal molecules also have ordered arrangement. We consider the liquid crystals as fluids made up of long rigid molecules.

There are several liquid crystalline phases: solid phase, the molecules have orientation and periodicity; liquid phase, the molecules have no orientation or periodicity; nematic phase, molecules have orientation but no periodicity; and smectic phase, molecules with orientation with some periodicity. Our main focus is the nematic liquid crystal and most of them are uniaxial (having one optic axis) and some of them are biaxial (in addition to orienting their long axis, they also orient along a secondary axis). In this paper, we study a wave equation modelling the uniaxial nematic liquid crystal in one space dimension with electric field applied. In the nematic phase, the orientation of the molecules can be described by a field of unit vector $\mathbf{n}(x,t) \in S^2$, the unit sphere.

The famous Oseen-Frank potential energy density W associated with the director field $\mathbf{n}$ is defined by

$$W(\mathbf{n}, \nabla\mathbf{n}) = \alpha|\mathbf{n} \times (\nabla \times \mathbf{n})|^2 + \beta(\nabla \cdot \mathbf{n})^2 + \gamma(\mathbf{n} \cdot \nabla \times \mathbf{n})^2,$$

where α, β and γ are positive elastic constants of the liquid crystal. α represents the splay phenomenon of the nematic liquid crystal, β represents the bend phenomenon, and γ represents the twist phenomenon. When the kinetic energy are neglected in studies of nematic liquid crystals, by variational principle, we obtain an elliptic partial differential equation [11]. When we include the kinetic energy on modelling the nematic liquid crystal in one space dimension without any fields applied, we can formulate it as a non-linear wave equation

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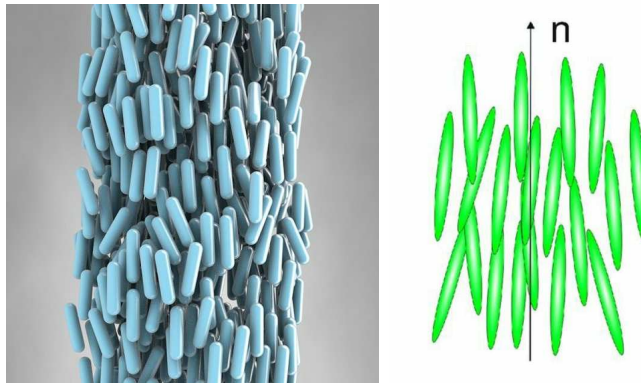
FIGURE 1. The director field **n**.
www.beautifulchemistry.net/liquid-crystals/
www.researchgate.net/figure/Molecular-ordering-and-tensor-material-properties-of-a-nematic-liquid-crystal

which is derived in [7]:

$$(1.1) \qquad\qquad u_{tt} - c(u)[c(u)u_x]_x = 0,$$

with smooth function u.

We study the nematic liquid crystal under the a constant electric field with the electric energy density described by

$$f_{electric} = -\frac{1}{2}\mathbf{P} \cdot \mathbf{E} = \frac{1}{2}\varphi\mathbf{E}^2 + \frac{1}{2}\eta(\mathbf{E} \cdot \mathbf{n})^2,$$

where $\mathbf{P}$ is the polarization, $\mathbf{E}$ is the electric field. We assume that the applied field is neither parallel nor perpendicular to $\mathbf{n}$. φ and η are positive constants related to permittivity and dielectric constants [12].

1.2. **Known results.**

For the equation (1.1), Glassey, Hunter, and Zheng [7] showed that the smooth solutions develop singularities in finite time. Also, Zhang and Zheng [19] studied that under weak conditions on the initial data which allow the solutions to have blow-up singularities and they established approximate solutions with estimates along precompactness using Young measure methods.

Our main reference is [5]. For the Cauchy problem for (1.1) with initial data $u(0, x) = u_0(x)$, $u_t(0, x) = u_1(x)$, Bressan and Zheng [5] proved the existence of a conservative weak solution by method of characteristics. They constructed conservative weak solution by introducing new sets of dependent and independent variables and showed that the solution can be obtained as the fixed point of a contraction transformation. See also [8]. Compared with [5], our energy equation has new terms from the applied electric field. These terms can be expressed as $G(v)$ where $G(v)$ is defined in (2.6). To solve this problem, we need to do some modification on the proof in [5] based on the observation that $v \in H^1$ and $G(v)$ is the lower order term in the energy equation.

For the Cauchy problem for (1.1) with initial data, Bressan, Chen and Zhang [3] proved the uniqueness of conservative solutions. Brassan and Huang [4] constructed dissipative solutions for $c' > 0$ relying on Kolmogorov's compactness theorem. Zhang and Zheng studied the existence and regularity properties of classical and weak solutions using the Young measure theory in [18] and proved the global existence of weak solutions in [20]. For C^3 initial data, Bressan and Chen [1] showed that the conservative solutions are piecewise smooth in t-x plane. In [2], Bressan and Chen constructed a metric that renders the flow uniformly Lipschitz continuous on bounded subsets of $H^1(\mathbb{R})$. Zhang and Zheng [21] studied the existence of global weak solutions to the initial value problem (1.1) with general initial data $(u(0), u_t(0)) = (u_0, u_1) \in W^{1,2} \times L^2$ with wave speed satisfying $c'(\cdot) \geq 0$ and $c'(u_0(\cdot)) > 0$.

For a wave system modelling nematic liquid crystals in one space dimension, Chen and Zheng [6] studied the global existence and singularity formation. Huang and Zheng [9] established the global existence of smooth solutions. Zhang and Zheng [13] constructed a weak global solutions to the Cauchy problem for a system of two variational wave equations on the real line and [14] showed the global weak solutions to the initial value problem for a complete system of variational wave equations modelling liquid crystals in one space dimension.

[10] shows that the weakly nonlinear unidirectional waves satisfying (1.1) are discribed asymptotically by

$$(1.2) \qquad (u_t + u^n u_x)_x = \frac{1}{2} n u^{n-1} (u_x)^2,$$

derived by Hunter and Saxton via weakly nonlinear geometric optics. In [15]-[17], Zhang and Zheng studied the global existence, uniqueness, and regularity of the dissipative and conservative solutions to $(1.2)(n = 1, 2)$ with L^2 initial data.

1.3. Some definitions and theorem from Partial Differential Equations by Evans. The definition of L^p space.

Let X be a measure space. Given a function f, we say $f \in L^p$ on X if f is Lebesgue measurable and if

$$\int_X |f|^p d\mu < \infty,$$

then f is L^p-integrable.

For $f \in L^p(\mu)$, we define the norm

$$\|f\|_{L^p} = \left(\int_X |f|^p d\mu \right)^{\frac{1}{p}}.$$

And call $\|f\|_{L^p}$ as the $L^p(\mu)$ norm of f.

The definition of Locally intergable function L^p_{loc}.

Let Ω be an open set in the $\mathbb{R}^n$ space and $f : \Omega \mapsto \mathbb{C}$ be a Lebesgue measurable function.

For given p with $1 \leq p \leq \infty$ and this p satisfies

$$\int_K |f|^p dx < \infty,$$

it belongs to the $L^P(K)$ for all compact subsets K of Ω, then f is called locally p-integrable. The set of all such function is denoted be L^p_{loc}:

$$L^p_{loc}(\Omega) = \{\, f : \Omega \mapsto \mathbb{C} \; measurable \mid f \in L^p(K), \;\; \forall K \subset \Omega, \;\; K \; is \; compact \,\}.$$

The definition of Hölder continuity.

We say function $f(x)$ is Hölder continuous with exponent α if there exists a constant $C > 0$ and α such that

$$|f(x) - f(y)| \leq C|x - y|^\alpha,$$

for all x and y in the domain of f.

The Definition of Sobolev space.

Suppose that Ω is an open set in $\mathbb{R}^n$, $k \in \mathbb{N}$, and $1 \leq p \leq \infty$. The Sobolev space $W^{k,p}(\Omega)$ consists of all locally integrable functions $f : \Omega \mapsto \mathbb{R}$ such that

$$\partial^\alpha f \in L^p(\Omega) \quad for \; 0 \leq |\alpha| \leq k.$$

We write $W^{k,p} = H^k(\Omega)$.

The Sobolev space $W^{k,p}(\Omega)$ is a Banach space when equipped with the norm

$$||f||_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \int_\Omega |\partial^\alpha f|^p dx \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$.

Sobolev embedding theorem

Suppose that $1 \leq p < n$ and $p \leq q \leq p^*$ where $p^* = \frac{np}{n-p}$ is the Sobolev conjugate of p. Then $W^{1,p}(\mathbb{R}^n) \mapsto L^q(\mathbb{R}^n)$ and

$$||f||_q \leq C||f||_{W^{1,p}}(\mathbb{R}^n) \quad for \; all \; f \in W^{1,p}(\mathbb{R}^n),$$

for some constant $C = C(n, p, q)$.

1.4. Main theorems.

Our main results are stated as follows. For the nematic liquid crystal under electric field, we obtain a Cauchy problem

$$(1.3) \qquad\qquad v_{tt} - c(v)(c(v)v_x)_x + \frac{1}{2}g(v) = 0,$$

with the initial data

$$(1.4) \qquad\qquad v(0, x) = v_0(x), \quad v_t(0, x) = v_1(x).$$

For the smooth function $c(\cdot)$, we assume that $c : \mathbb{R} \mapsto \mathbb{R}^+$ is a bounded and uniformly positive function.

Definition 1.1. The definition of weak solution.

We say that for all test function $\phi \in C_c^1$, the function $v \in H^1$ satisfies the following integral:

$$(1.5) \qquad \iint \phi_t v_t - [c(v)\phi]_x[c(v)v_x] - \frac{\phi}{2}g(v)dxdt = 0,$$

is a weak solution to the equation (1.3).

Definition 1.2. The definition of energy conservative weak solution.

For v_1 and v_0 defined in (1.4) and $G(v)$ is defined in (2.6), we define the ground state energy $\mathcal{E}_0$ as:

$$(1.6) \qquad \mathcal{E}_0 := \frac{1}{2}\int \left\{ v_1^2(x) + c^2(v_0(x))[v_0^2(x)]_x + G(v_0(x)) \right\}dx.$$

The function $v \in H^1$ is a energy conservative weak solution if it satisfying

$$(1.7) \qquad \mathcal{E}(t) := \frac{1}{2}\int \left\{ v_t^2(t,x) + c^2(v(t,x))v_x^2(t,x) + G(v(t,x)) \right\}dx = \mathcal{E}_0,$$

for almost every $t \in \mathbb{R}$.

Theorem 1.1. *Assume that $c: \mathbb{R} \mapsto [\mathcal{K}^{-1}, \mathcal{K}]$ is a smooth function for some $\mathcal{K} > 1$. $v_0(x)$ and $v_1(x)$ are stated in (1.4). Also assume that the initial data $v_0(x)$ is absolutely continuous, $(v_0(x))_x \in H^1$, and $v_1(x) \in H^1$. Then (1.3)-(1.4) can be considered as a Cauchy problem admitting a weak solution $v(t,x)$ defined for all $(t,x) \in \mathbb{R} \times \mathbb{R}$. Moreover, in the t-x plane, $v(x,t)$ is locally Hölder-$\frac{1}{2}$ continuous. For all $1 \leq p < 2$, the map $t \mapsto v(t,\cdot)$ is continuously differentiable with values in L_{loc}^p. The weak solution $v(t,\cdot)$ is Lipschitz continuous with respect to L^2 distance. So, for all $t, s \in \mathbb{R}$,*

$$(1.8) \qquad \|v(t,\cdot) - v(s,\cdot)\|_{L^2} \leq L|t-s|.$$

For all test function $\phi \in C_c^1$, the equation (1.3) satisfies (1.5).

Theorem 1.2. *A family of weak solutions to the Cauchy problem (1.3)-(1.4) can be obtained with the properties:*

$$(1.9) \qquad \mathcal{E}(t) \leq \mathcal{E}_0.$$

Let a sequence of initial condition satisfies:

$$\|(v_0^n(x))_x - (v_0(x))_x\|_{L^2} \to 0,$$

$$\|v_1^n(x) - v_1(x)\|_{L^2} \to 0.$$

Also, $u^n \to u$ uniformly on bounded subsets of the t-x plane and $v_0^n \to v_0$ on compact sets as $n \to \infty$.

Theorem 1.3. *There exists a continuous family of positive Radon measures $\{\mu_t : t \in \mathbb{R}\}$. This family of positive Radon measure is defined on the real line and it satisfies the following properties:*

(i) $\mu_t(\mathbb{R}) = \mathcal{E}_0$ *for any time* t.

(ii) With respect to Lebesgue measure, the absolutely continuous part of μ_t *has density* $\frac{1}{2}(v_t^2 + c^2(v)v_x^2 + G(v))$.

(iii) The singular part of μ_t *has measure zero on the set where* $c'(v) = 0$.

The paper is organized as follows. In section 2, we derive the energy equation and introduce a new set of dependent variables. Based on those dependent variables, we formulate a set of equations in terms of the new variables. This set of equations is equivalent to (1.3). In section 3, we use a transformation in a Banach space. In the transformation, we find the suitable weighted norm. This shows that there is a unique solution to the set of equations in terms of the new variables. In section 4, we show that the integral (1.5) holds and the Hölder-$\frac{1}{2}$ continuous condition holds. In section 5, we show that (1.9) holds and the Lipschitz condition on the map $t \mapsto v(t, \cdot)$ and provide a proof of Theorem 1.2. On section 6, we study the maps of $t \mapsto u_x(t, \cdot)$ and $t \mapsto u_t(t, \cdot)$, and complete the proof of Theorem 1.1. We provide a proof of Theorem 1.3 in section 7.

2. Variable Transformations

2.1. **Derivation of (1.3).** Equation (1.3) has some physical origins. In the context of nematic liquid crystals, we introduce the famous Oseen-Frank potential energy density W is given by

$$(2.1) \qquad W(\mathbf{n}, \nabla\mathbf{n}) = \alpha|\mathbf{n} \times (\nabla \times \mathbf{n})|^2 + \beta(\nabla \cdot \mathbf{n})^2 + \gamma(\mathbf{n} \cdot \nabla \times \mathbf{n})^2.$$

As stated in [12], in a electric field, the electric energy of the liquid crystal per unit volume is given by

$$(2.2) \qquad f_{electric} = -\frac{1}{2}\mathbf{P} \cdot \mathbf{E} = \frac{1}{2}\varphi\mathbf{E}^2 + \frac{1}{2}\eta(\mathbf{E} \cdot \mathbf{n})^2.$$

We discuss that when the electric energy is low when the applied electric field is normal to the liquid crystal director. And φ and η are some positive constants related to the permittivity so that (2.2) is equivalent to $|\mathbf{n} \cdot \mathbf{E}^\perp|^2 + 1$. And we denote $\mathbf{E}^\perp$ as a vector such that $\mathbf{E} \cdot \mathbf{E}^\perp = 0$ and $\mathbf{E}^\perp = (1, 0)$. So, the electric energy can be described as $|\mathbf{n} \cdot \mathbf{E}^\perp|^2$. By the property of the potential energy, the action can be describe as

$$S = \iint |\mathbf{n}_t|^2 - W(\mathbf{n}, \nabla\mathbf{n}) - |\mathbf{n} \cdot \mathbf{E}^\perp|^2 dx\, dt$$

.

By plug in $\mathbf{n} = (\cos u, \sin u)$, the action can be describe as:

$$(2.3) \qquad S = \iint u_t^2 - (c(u))^2 u_x^2 - (\cos u)^2 dx\, dt.$$

We let $v = u - \frac{\pi}{2}$ so that $\cos u = \sin v$. By the principle of least action,

$$\delta S = 0 = \iint \delta[v_t^2 - (c(v))^2 v_x^2 - (\sin v)^2] dx\, dt.$$

A straightforward computation shows that

(2.4)
$$v_{tt} - c(v)(c(v)v_x)_x + \frac{g(v)}{2} = 0,$$

where

(2.5)
$$g(v) = 2\sin v \cos v.$$

And define $G(v)$ as two times the anti derivative of $g(v)$,

(2.6)
$$G(v) = 2(\sin v)^2.$$

2.2. Derivation of the energy equation.

From (2.4), we can compute that

(2.7)
$$\int v_t v_{tt} - v_t c(v)[c(v)v_x]_x + \frac{g(v)}{2}v_t dx = 0$$
$$\int (\frac{1}{2}v_t^2)_t + (\frac{c^2(v)v_x^2}{2})_t + (\frac{1}{2}G(v))_t dx = 0.$$

And from (2.7), the energy equation can be described as

(2.8)
$$E := \frac{1}{2}\left(v_t^2 + c^2(v)v_x^2 + G(v)\right).$$

2.3. Variables transform.
In this section we derive identities that holds for smooth solutions. We first denote variables:

(2.9)
$$\begin{cases} R := v_t + c(v)v_x, \\ S := v_t - c(v)v_x. \end{cases}$$

Thus, we can write v_t and v_x as follows

(2.10)
$$\begin{cases} v_t = \dfrac{R+S}{2}, \\ v_x = \dfrac{R-S}{2c}. \end{cases}$$

By (1.3), the following identities are valid :

(2.11)
$$\begin{cases} S_t + cS_x = \dfrac{c'}{4c}(S^2 - R^2) - \dfrac{1}{2}g(v), \\ R_t - cR_x = \dfrac{c'}{4c}(R^2 - S^2) - \dfrac{1}{2}g(v), \end{cases}$$

by the following calculation

$$R_t - cR_x = (v_t + cv_x)_t - c(v_t + cv_x)_x$$
$$= -\frac{g(v)}{2} + \frac{c'}{4c}(R^2 - S^2).$$

We can compute $S_t + cS_x$ in the similar way to get (2.11) and denote energy and momentum as

(2.12)
$$E := \frac{1}{2}\left(v_t^2 + c^2(v)v_x^2 + G(v)\right) = \frac{R^2 + S^2}{4} + \frac{1}{2}G(v),$$

(2.13)
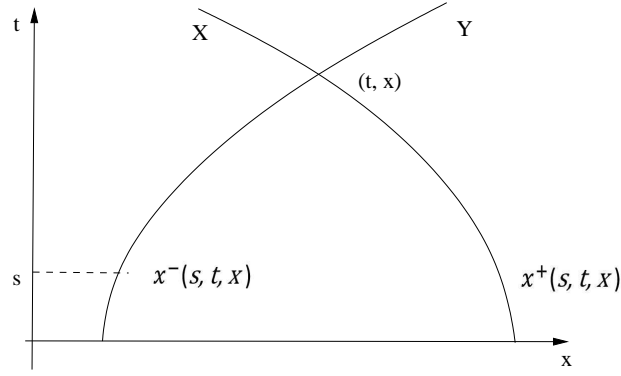$$M := -v_t v_x = \frac{S^2 - R^2}{4c}.$$

FIGURE 2. The characteristic curves.

The analysis of (1.3) has a main difficult that the possible breakdown of the regularity solutions. The quantities v_x and v_t can blow up in finite time even with smooth initial data. Thus we need to introduce a new set of dependent variables to deal with the possible unbounded value R and S:

$$(2.14) \qquad\qquad w := 2\arctan R, \qquad z := 2\arctan S.$$

Thus

$$(2.15) \qquad\qquad R = \tan\left(\frac{w}{2}\right), \qquad S = \tan\left(\frac{z}{2}\right).$$

By (2.11),

$$(2.16) \qquad w_t - cw_x = \frac{2}{1+R^2}(R_t - cR_x) = \frac{c'}{2c}\frac{R^2 - S^2}{1+R^2} - \frac{1}{1+R^2}g(v),$$

$$(2.17) \qquad z_t + cz_x = \frac{2}{1+S^2}(S_t + cS_x) = \frac{c'}{2c}\frac{S^2 - R^2}{1+S^2} - \frac{1}{1+S^2}g(v).$$

In order to reduce the equation to a semi-linear system, we need to have a further change of variables. The forward characteristics equation and the backward characteristics equation:

$$\dot{x}^+ = c(v), \qquad \dot{x}^- = c(v).$$

And we denote the characteristics lines pass through the point (t, x) as

$$s \to x^+(s, t, x), \qquad s \to x^-(s, t, x).$$

So we can use a new coordinate system (X, Y) to represent point (t, x) by

$$(2.18) \qquad\qquad X := \int_0^{x^-(0,t,x)} (1 + R^2(0, x))dx,$$

$$(2.19) \qquad\qquad Y := \int_{x^+(0,t,x)}^0 (1 + S^2(0, x))dx.$$

(2.18) and (2.19) implies that

$$(2.20) \qquad\qquad X_t - c(v)X_x = 0, \qquad Y_t + c(v)Y_x = 0,$$

$$(2.21) \qquad\qquad (X_x)_t - (c(v)X_x)_x = 0, \qquad (Y_x)_t + (c(v)Y_x)_x = 0.$$

Thus, given any smooth function f, by using (2.20),

$$(2.22) \qquad \begin{aligned} f_t + c(v)f_x &= 2c(v)X_x f_X, \\ f_t - c(v)f_x &= 2c(v)Y_x f_Y. \end{aligned}$$

From (2.20), $X_t + c(v)X_x = 2c(v)X_x$. To get (2.22) we compute directly

$$f_t + c(v)f_x = f_X X_t + f_Y Y_t + c(v)f_X X_x + c(v)f_Y Y_x = (X_t + c(v)X_x)f_X = 2c(v)X_x f_X,$$

$$f_t - c(v)f_x = f_X X_t + f_Y Y_t - c(v)f_X X_x - c(v)f_Y Y_x = (Y_t - c(v)Y_x)f_Y = 2c(v)Y_x f_Y.$$

Introducing new variables

$$(2.23) \qquad p := \frac{1+R^2}{X_x}, \qquad q := \frac{1+S^2}{-Y_x}.$$

From (2.23),

$$(2.24) \qquad \begin{aligned} \frac{1}{X_x} &= \frac{p}{1+R^2} = p\cos^2\left(\frac{w}{2}\right) = \frac{p(1+\cos w)}{2}, \\ \frac{1}{-Y_x} &= \frac{q}{1+S^2} = q\cos^2\left(\frac{z}{2}\right) = \frac{q(1+\cos z)}{2}. \end{aligned}$$

By applying (2.16)-(2.17) to (2.17),

$$w_t - cw_x = 2c\frac{1+S^2}{q}w_Y = \frac{c'}{2c}\frac{R^2-S^2}{1+R^2} + \frac{g(v)}{1+R^2},$$

$$z_t + cz_x = 2c\frac{1+R^2}{p}z_x = \frac{c'}{2c}\frac{S^2-R^2}{1+S^2} + \frac{g(v)}{1+S^2}.$$

Thus, w_Y and z_X can be write as

$$w_Y = \frac{c'}{4c^2}\frac{R^2-S^2}{1+R^2}\frac{q}{1+S^2} - \frac{q}{2c}\frac{1}{1+S^2}\frac{g(v)}{1+R^2},$$

$$z_X = \frac{c'}{4c^2}\frac{S^2-R^2}{1+S^2}\frac{p}{1+R^2} - \frac{p}{2c}\frac{1}{1+S^2}\frac{g(v)}{1+R^2}.$$

So

$$(2.25) \qquad \begin{cases} w_Y = \dfrac{c'}{8c^2}(\cos z - \cos w)q - \dfrac{q}{8c}g(v)(1+\cos z)(1+\cos w), \\[2mm] z_X = \dfrac{c'}{8c^2}(\cos w - \cos z)p - \dfrac{p}{8c}g(v)(1+\cos z)(1+\cos w). \end{cases}$$

By using (2.21) and (2.24),

$$\begin{aligned} p_t - cp_x &= \frac{1}{X_x}2R(R_t - cR_x) - \frac{1}{X_x^2}[(X_x)_t - c(X_x)_x](1+R^2) \\ &= \frac{c'}{2c}\frac{p}{1+R^2}[S(1+R^2) - R(1+S^2)] - \frac{p}{1+R^2}Rg(v), \\ q_t + cq_x &= \frac{1}{-Y_x}2S(S_t - cS_x) - \frac{1}{-Y_x^2}[(-Y_x)_t + c(-Y_x)_x](1+S^2) \\ &= \frac{c'}{2c}\frac{q}{1+S^2}[R(1+S^2) - S(1+R^2)] - \frac{q}{1+S^2}Sg(v). \end{aligned}$$

By applying (2.22),

$$p_t - cp_x = -2cY_x p_Y,$$

$$q_t + cq_x = 2cX_x q_X.$$

And thus,

$$p_Y = (p_t - cp_x)\frac{1}{-2cY_x} = (p_t - cp_x)\frac{1}{2c}\frac{q}{1 + S^2}$$

$$= \frac{c'}{8c^2}[\sin z - \sin w]pq - \frac{1}{8c}pq \sin w\, g(v)(1 + \cos z),$$

$$q_X = (q_t + cq_x)\frac{1}{2cX_x} = (q_t + cq_x)\frac{1}{2c}\frac{p}{1 + R^2}$$

$$= \frac{c'}{8c^2}[\sin w - \sin z]pq - \frac{1}{8c}pq \sin z\, g(v)(1 + \cos w).$$

So, the following identities hold:

(2.26)
$$\begin{cases} p_Y = \dfrac{c'}{8c^2}[\sin z - \sin w]pq - \dfrac{1}{8c}pq \sin w\, g(v)(1 + \cos z), \\[2mm] q_X = \dfrac{c'}{8c^2}[\sin w - \sin z]pq - \dfrac{1}{8c}pq \sin z\, g(v)(1 + \cos w). \end{cases}$$

Also, we plug in $f = v$ into the equation (2.22) and get

(2.27)
$$\begin{cases} v_X = (v_t + cv_x)\dfrac{1}{2c}\dfrac{p}{1 + R^2} = \dfrac{1}{2c}(\tan\dfrac{w}{2}\cos^2\dfrac{w}{2})p = p\dfrac{1}{4c}\sin w, \\[2mm] v_Y = (v_t - cv_x)\dfrac{1}{2c}\dfrac{q}{1 + S^2} = \dfrac{1}{2c}(\tan\dfrac{z}{2}\cos^2\dfrac{z}{2})q = q\dfrac{1}{4c}\sin z. \end{cases}$$

Combining (2.25), (2.26), and (2.27), we obtain a semi-linear hyperbolic system from the non-linear equation (1.3). This system uses X, Y as independent variables with smooth coefficients for the variables v, w, z, p, q

(2.28)
$$\begin{cases} w_Y = \dfrac{c'}{8c^2}(\cos z - \cos w)q - \dfrac{q}{8c}g(v)(1 + \cos z)(1 + \cos w), \\[2mm] z_X = \dfrac{c'}{8c^2}(\cos w - \cos z)p - \dfrac{p}{8c}g(v)(1 + \cos z)(1 + \cos w), \end{cases}$$

(2.29)
$$\begin{cases} p_Y = \dfrac{c'}{8c^2}[\sin z - \sin w]pq - \dfrac{1}{8c}pq \sin w\, g(v)(1 + \cos z), \\[2mm] q_X = \dfrac{c'}{8c^2}[\sin w - \sin z]pq - \dfrac{1}{8c}pq \sin z\, g(v)(1 + \cos w), \end{cases}$$

(2.30)
$$\begin{cases} v_X = \dfrac{p}{4c}\sin w, \\[2mm] v_Y = \dfrac{q}{4c}\sin z, \end{cases}$$

The system (2.28)-(2.30) should have non-characteristic boundary conditions related to (1.4). From (1.4), v_0 and v_1 determine the initial values of R and S at time $t = 0$. We denote the curve γ as the line in (X, Y) plane at time $t = 0$, say

$$Y = \varphi(X), \qquad X \in \mathbb{R}.$$

And $Y = \varphi(X)$ if and only if for some $x \in \mathbb{R}$,

$$X = \int_0^x (1 + R^2(0, x))dx, \qquad Y = \int_x^0 (1 + S^2(0, x))dx.$$

By the assumptions of the Theorem 1.1, $v_0 \in H^1, v_1 \in H^1$. This implies that $R \in H^1$ and $S \in H^1$. Moreover, in this case, we let

$$(2.31) \qquad \mathcal{E}_0 := \frac{1}{4} \int [R^2(0, x) + S^2(0, x)]dx < \infty.$$

Thus,

$$(2.32) \qquad X(x) := \int_0^x (1 + R^2(0, y))dy, \qquad Y(x) := \int_x^0 (1 + S^2(0, y))dy.$$

are absolutely continuous and well defined functions. Further more, by observing (2.32), X is increasing and Y is decreasing. So, we conclude that the map $X \mapsto \varphi(X)$ is continuous and decreasing. And from (2.31),

$$|X + \varphi(X)| \leq 4\mathcal{E}_0.$$

Since $(t, x) \in [0, \infty) \times (-\infty, \infty)$, so our new independent variables $(X, Y) \in \Omega^+$, and the domain is defined as

$$\Omega^+ := \{(X, Y) : Y \geq \varphi(X)\},$$

along the curve

$$\gamma := \{(X, Y) : Y = \varphi(X)\}.$$

We can have the following boundary data $(\bar{w}, \bar{z}, \bar{p}, \bar{q}, \bar{v}) \in L^\infty$,

$$(2.33) \qquad \begin{cases} \bar{w} = 2\arctan(R(0, x)), \\ \bar{z} = 2\arctan(S(0, x)), \end{cases}$$

$$(2.34) \qquad \begin{cases} \bar{p} \equiv 1, \\ \bar{q} \equiv 1, \end{cases}$$

$$(2.35) \qquad \bar{v} = v_0(x).$$

3. Construct the integral solution

We prove the global existence and uniqueness for the semi-linear system (2.28) - (2.30) in this section.

Theorem 3.1. *If the assumptions of Theorem 1.1 holds, then the semi-linear system (2.28) - (2.30) with the boundary conditions (2.33) - (2.35) has a unique solution for all $(X, Y) \in \mathbb{R} \times \mathbb{R}$.*

We construct the solution on the region Ω^+ which is the case that $Y \geq \varphi(X)$. The proof of the solution on the Ω^- which is the case that $Y \leq \varphi(X)$ can be construct in the similar way. We show the Lipschitz condition for the system (2.28) - (2.30). To make sure the solution
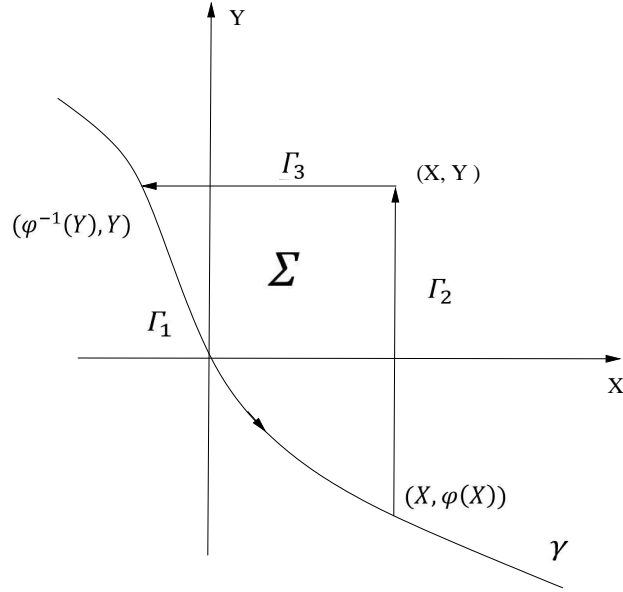
FIGURE 3. The closed curve Σ.

is defined in the region Ω^+, we need to construct some priori bounds. So that we can show that p, q are bounded. The Lipschitz condition can be derived as follows. From the energy conservation equations (1.7), (1.6) and the assumption that $v \in H^1$, we denote the following constants:

$$C_1 := \sup_{v \in \mathbb{R}} \left| \frac{c'(v)}{4c^2(v)} \right| < \infty,$$

(3.1)
$$K_1 := \sup_{t \geq 0} \int G(v) dx < \infty,$$

$$K_0 := \sup_{x,t} \left| g(v) \right| < \infty.$$

From (2.29),

$$
\begin{aligned}
q_X + p_Y = &\frac{1}{2} \left[G(v)q(1 + \cos z) \right]_X + \frac{1}{2} \left[G(v)p(1 + \cos w) \right]_Y \\
&- \frac{1}{2} G(v) \frac{c'}{8c} (\sin w - \sin z)(\cos z - \cos w) \left(\frac{1}{c} + 1 \right).
\end{aligned}
$$
(3.2)

We construct a closed curve Σ for every $(X, Y) \in \Omega^+$ with the vertical line segment connect (X, Y) with $(X, \varphi(X))$, the horizontal line segment connect (X, Y) with $(\varphi^{-1}(Y), Y)$, and a part of the boundary $\gamma = Y = \varphi(X)$ connecting $(X, \varphi(X))$ with $(\varphi^{-1}(Y), Y)$. The closed curve $\Sigma = \Gamma_1 + \Gamma_2 + \Gamma_3$. From (3.1), we compute $\iint q_X + p_Y dA = \int -p dX + \int q dY$ and denote that

(3.3)
$$\iint q_X + p_Y dA = \int -p dX + \int q dY,$$

(3.4)
$$Q_X := \frac{1}{2} \left[G(v)q(1 + \cos z) \right]_X,$$

$$(3.5) \qquad P_Y := \frac{1}{2} \left[G(v)p(1 + \cos w) \right]_Y,$$

$$(3.6) \qquad \xi := G(v)\frac{c'}{8c}(\sin w - \sin z)(\cos z - \cos w)\left(\frac{1}{c} + 1\right).$$

So

$$\iint_\Sigma p_y + q_X dX dY = \int_\Sigma -p dX + \int_\Sigma q dY$$

$$= \iint_\Sigma Q_X + P_Y - \frac{1}{2}\xi dX dY,$$

By Green's Theorem,

$$\int_\Sigma -p dX + \int_\Sigma q dY = \int_\Sigma -P dX + \int_\Sigma Q dY - \frac{1}{2}\iint_\Sigma \xi dX dY.$$

Thus,

$$\frac{1}{2}\iint_\Sigma \xi dX dY = \int_\Sigma p - P dX + \int_\Sigma Q - q dY,$$

$$= \int_\Sigma p - \frac{1}{2}G(v)p(1 + \cos w)dX + \int_\Sigma -q + \frac{1}{2}G(v)q(1 + \cos z)dX.$$

Since $\Sigma = \Gamma_1 + \Gamma_2 + \Gamma_3$ is a closed curve, so we compute the integral of Γ_1 directly and in the way $\Gamma_1 = -(\Gamma_2 + \Gamma_3)$.

$$(3.7) \qquad \int_{\Gamma_1} 1 - \frac{1}{2}G(v)(1 + \cos w)dX + \int_{\Gamma_1} -1 + \frac{1}{2}G(v)(1 + \cos z)dY,$$

and from (2.24)

$$dX = \frac{2}{1 + \cos w}dx, \quad dY = \frac{2}{1 + \cos z}dx.$$

Thus $(1 + \cos w)dX = 2dx$, and $(1 + \cos w)dY = 2dx$.

So (3.8) becomes

$$\int_{\Gamma_1} 1 - \frac{1}{2}G(v)(1 + \cos w)dX + \int_{\Gamma_1} -1 + \frac{1}{2}G(v)(1 + \cos z)dY$$

$$\leq 2(|X| + |Y| + 4\mathcal{E}_0) + K_1.$$

And also,

$$\int_{\Gamma_2} 1 - \frac{1}{2}G(v)(1 + \cos w)dX + \int_{\Gamma_2} -1 + \frac{1}{2}G(v)(1 + \cos z)dY$$

$$\leq 0 - Y + \varphi(X) + \frac{K_1}{2},$$

$$\int_{\Gamma_3} 1 - \frac{1}{2}G(v)(1 + \cos w)dX + \int_{\Gamma_3} -1 + \frac{1}{2}G(v)(1 + \cos z)dY$$

$$\leq \varphi^{-1}(Y) - X - 0 + \frac{K_1}{2}.$$

As a result,

$$(3.8) \qquad \int_{\varphi^{-1}(Y)}^X p(X', Y)dX' + \int_{\varphi(X)}^Y q(X, Y')dY' \leq 2(|X| + |Y| + 4\mathcal{E}_0) + K_1.$$

By observing the boundary conditions (2.33) - (2.35), $p, q > 0$. And by (2.29),

$$p_Y = \frac{1}{8c} pq \left\{ \frac{c'}{c} [\sin z - \sin w] - \sin w \, g(v)(1 + \cos z) \right\},$$

$$p(X, Y) = exp \left\{ \int_{\varphi(X)}^{Y} \frac{1}{8c} \frac{c'}{c} [\sin z - \sin w] - \sin w \, g(v)(1 + \cos z) q(X, Y') dY' \right\}$$

$$\leq exp \left\{ C_1 \int_{\varphi(X)}^{Y} q(X, Y') dY' \right\}$$

$$\leq exp \left\{ 2C_1(|X| + |Y| + 4\mathcal{E}_0) + C_1 K_1 \right\}.$$

Similarly, we have

$$q(X, Y) \leq exp\{ 2C_1(|X| + |Y| + 4\mathcal{E}_0) + C_1 K_1 \}.$$

Now, we show that on any bounded sets in X-Y plane, we can construct the solution for the system of the equations (2.28) - (2.30) with boundary condition (2.33) -(2.35) by the fixed point of a constructive map. For any $r > 0$, we can construct a bounded domain

$$\Omega_r := \{(X, Y) : Y \leq \varphi(X), X \leq r, Y \leq r\}.$$

And also introduce the function space :

(3.9) $$\Lambda_r := \{f : \Omega_r \mapsto \mathbb{R} : \|f\|_* := \operatorname*{ess\,sup}_{(X,Y) \in \Omega_r} e^{-K(X+Y)} |f(X, Y)| < \infty\}.$$

Where K is a suitably big constant and it will be determined later. And for $(w, z, p, q, v) \in \Lambda_r$, we construct a map $\tau(w, z, p, q, v) = (\tilde{w}, \tilde{z}, \tilde{p}, \tilde{q}, \tilde{v})$. And this map is define as follows.
(3.10)
$$\begin{cases} \tilde{w}(X, Y) = \bar{w}(X, \varphi(X)) + \int_{\varphi(X)}^{Y} \frac{c'}{8c^2}(\cos z - \cos w)q - \frac{q}{8c} g(v)(1 + \cos z)(1 + \cos w) dY, \\ \tilde{z}(X, Y) = \bar{z}(\varphi^{-1}(Y), Y) + \int_{\varphi^{-1}(Y)}^{X} \frac{c'}{8c^2}(\cos w - \cos z)p - \frac{p}{8c} g(v)(1 + \cos z)(1 + \cos w) dX, \end{cases}$$

(3.11) $$\begin{cases} \tilde{p}(X, Y) = 1 + \int_{\varphi(X)}^{Y} \frac{1}{8c} pq \left\{ \frac{c'}{c}[\sin z - \sin w] - \sin w \, g(v)(1 + \cos z) \right\} dY, \\ \tilde{q}(X, Y) = 1 + \int_{\varphi^{-1}(Y)}^{X} \frac{1}{8c} pq \left\{ \frac{c'}{c}[\sin w - \sin z] - \sin z \, g(v)(1 + \cos w) \right\} dX, \end{cases}$$

(3.12) $$\tilde{v}(X, Y) = \bar{v}(X, \varphi(X)) + \int_{\varphi(X)}^{Y} \frac{1}{4c} \sin z q dY.$$

We want to prove the uniform Lipschitz condition. First, we define

$$\Phi_r := \Lambda_r \times \Lambda_r \times \Lambda_r \times \Lambda_r \times \Lambda_r.$$

For some properly chosen distance $D : \Phi_r \times \Phi_r \mapsto \mathbb{R}$, we want to show that

$$D((\tilde{w}_1, \tilde{z}_1, \tilde{p}_1, \tilde{q}_1, \tilde{v}_1), (\tilde{w}_2, \tilde{z}_2, \tilde{p}_2, \tilde{q}_2, \tilde{v}_2)) < L \times D((w_1, z_1, p_1, q_1, v_1), (w_2, z_2, p_2, q_2, v_2)).$$

The Lipschitz constant L satisfies $L \leq 1$. In fact, we define the distance as:

$$D((\tilde{w}_1, \tilde{z}_1, \tilde{p}_1, \tilde{q}_1, \tilde{v}_1), (\tilde{w}_2, \tilde{z}_2, \tilde{p}_2, \tilde{q}_2, \tilde{v}_2)) := \max\{\|\tilde{w}_1 - \tilde{w}_2\|_* , \ \|\tilde{z}_1 - \tilde{z}_2\|_* ,$$

$$\|\tilde{p}_1 - \tilde{p}_2\|_* , \ \|\tilde{q}_1 - \tilde{q}_2\|_* , \ \|\tilde{v}_1 - \tilde{v}_2\|_*\},$$

and the norm $\|\cdot\|_*$ is defined in (3.9).

We first proving the Lipschitz condition of $\|\tilde{w}_1 - \tilde{w}_2\|$.

$$\tilde{w}_1 - \tilde{w}_2 = \int_{\varphi(X)}^Y \frac{c'}{8c^2}(\cos z_1 - \cos w_1)q_1 - \frac{q_1}{8c}g(v)(1+\cos z_1)(1+\cos w_1))$$

$$- \frac{c'}{8c^2}(\cos z_1 - \cos w_1)q_1 + \frac{q_1}{8c}g(v)(1+\cos z_1)(1+\cos w_1)dY$$

$$= \int_{\varphi(X)}^Y \frac{c'}{8c^2}(\cos z_1 - \cos w_1)q_1 - \frac{c'}{8c^2}(\cos z_2 - \cos w_2)q_2 dY$$

$$+ \int_{\varphi(X)}^Y \frac{q_2}{8c}g(v)(1+\cos z_2)(1+\cos w_2) - \frac{q_1}{8c}g(v)(1+\cos z_1)(1+\cos w_1)dY.$$

And we denote that

$$(3.13) \qquad \int_{\varphi(X)}^Y \frac{c'}{8c^2}(\cos z_1 - \cos w_1)q_1 - \frac{c'}{8c^2}(\cos z_2 - \cos w_2)q_2 dY,$$

and

$$(3.14) \qquad \int_{\varphi(X)}^Y \frac{q_2}{8c}g(v)(1+\cos z_2)(1+\cos w_2) - \frac{q_1}{8c}g(v)(1+\cos z_1)(1+\cos w_1)dY.$$

We compute (3.13),

$$(3.13) = \int_{\varphi(X)}^Y \frac{c'}{8c^2}(\cos z_1 - \cos w_1)q_1 + \frac{c'}{8c^2}(\cos z_1 - \cos w_1)q_2$$

$$- \frac{c'}{8c^2}(\cos z_1 - \cos w_1)q_2 + \frac{c'}{8c^2}(\cos z_2 - \cos w_2)q_2 dY$$

$$= \int_{\varphi(X)}^Y \frac{c'}{8c}(q_1 - q_2)(\cos z_1 - \cos w_1) + \frac{c'}{8c}(\cos w_2 - \cos w_1) + \frac{c'}{8c}q_2(\cos z_1 - \cos z_2)dY$$

$$\leq \int_{\varphi(X)}^Y \frac{c'}{8c}2(q_1 - q_2) + \frac{c'}{8c}q_2(w_2 - w_1) + \frac{c'}{8c}q_2(z_1 - z_2)dY$$

$$\leq \frac{c'}{4c^2}\|q_1 - q_2\|e^{K(X+Y)}|_{\varphi(X)}^Y$$

$$+ \frac{c'}{8c^2}\|z_1 - z_2\|\frac{1}{2C_1 + K}e^{(2C_1+K)(X+Y)+8C_1E_0+K_1C_1}|_{\varphi(X)}^Y$$

$$+ \frac{c'}{8c^2}\|w_2 - w_1\|\frac{1}{2C_1 + K}e^{(2C_1+K)(X+Y)+8C_1E_0+K_1C_1}|_{\varphi(X)}^Y.$$

We denote some constant that only depends on K,

$$\xi_1 := \frac{c'}{4c^2}e^{K(X+Y)}|_{\varphi(X)}^Y,$$

$$\xi_2 := \frac{c'}{8c^2}\frac{1}{2C_1 + K}e^{(2C_1+K)(X+Y)+8C_1E_0+K_1C_1}|_{\varphi(X)}^Y.$$

Now, we compute (3.14),

$$(3.14) = \int_{\varphi(X)}^Y \frac{q_2}{8c}g(v)(1+\cos z_2)(1+\cos w_2) - \frac{q_1}{8c}g(v)(1+\cos z_2)(1+\cos w_2)$$

$$+ \frac{q_1}{8c} g(v)(1 + \cos z_2)(1 + \cos w_2) - \frac{q_1}{8c} g(v)(1 + \cos z_1)(1 + \cos w_1) dY$$

$$= \int_{\varphi(X)}^{Y} \frac{1}{8c} g(v)(1 + \cos z_2)(1 + \cos w_2)(q_2 - q_1)$$

$$+ \frac{q_1}{8c} g(v)(1 + \cos z_2)(\cos w_2 - \cos w_1) + \frac{q_1}{8c} g(v)(1 + \cos w_1)(\cos z_2 - \cos z_1)$$

$$\leq \frac{1}{8c} K_0 \frac{Y}{\varphi(X)} 4(q_2 - q_1) dY + \frac{1}{4c} K_0 \int_{\varphi(X)}^{Y} q_1(w_2 - w_1) dY$$

$$+ \frac{1}{4c} K_0 \int_{\varphi(x)}^{Y} q_1(w_2 - w_1) dY$$

$$\leq \frac{1}{2c} K_0 \|q_2 - q_1\| \frac{1}{K} e^{K(X+Y)}\big|_{\varphi(X)}^{Y} + \frac{1}{4c} K_0 \|w_2 - w_1\| \frac{1}{2C_1 + K} e^{(2C_1+K)(X+Y)+8C_1 E_0 + K_1 C_1}\big|_{\varphi(X)}^{Y}$$

$$+ \frac{1}{4c} K_0 \|z_2 - z_1\| \frac{1}{2C_1 + K} e^{(2C_1+K)(X+Y)+8C_1 E_0 + K_1 C_1}\big|_{\varphi(X)}^{Y}.$$

We denote some constant that only depends on K,

$$\xi_3 := \frac{1}{2c} K_0 \frac{1}{K} e^{K(X+Y)}\big|_{\varphi(X)}^{Y},$$

$$\xi_4 := \frac{1}{4c} K_0 \frac{1}{2C_1 + K} e^{(2C_1+K)(X+Y)+8C_1 E_0 + K_1 C_1}\big|_{\varphi(X)}^{Y}.$$

And thus, we have

$$\|\tilde{w}_1 - \tilde{w}_2\| = e^{-K(X)}|\tilde{w}_1 - \tilde{w}_2| = e^{-K(X)}|(3.15) + (3.16))|$$

$$\leq e^{-K(X)}[(\xi_1 + \xi_3)\|q1 - q_2\| + (\xi_2 + \xi_4)\|w_1 - w_2\| + (\xi_2 + \xi_4)\|z_1 - z_2\|]$$

$$\leq e^{-K(X)}(\xi_1 + \xi_3)\|q1 - q_2\| + e^{-K(X)}(\xi_2 + \xi_4)\|w_1 - w_2\| + e^{-K(X)}(\xi_2 + \xi_4)\|z_1 - z_2\|$$

We then denote that

$$L_{1,3} := e^{-K(X)}(\xi_1 + \xi_3),$$

$$L_{2,4} := e^{-K(X)}(\xi_2 + \xi_4).$$

So that

$$\|\tilde{w}_1 - \tilde{w}_2\| \leq L_{1,3}\|q1 - q_2\| + L_{2,4}\|w_1 - w_2\| + L_{2,4}\|z_1 - z_2\|.$$

Choose K large enough so that $0 < L_{1,3}, L_{2,4} < 1$ And the process of proving the Lipschitz condition of $\|\tilde{z}_1 - \tilde{z}_2\|$ is identical to the process above.

Now we are proving the Lipschitz condition on $\|\tilde{p}_1 - \tilde{p}_2\|$,

$$\tilde{p}_1 - \tilde{p}_2 = \int_{\varphi(X)}^{Y} \frac{c'}{8c^2}[\sin z_1 - \sin w_1]p_1 q_1 - \frac{1}{8c} p_1 q_1 \sin w_1 \, g(v)(1 + \cos z_1)$$

$$- \frac{c'}{8c^2}[\sin z_2 - \sin w_2]p_2 q_2 + \frac{1}{8c} p_2 q_2 \sin w_2 \, g(v)(1 + \cos z_2) dY$$

$$= \int_{\varphi(X)}^{Y} \frac{c'}{8c^2}[\sin z_1 - \sin w_1]p_1q_1 - \frac{c'}{8c^2}[\sin z_2 - \sin w_2]p_2q_2 dY$$

$$+ \int_{\varphi(X)}^{Y} \frac{1}{8c}p_2q_2 \sin w_2\, g(v)(1 + \cos z_2) - \frac{1}{8c}p_1q_1 \sin w_1\, g(v)(1 + \cos z_1) dY.$$

We denote that

$$(3.15) \qquad \int_{\varphi(X)}^{Y} \frac{c'}{8c^2}[\sin z_1 - \sin w_1]p_1q_1 - \frac{c'}{8c^2}[\sin z_2 - \sin w_2]p_2q_2 dY,$$

and

$$(3.16) \qquad \int_{\varphi(X)}^{Y} \frac{1}{8c}p_2q_2 \sin w_2\, g(v)(1 + \cos z_2) - \frac{1}{8c}p_1q_1 \sin w_1\, g(v)(1 + \cos z_1) dY.$$

Then compute (3.15) and get,

$$(3.15) = \int_{\varphi(X)}^{Y} \frac{c'}{8c^2}[\sin z_1 - \sin w_1]p_1q_1 dY - \frac{c'}{8c^2}[\sin z_1 - \sin w_1]p_1q_2$$

$$+ \frac{c'}{8c^2}[\sin z_1 - \sin w_1]p_1q_2 - \frac{c'}{8c^2}[\sin z_2 - \sin w_2]p_2q_2 dY$$

$$= \int_{\varphi(X)}^{Y} \frac{c'}{8c^2}[\sin z_1 - \sin w_1]p_1(q_1 - q_2) + \frac{c'}{8c^2}[\sin z_1 - \sin w_1]q_2(p_1 - p_2)$$

$$+ \frac{c'}{8c^2}[\sin w_2 - \sin w_1]p_2q_2 + \frac{c'}{8c^2}[\sin z_1 - \sin z_2]p_2q_2 dY$$

$$\leq \frac{c'}{4c^2} \int_{\varphi(X)}^{Y} p_1|q_1 - q_2|e^{-K(X+Y)}e^{K(X+Y)}dY + \frac{c'}{4c^2} \int_{\varphi(X)}^{Y} q_2|p_1 - p_2|e^{-K(X+Y)}e^{K(X+Y)}dY$$

$$+ \frac{c'}{8c^2} \int_{\varphi(X)}^{Y} |w_2 - w_1|e^{-k(X+Y)}e^{k(X+Y)}p_2q_2 dY + \frac{c'}{8c^2} \int_{\varphi(X)}^{Y} |z_1 - z_2|e^{-k(X+Y)}e^{k(X+Y)}p_2q_2 dY$$

$$\leq \|q_1 - q_2\| \frac{c'}{4c^2} \frac{1}{2C_1 + K} e^{(2C_1+K)(X+Y)+8C_1E_0+k_1C_1}|_{\varphi(X)}^{Y}$$

$$+ \|p_1 - p_2\| \frac{c'}{4c^2} \frac{1}{2C_1 + K} e^{(2C_1+K)(X+Y)+8C_1E_0+k_1C_1}|_{\varphi(X)}^{Y}$$

$$+ \|w_2 - w_1\| \frac{c'}{8c^2} \frac{1}{4C_1 + K} e^{(4C_1+K)(X+Y)+16C_1E_0+2k_1C_1}|_{\varphi(X)}^{Y}$$

$$+ \|z_1 - z_2\| \frac{c'}{8c^2} \frac{1}{4C_1 + K} e^{(4C_1+K)(X+Y)+16C_1E_0+2k_1C_1}|_{\varphi(X)}^{Y}.$$

We denote that

$$\psi_1 = \frac{c'}{4c^2} \frac{1}{2C_1 + K} e^{(2C_1+K)(X+Y)+8C_1E_0+k_1C_1}|_{\varphi(X)}^{Y},$$

$$\psi_2 = \frac{c'}{8c^2} \frac{1}{4C_1 + K} e^{(4C_1+K)(X+Y)+16C_1E_0+2k_1C_1}|_{\varphi(X)}^{Y}.$$

Now,we compute (3.16),

$$(3.16) = \int_{\varphi(X)}^{Y} \frac{1}{8c}p_2q_2 \sin w_2\, g(v)(1 + \cos z_2) - \frac{1}{8c}p_1q_2 \sin w_2\, g(v)(1 + \cos z_2)$$

$$+ \frac{1}{8c}p_1q_2 \sin w_2\, g(v)(1 + \cos z_2) - \frac{1}{8c}p_1q_1 \sin w_2\, g(v)(1 + \cos z_2)$$

$$+ \frac{1}{8c} p_1 q_1 \sin w_2 \, g(v)(1 + \cos z_2) - \frac{1}{8c} p_1 q_1 \sin w_1 \, g(v)(1 + \cos z_2)$$

$$+ \frac{1}{8c} p_1 q_1 \sin w_1 \, g(v)(1 + \cos z_2) - \frac{1}{8c} p_1 q_1 \sin w_1 \, g(v)(1 + \cos z_1) dY$$

$$= \int_{\varphi(X)}^{Y} \frac{1}{8c} q_2 \sin w_2 \, g(v)(1 + \cos z_2)(p_2 - p_1) + \frac{1}{8c} p_1 \sin w_2 \, g(v)(1 + \cos z_2)(q_2 - q_1)$$

$$+ \frac{1}{8c} p_1 q_1 \, g(v)(1 + \cos z_2)(\sin w_2 - \sin w_1) + \frac{1}{8c} p_1 q_1 \, g(v) \sin w_1(\cos z_2 - \cos z_1) dY$$

$$\leq \frac{K_0}{4c} \int_{\varphi(X)}^{Y} q_2(p_2 - p_1) e^{-K(X+Y)} e^{K(X+Y)} dY + \frac{K_0}{4c} \int_{\varphi(X)}^{Y} p_1(q_2 - q_1) e^{-K(X+Y)} e^{K(X+Y)} dY$$

$$+ \frac{K_0}{4c} \int_{\varphi(X)}^{Y} p_1 q_1(w_2 - w_1) e^{-K(X+Y)} e^{K(X+Y)} dY + \frac{K_0}{8c} \int_{\varphi(X)}^{Y} p_1 q_2(z_2 - z_1) e^{-K(X+Y)} e^{K(X+Y)} dY$$

$$\leq \|p_2 - p_1\| \frac{K_0}{4c} \frac{1}{2C_1 + K} e^{(2C_1 + K)(X+Y) + 8C_1 E_0 + C_1 K} |_{\varphi(X)}^{Y}$$

$$+ \|q_2 - q_1\| \frac{K_0}{4c} \frac{1}{2C_1 + K} e^{(2C_1 + K)(X+Y) + 8C_1 E_0 + C_1 K} |_{\varphi(X)}^{Y}$$

$$+ \|w_2 - w_1\| \frac{K_0}{4c} \frac{1}{2C_1 + K} e^{(4C_1 + K)(X+Y) + 16C_1 E_0 + 2C_1 K} |_{\varphi(X)}^{Y}$$

$$+ \|z_2 - z_1\| \frac{K_0}{8c} \frac{1}{2C_1 + K} e^{(4C_1 + K)(X+Y) + 16C_1 E_0 + 2C_1 K} |_{\varphi(X)}^{Y}.$$

We denote some constant that only dependent on the choose of K,

$$\psi_3 := \frac{K_0}{4c} \frac{1}{2C_1 + K} e^{(2C_1 + K)(X+Y) + 8C_1 E_0 + C_1 K} |_{\varphi(X)}^{Y},$$

$$\psi_4 := \frac{K_0}{4c} \frac{1}{2C_1 + K} e^{(4C_1 + K)(X+Y) + 16C_1 E_0 + 2C_1 K} |_{\varphi(X)}^{Y},$$

$$\psi_5 := \frac{K_0}{8c} \frac{1}{2C_1 + K} e^{(4C_1 + K)(X+Y) + 16C_1 E_0 + 2C_1 K} |_{\varphi(X)}^{Y}.$$

Thus, we can compute that:

$$\|\tilde{p}_1 - \tilde{p}_2\| = e^{-K(X+Y)} |p_1 - p_2| = e^{-K(X+Y)} |(3.17) + (3.18)|$$

$$\leq e^{-K(X+Y)} [(\psi_3 + \psi_1)\|q_2 - q_1\| + (\psi_3 + \psi_1)\|p_2 - p_1\|$$

$$+ (\psi_4 + \psi_2)\|w_2 - w_1\| + (\psi_2 + \psi_5)\|z_2 - z_1\|]$$

$$\leq e^{-K(X+Y)}(\psi_3 + \psi_1)\|q_2 - q_1\| + e^{-K(X+Y)}(\psi_3 + \psi_1)\|p_2 - p_1\|$$

$$+ e^{-K(X+Y)}(\psi_4 + \psi_2)\|w_2 - w_1\| + e^{-K(X+Y)}(\psi_5 + \psi_2)\|z_2 - z_1\|.$$

We denote some constant that only dependent on the choose of K,

$$L_{3,1} := e^{-K(X+Y)}(\psi_3 + \psi_1),$$

$$L_{4,2} := e^{-K(X+Y)}(\psi_4 + \psi_2),$$

$$L_{5,2} := e^{-K(X+Y)}(\psi_5 + \psi_2).$$

We choose the K large enough so that $0 < L_{3,1}, L_{4,2}, L_{5,2} < 1$ And the process of proving the Lipschitz condition of $\|\tilde{q}_1 - \tilde{q}_2\|$ is identical to the process above.

We now prove that Lipschitz condition of $\tilde{v}$.

$$\tilde{v}_1 - \tilde{v}_2 = \int_{\varphi(X)}^{Y} \frac{\sin z_1}{4c} q_1 - \frac{\sin z_2}{4c} q_2 dY$$

$$= \int_{\varphi(X)}^{Y} \frac{\sin z_1}{4c}(q_1 - q_2)dY + \int_{\varphi(X)}^{Y} \frac{1}{4c} q_2(\sin z_1 - \sin z_2)dY,$$

$$|\tilde{x}_1 - \tilde{x}_2| \le \int_{\varphi(X)}^{Y} \frac{1}{4c}|q_1 - q_2|dY + \int_{\varphi(X)}^{Y} \frac{1}{4c} q_2 1 \times |z_1 - z_2|dY$$

$$\le \|q_1 - q_2\| \frac{1}{4c} \frac{1}{K} e^{K(X+Y)}|_{\varphi(X)}^{Y} + \|z_1 - z_2\| \frac{1}{4c} \frac{1}{2C_1 + K} e^{(2C_1+K)(X+Y)+8C_1E_0+C_1K_1}|_{\varphi(X)}^{Y}.$$

We denote that

$$\eta_1 := \frac{1}{4c} \frac{1}{K} e^{K(X+Y)}|_{\varphi(X)}^{Y},$$

$$\eta_2 := \frac{1}{4c} \frac{1}{2C_1 + K} e^{(2C_1+K)(X+Y)+8C_1E_0+C_1K_1}|_{\varphi(X)}^{Y}.$$

Thus we have

$$\|\tilde{v}_1 - \tilde{v}_2\| = e^{-K(X+Y)}|\tilde{v}_1 - \tilde{v}_2|$$

$$\le e^{-K(X+Y)}\eta_1\|q_1 - q_2\| + e^{-K(X+Y)}\eta_2\|z_1 - z_2\|.$$

And we denote that

$$L_1 := e^{-K(X+Y)}\eta_1,$$

$$L_2 := e^{-K(X+Y)}\eta_2.$$

We choose K Large enough such that $0 < L_1, L_2 < 1$.

The computation shows that $L = \dfrac{C(\mathcal{E}_0, \mathcal{K})}{K}$, where $C(\mathcal{E}_0, \mathcal{K})$ is a constant depends on $\mathcal{E}_0$ and $\mathcal{K}$. By choosing K sufficiently large, we can guarantee $L < 1$. Hence, the uniform Lipschitz condition is proved. By the fixed point theorem, the solution in the X-Y plane exists and is unique. $\square$

If the initial data in (1.2) are smooth, then the solutions of (2.28) - (2.30) with boundary condition (2.33) - (2.35) are smooth functions with variables (X, Y). Also, if there is a sequence of smooth functions $(v_0^m(x), v_1^m(x))_{m\ge 1}$ with the following conditions:

$$v_0^m(x) \to v_0(x) , v_1^m(x) \to v_1(x), (v_0^m(x))_x \to (v_0(x))_x,$$

uniformly on a compact subset of $\mathbb{R}$. Then

$$(p^m, q^m, w^m, z^m, v^m) \to (p, q, w, z, v),$$

uniformly on some bounded subsets of X-Y plane.

4. WEAK SOLUTIONS

In this section, we construct a map $v(X, Y) \to v(t, x)$. That is to write (X, Y) in terms of (t, x) so we obtain a solution to the Cauchy problem (1.3), (1.4). The map $(X, Y) \mapsto (t, x)$ can be obtain in the following way. We plug in $f = x$ and $f = t$ into the equation (2.22), and get

$$(4.1) \qquad \begin{cases} c = 2cX_x x_X, \\ -c = -2cY_x x_Y, \\ 1 = 2cX_x t_X, \\ 1 = -2cY_x t_Y. \end{cases}$$

And by applying (2.24) we have

$$(4.2) \qquad \begin{cases} X_x = \dfrac{2}{(1 + \cos w)p}, \\ Y_x = \dfrac{-2}{(1 + \cos z)q}, \\ X_t = \dfrac{2c}{(1 + \cos w)p}, \\ Y_t = \dfrac{-2c}{(1 + \cos z)q}. \end{cases}$$

We assume that the partial derivatives above valid for points that $w, z \neq -\pi$. Thus, we have

$$(4.3) \qquad \begin{cases} x_X = \dfrac{1}{2X_x} = \dfrac{(1 + \cos w)p}{4}, \\ x_Y = \dfrac{1}{2Y_x} = \dfrac{-(1 + \cos z)q}{4}, \end{cases}$$

$$(4.4) \qquad \begin{cases} t_X = \dfrac{1}{2cX_x} = \dfrac{(1 + \cos w)p}{4c}, \\ t_Y = \dfrac{1}{-2cY_x} = \dfrac{(1 + \cos z)q}{4c}. \end{cases}$$

A computation shows that $x_{XY} = x_{YX}$ and $t_{XY} = t_{YX}$

$$\begin{aligned} x_{XY} &= \frac{(1 + \cos w)p_Y}{4} - \frac{p \sin w \, w_Y}{4} \\ &= \frac{c'pq}{32c}[\sin z - \sin w + \sin(z - w)], \\ x_{YX} &= \frac{(1 + \cos z)q_X}{4} - \frac{q \sin z \, z_Y}{4} \\ &= \frac{c'pq}{32c}[\sin z - \sin w + \sin(z - w)]. \end{aligned}$$

So, $x_{XY} = x_{YX}$.

And similarly, we can compute that $t_{XY} = t_{YX}$. Thus, the two equation in (4.3) are equivalent: $x_{XY} = x_{YX}$. And the two equation in (4.4) are equivalent since $t_{XY} = t_{YX}$. We

can recover the solution in terms of (t, x) with function $x = x(X, Y)$ by integrating one of the equation in (4.3). Also, we can write $t = t(X, Y)$ by integrating one of the equation in (4.4).

Next, we prove that the function v is a weak solution to (1.3). From (1.5), we want to show that

$$0 = \iint \phi_t v_t - [c(v)\phi]_x [c(v)v_x] - \frac{\phi}{2} g(v) dx dt.$$

In fact, it is equivalent to prove:

$$0 = \iint (v_t + cv_x)[\phi_t - (c(v)\phi)_x] + (v_t - cv_x)[\phi_t + (c(v)\phi)_x] - \phi g(v) dx dt$$

$$= \iint -(\frac{\sin w}{2} p)_Y \phi - (\frac{\sin z}{2} q)_X \phi + \frac{c'pq}{8c}[\sin w \frac{1 + \cos z}{2} - \sin z \frac{1 + \cos w}{2}]\phi(\tan \frac{z}{2} - \tan \frac{w}{2}) dX dY$$

$$- \iint \phi g(v) dx dt$$

$$:= \text{I} + \text{II}.$$

We define I and II later, and where in the last step, we have used (4.2),

$$dx dt = \begin{vmatrix} \dfrac{dx}{dX} & \dfrac{dx}{dY} \\ \dfrac{dt}{dX} & \dfrac{dt}{dY} \end{vmatrix} dX dY = \frac{pq}{2c(1 + R^2)(1 + S^2)} dX dY.$$

And used the following identities derived from (2.30),

$$(4.5) \qquad \begin{cases} \dfrac{1}{1 + R^2} = \dfrac{1 + \cos w}{2}, \\ \dfrac{1}{1 + S^2} = \dfrac{1 + \cos z}{2}, \end{cases}$$

$$(4.6) \qquad \begin{cases} \dfrac{R}{1 + R^2} = \dfrac{\sin w}{2}, \\ \dfrac{S}{1 + S^2} = \dfrac{\sin z}{2}. \end{cases}$$

We denote I and II as follows

$$(4.7) \qquad \begin{aligned} \text{I} = \iint &-\left(\frac{\sin w}{2} p\right)_Y \phi - \left(\frac{\sin z}{2} q\right)_X \phi \\ &+ \frac{c'pq}{8c}\left[\sin w \frac{1 + \cos z}{2} - \sin z \frac{1 + \cos w}{2}\right] \phi \left(\tan \frac{z}{2} - \tan \frac{w}{2}\right) dX dY, \end{aligned}$$

and

$$(4.8) \qquad \text{II} = \iint \phi g(v) dx dt.$$

A computation on I with (2.28) - (2.30) shows that

$$\text{I} = \iint -\left(\frac{\cos w}{2} w_Y p + \frac{\sin w}{2} p_Y\right)\phi - \left(\frac{\cos z}{2} z_X q + \frac{\sin z}{2} q_X\right)\phi + \frac{c'pq}{8c^2}[\cos(w + z) - 1]\phi dX dY$$
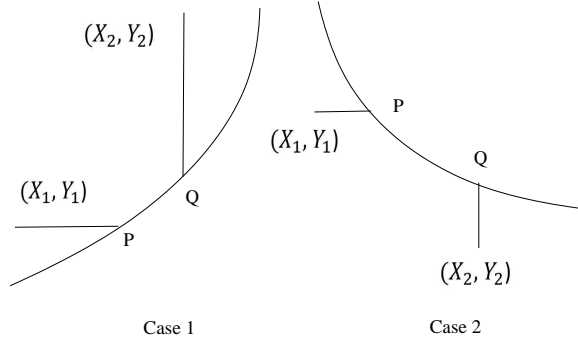
$$= \iint \frac{pq}{16c} \phi g(v)(\cos w + \cos z + 2 + 2 \cos z \cos w$$

FIGURE 4. The paths of integration.

$$+ \cos^2 w \cos z + \cos w \cos^2 z + \sin^2 w \cos z + \sin^2 z \cos w) dXdY$$

$$= \iint \frac{pq}{8c} \phi \, g(v)(1 + \cos z + \cos w + \cos z \cos w) dXdY.$$

A computation on II shows that

$$\text{II} = \iint \phi \, g(v) \frac{pq}{2c(1+s^2)(1+R^2)} dXdY$$

$$= \iint \frac{pq}{8c} \phi \, g(v)(1 + \cos z + \cos w + \cos z \cos w) dXdY.$$

Clearly, I = II. Thus the integral (1.5) holds since

$$0 = \iint - \left(\frac{\sin w}{2} p \right)_Y \phi - \left(\frac{\sin z}{2} q \right)_X \phi$$

$$+ \frac{c'pq}{8c} \left[\sin w \frac{1 + \cos z}{2} - \sin z \frac{1 + \cos w}{2} \right] \phi \left(\tan \frac{z}{2} - \tan \frac{w}{2} \right) dXdY - \iint \phi \, g(v) dxdt$$

$$0 = \text{I} - \text{II},$$

where I is defined in (4.7), and II is defined in (4.8).

Next, we define v as a function in terms of the original variables (t, x). We invert the map $(X, Y) \mapsto (t, x)$ and then we have $v(t, x) = v(X(t, x), Y(t, x))$. Given arbitrary (t^*, x^*) in the t-x plane, we choose arbitrary point (X^*, Y^*) in X-Y plane such that $t^* = t(X^*, Y^*)$ and $x^* = x(X^*, Y^*)$. We define that $v(t^*, x^*) = v(X^*, Y^*)$ and assume that there are two different points $(t(X_1, Y_1), x(X_1, Y_1)) = (t(X_2, Y_2), x(X_2, Y_2)) = (t^*, x^*)$. We consider two cases: case 1: $X_1 \leq X_2, Y_1 \leq Y_2$, and case 2: $X_1 \leq X_2, Y_1 \geq Y_2$.   Case 1: $X_1 \leq X_2, Y_1 \leq Y_2$. We consider the set

$$\Gamma_{x^*} := \{(X, Y) : x(X, Y) \leq x^*\}.$$

We denote $\partial \Gamma_{x^*}$ as the boundary of Γ_{x^*}. By (4.3), we observe that x is increasing with X increasing and x is decreasing with Y increasing. Thus, this boundary can be write as a Lipschitz continuous function denoted as $X - Y = \phi(X - Y)$.   We construct the Lipschitz continuous curve γ with the following properties:

- a horizontal line segment connecting (X_1, Y_1) with a point $P = (X_P, Y_P) \in \partial\Gamma_{x^*}$ and $Y_P = Y_1$.
- a vertical line segment connecting (X_2, Y_2) with a point $Q = (X_Q, Y_Q) \in \partial\Gamma_{x^*}$ and $X_Q = X_2$.
- a part of $\partial\Gamma_{x^*}$.

Thus, we obtain a Lipschitz continuous parametrization of the curve $\gamma : [\xi_1, \xi_2] \mapsto \mathbb{R} \times \mathbb{R}$ where the parameter $\xi = X + Y$. By observing, the map $(X, Y) \mapsto (t, x)$ is constant along the curve γ. And (4.3) - (4.4) implies that

$$(4.9) \qquad (1 + \cos w)X_\xi = (1 + \cos z)Y_\xi = 0,$$

From (4.9),

$$(4.10) \qquad \sin w X_\xi = \sin z Y_\xi = 0.$$

Thus, by (4.10)

$$v(X_2, Y_2) - v(X_1, Y_1) = \int_\gamma (v_X dX + v_Y dY)$$
$$= \int_{\xi_1}^{\xi_2} \left(\frac{p \sin w}{4c} X_\xi - \frac{q \sin z}{4c} Y_\xi\right) d\xi = 0.$$

So our claim for case 1 is proved.

Case 2: $X_1 \le X_2, Y_1 \ge Y_2$. We consider the set:

$$\Gamma_{t^*} := \{(X, Y) : t(X, Y) \le t^*\}.$$

And we do the same process as we did in case 1. Construct γ connecting (X_1, X_2) and (X_2, Y_2) as Figure 3 case 2 indicates.

Next, we prove the function $v(t, x) = v(X(t, x), Y(t, x))$ is Hölder-$\frac{1}{2}$ continuous on the bounded sets. To prove this, we need to consider characteristic curve such that $t \mapsto x^+(t)$ with $\bar{x}^+ = c(v)$. For some fixed $\bar{Y}$, this can be parametrized by the function $X \mapsto (t(X, \bar{Y}), x(X, \bar{Y}))$. By (2.20), (2.22), (2.24) and (2.30),

$$\int_0^\tau [v_t + c(v)v_x]^2 dt = \int_{X_0}^{X_\tau} (2cX_x v_X)^2 \frac{1}{2X_t} dX$$
$$= \int_{X_0}^{X_\tau} \frac{p}{2c} \sin^2\left(\frac{w}{2}\right) dX \le \int_{X_0}^{X_\tau} \frac{p}{2c} dX \le C_\tau.$$

Thus, we obtain that

$$(4.11) \qquad \int_0^\tau [v_t + c(v)v_x]^2 dt \le C_\tau.$$

Similarly, we integrate along backward characteristics curves $t \mapsto x^-(t)$ and find out that

$$(4.12) \qquad \int_0^\tau [v_t - c(v)v_x]^2 dt \le C_\tau.$$

Thus, since the speed of the characteristic curve is $+c(v)$ or $-c(v)$ and $c(v)$ is uniformly positive bounded. With the bounds (4.11) and (4.12), the function $v(t,x)$ is Hölder-$\frac{1}{2}$ continuous. $\square$

5. CONSERVED QUANTITIES

This section provides a proof of Theorem 1.2. Recalling (2.12) and (2.13), a straightforward computation shows that

$$E_t = \left(\frac{1}{2}v_t^2 + \frac{1}{2}c^2 v_x^2 + \frac{G(v)}{2} \right)_t,$$

$$= v_{tt}v_t + cc'v_t v_x^2 + c^2 v_x v_{xt} + \frac{1}{2}(v + v^3)v_t,$$

$$(c^2 M)_x = (-c^2 v_t v_x)_x = -2cc'v_x^2 v_t - c^2 v_{tx}v_x - c^2 v_t v_{xx},$$

$$E_t + (c^2 M)_x = v_t(v_{tt} - cc'v_x - c^2 v_{xx} + \frac{1}{2}g(v)) = 0,$$

and

$$M_t = -v_{tt}v_x - v_t v_{xt},$$

$$E_x = v_t v_{tx} + cc'v_x v_x^2 + c^2 v_x v_{xx} + \frac{1}{2}g(v)v_x,$$

$$M_t + E_x = -v_x \left(v_{tt} - cc'v_x^2 - c^2 v_{xx} - \frac{1}{2}g(v) \right) = 0.$$

Thus,

$$(5.1) \qquad \begin{cases} E_t + (c^2 M)_x = 0, \\ M_t + E_x = 0. \end{cases}$$

Also,

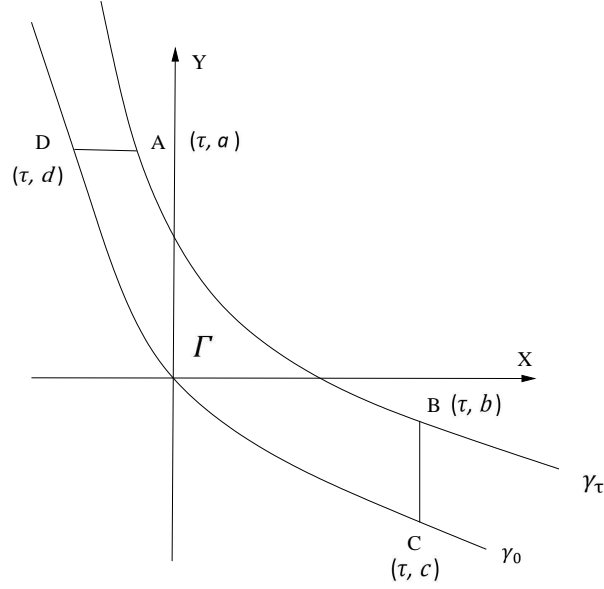$$(5.2) \qquad dx = \frac{(1 + \cos w)p}{4}dX - \frac{(1 + \cos z)q}{4}dY,$$

$$(5.3) \qquad dt = \frac{(1 + \cos w)p}{4c}dX + \frac{(1 + \cos z)q}{4c}dY,$$

which is closed. We want to show that $E dx - (c^2 M)dt$, $M dx - E dt$ are closed. Recalling (2.28) - (2.30), we write $E dx - (c^2 M)dt$, $M dx - E dt$ in terms of X, Y, and show that they are closed.

$$E dx - (c^2 M)dt =$$
$$(5.4) \qquad \left[\frac{(1 - \cos w)}{8} + \frac{(1 + \cos w)}{8}G(v) \right] p dX - \left[\frac{(1 - \cos z)}{8} + \frac{1 + \cos z}{8}G(v) \right] q dY,$$

$$M dx + E dt =$$
$$\left\{ \frac{(1 - \cos w)}{8c} + \frac{(1 + \cos w)}{8c}G(v) \right\} p dX + \left\{ \frac{1 - \cos z}{8c} + \frac{1 + \cos z}{8c}G(v) \right\} q dY.$$

FIGURE 5. The region Γ.

And we compute that

$$\left\{\left[\frac{(1-\cos w)}{8} + \frac{(1+\cos w)}{8}G(v)\right]p\right\}_Y$$

$$= \frac{-\sin w\, G(v)p}{8}\frac{c'}{8c^2}(\cos z - \cos w)q + \frac{(1+\cos w)\,g(v)p}{16}\frac{1}{4c}\sin zq$$

$$+ \frac{1+\cos w}{8}G(v)\frac{c'}{8c^2}[\sin z - \sin w]pq$$

$$= -\left\{\left[\frac{(1-\cos z)}{8} + \frac{1+\cos z}{8}G(v)\right]q\right\}_X,$$

and

$$\left\{\left[\frac{(1-\cos w)}{8c} + \frac{(1+\cos w)}{8c}G(v)\right]p\right\}_Y$$

$$= \frac{\sin wp}{64c^3}c'(\cos z - \cos w)q - \frac{\sin wpq}{64c^2}g(v)(1+\cos z)(1+\cos w)$$

$$+ \frac{(1-\cos w)c'}{64c^3}[\sin z - \sin w]pq - \frac{(1-\cos w)pq}{64c^2}\sin w\, g(v)(1+\cos z)$$

$$- \frac{\sin w}{8c}pG(v)\frac{c'}{8c^2}(\cos z - \cos w)q + \frac{(1+\cos w)}{8c}G(v)$$

$$\frac{c'}{8c^2}[\sin z - \sin w]pq + \frac{1+\cos w}{32c}p\, g(v)\frac{1}{c}\sin zq$$

$$= \left\{\left[\frac{1-\cos z}{8c} + \frac{1+\cos z}{8c}G(v)\right]q\right\}_X.$$

Thus $\{Edx - (c^2M)dt\}$, $\{Mdx - Edt\}$ are closed.

To prove the inequality (1.9), We fixed some $\tau > 0$, and the case $\tau < 0$ is identical. We

assume that for an arbitrary large $r > 0$. We define the set

(5.5) $$\Gamma := \{(X, Y) : 0 \le t(X, Y) \le \tau, X \le r, Y \le r\}.$$

We form the map $(X, Y) \mapsto (t, x)$ in the following pattern:

$$A \mapsto (\tau, a), \quad B \mapsto (\tau, b), \quad C \mapsto (0, c), \quad D \mapsto (0, d),$$

such that $a < b$ and $c > d$. Then, we can integrate the (5.4) along $\partial\Gamma$, the boundary of Γ.

$$\int_{AB} \left\{ \frac{(1 - \cos w)p}{8} + \frac{(1 + \cos w)p}{8} G(v) \right\} dX - \left\{ \frac{(1 - \cos z)q}{8} + \frac{(1 + \cos z)q}{8} G(v) \right\} dY$$

$$= \int_{DC} \left\{ \frac{(1 - \cos w)p}{8} + \frac{(1 + \cos w)p}{8} G(v) \right\} dX - \left\{ \frac{(1 - \cos z)q}{8} + \frac{(1 + \cos z)q}{8} G(v) \right\} dY$$

$$- \int_{DA} \left\{ \frac{(1 - \cos w)p}{8} + \frac{(1 + \cos w)p}{8} G(v) \right\} dX$$

$$- \int_{CB} \left\{ \frac{(1 - \cos z)q}{8} + \frac{(1 + \cos z)q}{8} G(v) \right\} dY$$

$$\le \int_{DC} \left\{ \frac{(1 - \cos w)p}{8} + \frac{(1 + \cos w)p}{8} G(v) \right\} dX - \left\{ \frac{(1 - \cos z)q}{8} + \frac{(1 + \cos z)q}{8} G(v) \right\} dY$$

$$\le \int_d^c \frac{1}{2} \left[v_t^2(0, x) + c^2(v(0, x))v_x^2(0, x) + \frac{1}{2} g(v(0, x)) \right] dx.$$

Also,

$$\int_a^b \frac{1}{2} \left[v_t^2(0, x) + c^2(v(0, x))v_x^2(0, x) + \frac{1}{2} g(v(0, x)) \right] dx$$

$$= \int_{AB \cap \{\cos w \ne -1\}} \left\{ \frac{(1 - \cos w)p}{8} + \frac{(1 + \cos w)p}{8} G(v) \right\} dX$$

$$- \left\{ \frac{(1 - \cos z)q}{8} + \frac{(1 + \cos z)q}{8} G(v) \right\} dY$$

$$\le \mathcal{E}_0.$$

Let $r \to \infty$, $a \to -\infty$ and $b \to +\infty$. We conclude that $\mathcal{E}(t) \le \mathcal{E}_0$. Thus, the inequity (1.9) is proved.

Now, we prove the Lipschitz condition on the map $t \mapsto v(t, \cdot)$ in the L^2 distance. First, for any fixed time τ, we define $\mu_\tau := \mu_\tau^- + \mu_\tau^+$ and μ_τ is the positive measure on the real lines. We define μ_τ^-, μ_τ^+ as follows.
we define $\Gamma_\tau := \{(X, Y) : t(X, Y) \le \tau\}$ and let γ_τ be the boundary of Γ_τ.

For any open interval $]a, b[$, we define $A = (X_A, Y_A), B = (X_B, Y_B)$ be points on the γ_τ such that

$x(A) = a$, and $X_P - Y_P \le X_A - Y_A$ for all points $P \in \gamma_\tau$ and $x(P) \le a$,

$x(B) = b$, and $X_B - Y_B \le X_P - Y_P$ for all points $P \in \gamma_\tau$ and $x(P) \ge b$.

Then we have

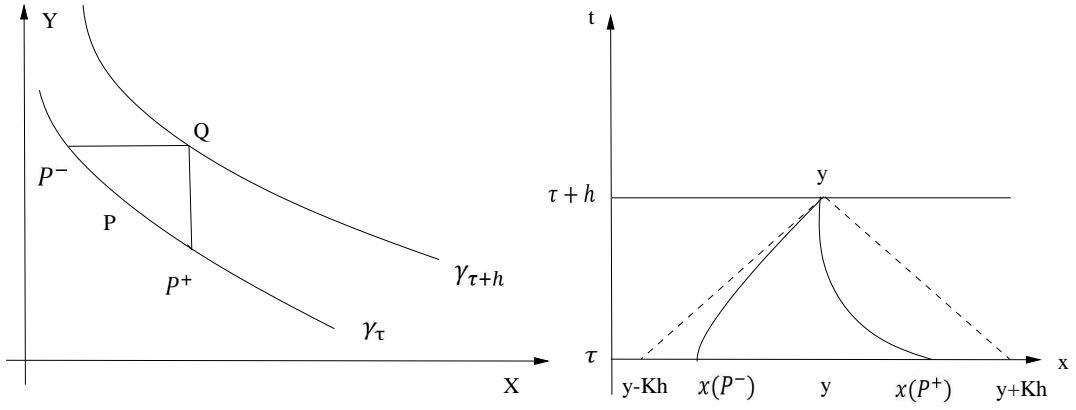(5.6) $$\mu_\tau := \mu_\tau^-(]a, b[) + \mu_\tau^+(]a, b[),$$

FIGURE 6. Proving Lipschitz condition.

and define that in the general case

$$(5.7) \qquad \mu_\tau^-(]a,b[) := \int_{AB} \left\{ \frac{(1-\cos w)p}{8} + \frac{(1+\cos w)p}{8} G(v) \right\} dX,$$

$$(5.8) \qquad \mu_\tau^+(]a,b[) := \int_{AB} -\left\{ \frac{(1-\cos z)q}{8} + \frac{(1+\cos z)q}{8} G(v) \right\} dY.$$

In the smooth case:

$$(5.9) \qquad \mu_\tau^-(]a,b[) := \frac{1}{4} \int_a^b R^2(\tau,x) dx,$$

$$(5.10) \qquad \mu_\tau^+(]a,b[) := \frac{1}{4} \int_a^b S^2(\tau,x) dx.$$

Clearly, μ^+ and μ^- are bounded positive measure. For all τ, we have $\mu_\tau(\mathbb{R}) = \mathcal{E}_0$ by (5.4). By (5.9)- (5.10) and (2.13) we compute that

$$\int_a^b c^2 u_x^2 dx = \int_a^b \frac{c^2(R-S)^2}{4c^2} dx = \int_a^b \frac{R^2 - 2RS + S^2}{4} dx$$
$$\leq \int_a^b \frac{R^2 + S^2}{2} dx = 2\mu(]a,b[).$$

Thus, for arbitrary a, b with $a < b$,

$$(5.11) \qquad |v(\tau,b) - v(\tau,a)|^2 \leq |b-a| \int_a^b v_x(\tau,y) dy \leq |b-a| 2\mathcal{K}^2 \mu_\tau(]a,b[).$$

For given $y \in R$ and $h > 0$, our goal is to estimate the $|v(\tau+h,y) - v(\tau,y)|$. We first denote that $\Gamma_{\tau+h}$ as the set $\Gamma_{\tau+h} := \{(X,Y) : t(X,Y) \leq \tau+h\}$ and denote that $\gamma_{\tau+h}$ to be the boundary of the set $\Gamma_{\tau+h}$.

Let $P = (P_X, P_Y)$ be points on $\gamma_{\tau+h}$ (as the figure 5(a) shows) such that $x(P) = y$, and $X_{\tilde{P}} - Y_{\tilde{P}} \leq X_P - Y_P$ for all $\tilde{P} \in \gamma_\tau$, $x(\tilde{P}) \leq x(P)$.

Let $Q = (Q_X, Q_Y)$ be points on $\gamma_{\tau+h}$ such that $x(Q) = y$ and $X_{\tilde{Q}} - Y_{\tilde{Q}} \leq X_Q - Y_Q$ for all $\tilde{Q} \in \gamma_{\tau+h}$, $x(\tilde{Q}) \leq x(Q)$.

So $X_P \leq X_Q$ and $Y_P \leq Y_Q$. Let $P^+ = (X_Q, Y^+) \in \gamma_\tau$ and $P^- = (X^-, Y_Q) \in \gamma_\tau$.

As shown in the figure 5, since the point $(\tau, x(P^+))$ lies on some characteristic curve with the speed $c(v) \leq \mathcal{K}$ and go through the point $(\tau + h, y)$, so $x(P^+) \in]y, y + \mathcal{K}h[$. Also, $x(P^-) \in]y - \mathcal{K}h, y[$, since point (τ, y) lies on some characteristic curve with the speed $-c(v) \geq -\mathcal{K}$ and go through the point $(\tau + h, y)$.

Thus, and by (2.30), we can compute that

$$
\begin{aligned}
|v(Q) - v(P^+)| &\leq \int_{Y^+}^{Y_Q} |v_Y(X_Q, Y)| dY \\
&= \int_{Y^+}^{Y_Q} \left(\frac{1 + \cos z}{4c} q\right)^{\frac{1}{2}} \left(\frac{1 - \cos z}{4c} q\right)^{\frac{1}{2}} dY \\
&\leq \left(\int_{Y^+}^{Y_Q} \frac{1 + \cos z}{4c} q dY\right)^{\frac{1}{2}} \left(\int_{Y^+}^{Y_Q} \frac{1 - \cos z}{4c} q dY\right)^{\frac{1}{2}} \\
&\leq \left(\int_{Y^+}^{Y_Q} \frac{1 + \cos z}{4c} q dY + \frac{1 + \cos w}{4c} p dX\right)^{\frac{1}{2}} \left(\int_{Y^+}^{Y_Q} \frac{1 - \cos z}{4c} q dY + \frac{1 - \cos w}{4c} p dY\right)^{\frac{1}{2}} \\
&\leq \left(\int_{\tau}^{\tau + h} 1 dt\right)^{\frac{1}{2}} \left(\int_{P^-}^{P^+} \frac{1 - \cos z}{4c} q dY + \frac{1 - \cos w}{4c} p dY\right)^{\frac{1}{2}} \\
&\leq h^{\frac{1}{2}} \left(\int_{P^-}^{P^+} \frac{1 - \cos z}{4c} q dY + \frac{1 - \cos w}{4c} p dY\right)^{\frac{1}{2}}.
\end{aligned}
$$

Thus

$$
(5.12) \qquad \left|v(Q) - v(P^+)\right| \leq h^{\frac{1}{2}} \left(\int_{P^-}^{P^+} \frac{1 - \cos z}{4c} q dY + \frac{1 - \cos w}{4c} p dY\right)^{\frac{1}{2}}.
$$

So, by (5.11) and (5.12) we compute that

$$
\begin{aligned}
|v(\tau + h, x) - v(\tau, x)|^2 &= |v(\tau + h, x) - v(t(P^+), x(P^+)) + v(t(P^+), x(P^+)) - v(\tau, x)|^2 \\
&\leq 2\{v(\tau + h, x) - v(t(P^+), x(P^+))\}^2 + 2\{v(t(P^+), x(P^+)) - v(\tau, x)\}^2 \\
&\leq 2\{v(Q) - v(P^+)\}^2 + 2\{v(P^+) - v(P)\}^2 \\
&\leq 2 \left[h^{\frac{1}{2}} \left(\int_{P^-}^{P^+} \frac{1 - \cos z}{4c} q dY + \frac{1 - \cos w}{4c} p dY\right)^{\frac{1}{2}}\right]^2 \\
&\quad + 2 \left[2\mathcal{K}^2(\mathcal{K}h)\mu_\tau(]x, x + h[)\right] \\
&\leq 4h\mu_\tau(]x - \mathcal{K}h, x + \mathcal{K}h[) + 4\mathcal{K}^3 h \mu_\tau(]x, x + h[) \\
&\leq 4h\mu_\tau(]x - \mathcal{K}h, x + \mathcal{K}h[)(1 + \mathcal{K}^3).
\end{aligned}
$$

Thus, for all $h > 0$,

$$
\|v(\tau + h, \cdot) - v(\tau, \cdot)\|_{L^2} = \left\{\int |v(\tau + h, x) - v(\tau, x)|^2 dx\right\}^{\frac{1}{2}}
$$

$$\leq \left\{ \int 4(1+\mathcal{K}^3)h\mu_\tau(]x-\mathcal{K}h, x+\mathcal{K}h[) \right\}^{\frac{1}{2}}$$

$$\leq \left\{ 4(\mathcal{K}^3+1)h^2\mu_\tau(\mathbb{R}) \right\}^{\frac{1}{2}}$$

$$\leq h[4(\mathcal{K}^3+1)\mathcal{E}_0]^{\frac{1}{2}}$$

$$\leq |\tau+h-\tau|L,$$

(5.13) $$\|v(\tau+h,\cdot)-v(\tau,\cdot)\|_{L^2} \leq h[4(\mathcal{K}^3+1)\mathcal{E}_0]^{\frac{1}{2}},$$

where $L = [4(\mathcal{K}^3+1)\mathcal{E}_0]^{\frac{1}{2}}$ is the Lipschitz constant. So, this proves the uniform Lipschitz continuous of the maps $t \mapsto v(t,\cdot)$. $\qquad\square$

6. Regularity of trajectories

In this section, we show that continuity of functions $t \mapsto v_t(t,\cdot)$ and $t \mapsto v_x(t,\cdot)$ as functions with function value in L^2. It completes the proof of Theorem 1.1.

We consider the that the initial data $(v_0)_x$ and v_1 are smooth functions with compact support. In this situation, the solution $v(X,Y)$ is smooth on the X-Y plane. Fix some time τ and denote that $\Gamma_\tau := \{(X,Y) : t(X,Y) \leq \tau\}$. γ_τ is the boundary of set Γ_τ. Then we claim that

(6.1) $$\frac{d}{dt}v(t,\cdot)|_{t=\tau} = v_t(\tau,\cdot).$$

By (2.21), (2.24), and (2.30),

$$v_t(\tau,x) := v_X X_t + v_Y Y_t$$

(6.2) $$= \frac{\sin w}{4c}p\frac{2c}{p(1+\cos w)} + \frac{\sin z}{4c}q\frac{2c}{q(1+\cos z)}$$

$$= \frac{\sin w}{2(1+\cos w)} + \frac{\sin z}{2(1+\cos z)}.$$

(6.2) define the value of $v_t(\tau,\cdot)$ at almost all the point of $x \in \mathbb{R}$. By the inequity (1.9) and $c(v) \geq \mathcal{K}^{-1}$,

(6.3) $$\int_{\mathbb{R}} |v_t(\tau,x)|^2 dx \leq \mathcal{K}^2 \mathcal{E}(\tau) \leq \mathcal{K}^2 \mathcal{E}_0.$$

Next, to prove (6.1), given $\epsilon > 0$, there exists finitely many disjoint intervals $[a_i, b_i]$ subsets of $\mathbb{R}$ with $i = 1, 2 ... N$. We call the $A_i, B_i \in \gamma_\tau$ with $x(A_i) = a_i$, $x(B_i) = b_i$. Then at every point P in the arcs $A_i B_i$ while $1 + \cos(w(P)) > \epsilon$ and $1 + \cos(z(P)) > \epsilon$,

$$\min\{1 + \cos(w(P)), 1 + \cos(z(P))\} \leq 2\epsilon.$$

We call that $J := \bigcup_{1 \leq i \leq N}[a_i, b_i]$ as the points P along the curve γ_τ that does not contain in any of the arcs $A_i B_i$ and denote that $J' := \mathbb{R} \backslash J$. Since $v(t,x)$ is smooth in a neighbourhood of the set $\{\tau\} \times J'$ and by the differentiability of v and apply the Minkowski's inequality,

$$\lim_{h \to 0} \frac{1}{h} \left\{ \int_{\mathbb{R}} |v(\tau+h,x)-v(\tau,x)-hv_t(\tau,x)|^p dx \right\}^{\frac{1}{p}}$$

$$\leq \lim_{h \to 0} \frac{1}{h} \left\{ \int_j |v(\tau + h, x) - v(\tau, x)|^p dx \right\}^{\frac{1}{p}} + \left\{ \int_J |v_t(\tau, x)|^p dx \right\}^{\frac{1}{p}}.$$

Now, we estimate the measure of the bad set J. Since $(1 + \cos w) < 2\epsilon(1 - \cos w)$ and $(1 + \cos z) < 2\epsilon(1 - \cos z)$,

$$meas(J) = \int_J dx = \sum_i \int_{A_i B_i} \frac{(1 + \cos w)p}{4} dX - \frac{(1 + \cos z)q}{4} dY$$

$$\leq 2\epsilon \sum_i \int_{A_i B_i} \frac{(1 - \cos w)p}{4} dX - \frac{(1 - \cos z)q}{4} dY$$

$$\leq 2\epsilon \int_{\gamma_\tau} \frac{(1 - \cos w)p}{4} dX - \frac{(1 - \cos z)q}{4} dY$$

$$\leq 2\epsilon \mathcal{E}_0.$$

Using Hölder's inequality with exponents $\frac{2}{p}$ and q, we choose $q = \frac{2}{2 - p}$ so that $\frac{p}{2} + \frac{1}{q} = 1$. By (5.13),

$$\int_J |v(\tau + h, x) - v(\tau, x)|^p dx \leq meas(J)^{\frac{1}{q}} \left\{ \int_J |v(\tau, x) - v(\tau, x)|^2 dx \right\}^{\frac{p}{2}}$$

$$\leq [2\epsilon \mathcal{E}_0]^{\frac{1}{q}} + \left\{ h[4(\mathcal{K}^3 + 1)\mathcal{E}_0]^{\frac{1}{2}} \right\}^p.$$

Thus,

$$\limsup_{h \to 0} \frac{1}{h} \left\{ \int_J |v(\tau + h, x) - v(\tau, x)|^p dx \right\}^{\frac{1}{p}}$$

$$\leq [2\epsilon \mathcal{E}_0]^{\frac{1}{pq}} + h[4(\mathcal{K}^3 + 1)\mathcal{E}_0]^{\frac{1}{2}}.$$

Similarly, and by (6.3) we estimate that

$$\int_J |v_t(\tau, x)|^p dx \leq [meas(J)]^{\frac{1}{q}} \left\{ \int_J |v_t(\tau, x)|^2 dx \right\}^{\frac{p}{2}}.$$

Thus,

$$\left\{ \int_J |v_t(\tau, x)|^p dx \right\}^{\frac{1}{p}} \leq [2\epsilon \mathcal{E}_0]^{\frac{1}{pq}} [\mathcal{K}^2 \mathcal{E}_0]^{\frac{1}{2}}.$$

Since $\epsilon > 0$ is arbitrary, so we conclude that

$$\lim_{h \to 0} \frac{1}{h} \left\{ \int_{\mathbb{R}} |v(\tau + h, x) - v(\tau, x) - h v_t(\tau, x)|^p dx \right\}^{\frac{1}{p}} = 0.$$

Next, we prove the continuity of the map $t \mapsto v_t$. First, we fix $\epsilon > 0$ and consider disjoint intervals $[a_i, b_i]$ subsets of $\mathbb{R}$ with $i = 1, 2...N$. We call the $A_i, B_i \in \gamma_\tau$ with $x(A_i) = a_i$, $x(B_i) = b_i$. Since v is a smooth function on the neighbourhood of $\{\tau\} \times J'$. By Hölder's inequality and Minkowski's inequality, we estimate that

$$\limsup_{h \to 0} \int_J |v_t(\tau + h, x) - v_t(\tau, x)|^p dx$$

$$\leq \limsup_{h \to 0} \int_J |v_t(\tau + h, x) - v_t(\tau, x)|^p dx$$

$$\leq \limsup_{h \to 0} [meas(J)]^{\frac{1}{q}} \left\{ \int_J |v_t(\tau + h, x) - v_t(\tau, x)|^2 dx \right\}^{\frac{q}{2}}$$

$$\leq \limsup_{h \to 0} [2\epsilon \mathcal{E}_0]^{\frac{1}{q}} \left\{ \|v_t(\tau + h, \cdot)\|_{L^2} + \|v_t(\tau, \cdot)\|_{L^2} \right\}^{\frac{q}{2}}$$

$$\leq [2\epsilon \mathcal{E}_0]^{\frac{1}{q}} [4\mathcal{E}_0]^p.$$

Since the $\epsilon > 0$ is arbitrary, so we prove the continuity.

For general initial data $(v_0)_x$, $v_1 \in H^1$, we consider a sequence of initial data $v_0^n \to v_0$, $(v_0^n)_x \to (v_0)_x$, and $v_1^n \to v_1$ in $\in H^1$ for all $n \in \mathbb{N}$, $(v_0^n)_x$, v_0^n, $v_1^n \in C_c^\infty$. The continuity of the map $t \mapsto v_x(t, \cdot)$ with values in L^p and $1 \leq p < 2$ can be proved in the same way as above.

7. Energy conservation

In this section, we provide proof of Theorem 1.3. First, we define the wave interaction potential as

$$(7.1) \qquad \Lambda(t) := (\mu_t^- \otimes \mu_t^+)\{(x, y) : x > y\},$$

where the μ_t^- and μ_t^+ are defined in (5.7) and (5.8). And since μ_t^- and μ_t^+ are absolutely continuous in Lebesgue measure, so (5.9) and (5.10) holds and (7.1) implies that

$$(7.2) \qquad \Lambda(t) = \frac{1}{4} \iint_{x > y} R^2(t, x) S^2(t, x) dx dy.$$

Lemma 7.1. *There exists a Lipschitz constant L_0 such that*

$$\Lambda(t) - \Lambda(s) \leq L_0(t - s),$$

with $t > s > 0$. So the map $t \mapsto \Lambda(t)$ has bounded variation.

The Lemma is proved later in this section.

To prove Theorem 1.3, we need to consider three sets

$$\Omega_1 := \{(X, Y) : w(X, Y) = -\pi, \ z(X, Y) \neq -\pi, \ c'(v(X, Y)) \neq 0\},$$

$$\Omega_2 := \{(X, Y) : w(X, Y) \neq \pi, \ z(X, Y) = -\pi, \ c'(v(X, Y)) \neq 0\},$$

$$\Omega_3 := \{(X, Y) : w(X, Y) = -\pi, \ z(X, Y) = -\pi, \ c'(v(X, Y)) \neq 0\}.$$

From (2.28), and since $w_Y \neq 0$ on Ω_1 and $z_X \neq 0$ on Ω_2, so that $meas(\Omega_1) = 0$ and $meas(\Omega_2) = 0$.

We define Ω_3^* be the set of Lebesgue points of Ω_3 and want to show that

$$(7.3) \qquad meas(\{t(X, Y) : (X, Y) \in \Omega_3^*\}) = 0$$

First, we fix point $P^* \in \Omega_3^*$ and $P^* := (X^*, Y^*)$ and claim that for $h, k > 0$,

$$(7.4) \qquad \lim_{h, k \to 0^+} \frac{\Lambda(\tau - h) - \Lambda(\tau + k)}{h + k} = +\infty.$$

For arbitrary $\epsilon > 0$, ϵ arbitrary small, we can find $\delta > 0$ such that for any square Q with length $l \leq \delta$ center at P^*, there exists a vertical segment σ satisfying $meas(\Omega_3 \cup \sigma) \geq (1 - \epsilon)l$,

and a horizontal segment σ' satisfying $meas(\Omega_3 \cup \sigma') \geq (1-\epsilon)l$.

We define that

$$(7.5) \qquad t^+ := \max\left\{t(X,Y) : (X,Y) \in \sigma \cup \sigma'\right\},$$

$$(7.6) \qquad t^- := \min\left\{t(X,Y) : (X,Y) \in \sigma \cup \sigma'\right\}.$$

By (4.4), for some constant $c_0 > 0$

$$(7.7) \qquad t^+ - t^- \leq \int_\sigma \frac{(1+\cos w)p}{4c}dX + \int_{\sigma'} \frac{(1+\cos z)q}{4c}dY \leq c_0(\epsilon l)^2.$$

(7.7) is Lipschitz continuous and vanished outside of a set of measure ϵl. Also, for some constant $c_1, c_2 > 0$,

$$(7.8) \qquad \Lambda(t^-) - \Lambda(t^+) \geq c_1(1-\epsilon)^2 l^2 - c_2(t^+ + -t^-).$$

Since the choose of $\epsilon > 0$ is arbitrary, so this implies (7.4). And by the Lemma 1, the map $t \mapsto \Lambda$ has bounded variation, so (7.4) implies (7.3).

Thus, the singular part of the μ_t is not trivial only if the set $\Omega_4 := \{P \in \gamma_t : w(P) = -\pi, z(P) = -\pi\}$ has positive one-dimensional measure. By the above analysis, this is restricted to a set where $c' \neq 0$ and only happens for a set of time with measure zero.

Proof of Lemma 1.

From (2.11),

$$(7.9) \qquad \begin{cases} (R^2)_t - (cR^2)_x = \frac{c'}{2c}(R^2 S - S^2 R) - R\,g(v), \\ (S^2)_t + (cS^2)_x = -\frac{c'}{2c}(R^2 S - S^2 R) - S\,g(v). \end{cases}$$

We first provide an argument valid for $v = v(t,x)$ is smooth. (7.9) implies that

$$\frac{d}{dt}(4\Lambda(t)) = \frac{d}{dt}\iint_{x>y} R^2(t,x)S^2(t,y)dxdy$$

$$= \iint 2R(t,x)S^2(t,y)cR_x(t,x) + 2S(t,y)R^2(t,x)cS_x(t,y)$$

$$+ 2S(t,y)R^2(t,x)\frac{c'}{4c}(S^2(t,x) - R^2(t,x)) + 2R(t,x)S^2(t,y)\frac{c'}{4c}(R^2(t,x) - S^2(t,x))$$

$$- 2R(t,x)S^2(t,y)\frac{1}{2}g(v(t,x)) - 2S(t,y)R^2(t,x)\frac{1}{2}g(v(t,y))dxdy$$

$$\leq \iint c(S^2 R^2)_x + \frac{c'}{2c}(R^2 - S^2)(RS^2 - S^2 R)$$

$$- R(t,x)S^2(t,y)g(v(t,x)) - R^2(t,x)S(t,y)g(v(t,y))dxdy$$

$$\leq -2\int cR^2 S^2 dx + \int (R^2 + S^2)dx \cdot \int \frac{c'}{2c}|R^2 S - S^2 R|dx$$

$$- \iint R(t,x)S^2(t,y)g(v(t,x)) + R^2(t,x)S(t,y)g(v(t,y))dxdy.$$

And estimate the last term from the above calculation,

$$\left|\iint R(t,x)S^2(t,y)g(v(t,x))dxdy\right|$$

$$\leq \int S^2(t,y)dy \int |R(t,x)g(v(t,x))|\,dx$$

$$\leq \mathcal{E}_0 \|R(t,x)\|_{L^2}\|g(v(t,x))\|_{L^2}$$

$$\leq \mathcal{E}_0\mathcal{E}_0^{\frac{1}{2}}\|v(t,x)\|_{L^2}\|\frac{g(v(t,x))}{v(t,x)}\|_{L^\infty}$$

$$\leq \mathcal{E}_0^2.$$

Similarly,

$$\left|\iint R^2(t,x)S(t,y)g(v(t,y))dxdy\right|$$

$$\leq \mathcal{E}_0^2.$$

Thus

(7.10) $$\frac{d}{dt}(4\Lambda(t)) \leq -2\mathcal{K}^{-1}\iint R^2 S^2 dx + 4\mathcal{E}_0\left\|\frac{c'}{2c}\right\|_{L^\infty}\int |R^2 S - S^2 R|dx + 2\mathcal{E}_0^2,$$

Where $\vartheta(1)$ is defined as a quantity and its absolute value has a uniform bound depending only on $c(v)$. Also, the map $t \mapsto \Lambda(t)$ has bounded variation on any bounded interval. The smooth case is proved. The following provides a proof of Lemma 1 in general cases. For every $\epsilon > 0$, there exists a constant K_ϵ satisfying that for all w,z,

(7.11)
$$|\sin z(1-\cos w) - \sin w(1-\cos z)|$$
$$\leq K_\epsilon\left[\tan^2\left(\frac{w}{2}\right) + \tan^2\left(\frac{z}{2}\right)\right](1+\cos w)(1+\cos z) + \epsilon(1-\cos w)(1-\cos z).$$

For fixed $0 \leq s < t$, consider the sets Γ_s and Γ_t as we defined in (5.5) and define $\Gamma_{st} := \Gamma_t\backslash\Gamma_s$. Recall that

$$dxdt = \frac{pq}{8c}(1+\cos w)(1+\cos z)dXdY.$$

We write that

(7.12) $$\int_s^t\int_{-\infty}^{+\infty}\frac{1}{4}(R^2 - S^2)dxdt = (t-s)\mathcal{E}_0,$$

(7.13)
$$\int_s^t\int_{-\infty}^{+\infty}\frac{1}{4}(R^2 - S^2)dxdt = \iint_{\Gamma_{st}}\frac{pq}{32c}(1+\cos w)(1+\cos z)\left[\tan^2(\frac{z}{2}) + \tan^2(\frac{w}{2})\right]dXdY.$$

(7.12) holds only on the case that $v(t,x)$ is smooth while (7.13) holds for all cases. Combine (5.4), (5.7), (5.8) and apply (7.11)-(7.13), we obtain that

$$\Lambda(t) - \Lambda(s) \leq \iint_{\Gamma_{st}}\frac{(1-\cos w)(1-\cos z)pq}{64}dXdY$$

$$+ \mathcal{E}_0\iint_{\Gamma_{st}}\frac{c'pq}{64c^2}[\sin z(1-\cos w) - \sin w(1-\cos z)]dXdY$$

$$+ \mathcal{E}_0\iint_{\Gamma_{st}}\left\{-\frac{pq}{32c^2}g(v)(1+\cos z)\sin w - \frac{\sin w\,G(v)p}{8}\frac{c'}{8c}(\cos z - \cos w)q\right.$$

$$\left.+ \frac{(1+\cos w)g(v)p}{8}\frac{1}{4c}\sin zq + \frac{(1+\cos w)}{32}\frac{c'}{c^2}G(v)[\sin z - \sin w]pq\right\}dXdY$$

$$\leq \frac{1}{64} \iint_{\Gamma_{st}} (1 - \cos w)(1 - \cos z) pq \, dX dY$$

$$+ \mathcal{E}_0 \iint_{\Gamma_{st}} \frac{c'}{64c^2} pq \Big\{ K_\epsilon [\tan^2(\frac{w}{2}) + \tan^2(\frac{z}{2})](1 + \cos w)(1 + \cos z)$$

$$+ \epsilon(1 - \cos w)(1 - \cos z) \Big\} dX dY + \mathcal{E}_0 \iint_{\Gamma_{st}} \Big\{ -\frac{pq}{32c^2} g(v)(1 + \cos z) \sin w$$

$$- \frac{\sin w \, G(v) p}{8} \frac{c'}{8c}(\cos z - \cos w) q + \frac{(1 + \cos w) g(v) p}{8} \frac{1}{4c} \sin z q$$

$$+ \frac{(1 + \cos w)}{32} \frac{c'}{c^2} G(v)[\sin z - \sin w] pq \Big\} dX dY$$

$$\leq K(t - s),$$

for a suitable constant K. Thus, Lemma 1 is proved. $\qquad\square$

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