THE FUNDAMENTAL GROUP OF THE COMPLEMENT OF BRAIDED SURFACES

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1. Introduction

Let S be an algebraic curve in $\mathbb{C}^2$. In the 1930s, the algorithm for computing the fundamental group of the complement $\pi_1(\mathbb{C}^2 \setminus S)$ was developed by van Kampen [5]. Refinements of van Kampen's algorithm were then given by Chisini, Chéniot, Abelson, and Chang, amongst many others. In the early '80s, Moishezon [10] introduced the idea of braid monodromy and used it to recover van Kampen's presentation. Furthermore, Libgober [7] proved that the two-dimensional complex associated to the braid monodromy presentation is homotopy equivalent to $\mathbb{C}^2 \setminus S$. The study of the fundamental groups of the complements to algebraic curves is important because it tells us information on how to construct complex algebraic surfaces. For example, Zariski [14] showed the existence of two curves of degree 6, both with six cusps, such that their fundamental groups are not isomorphic. This difference is due to the placement of the cusps; one curve has all six cusps lying on a conic while the other does not have this property. Artal-Bartolo [1] first introduced this notion of a Zariski pair, which is a pair of curves S_1 and S_2 such that their singularities are topologically equivalent but their embeddings into $\mathbb{CP}^2$ are not.

Until recently, most of the curves that were considered for fundamental group calculations only had simple singularities. In the late '80s, Moishezon and Teicher [11] gave an algorithm to compute the local braid monodromy generated by branch points, cusp points, and transverse multi-intersection points. Then, in 2004, Kaplan, Liberman, and Teicher [6] expanded this list by computing the local braid monodromy for ten more types of singularities. In this thesis, we focus on five types of singularities: branch points, tangent points, transverse multi-intersection points, cusp points, and cusp points intersecting tangent lines. Given a polynomial S representing an arrangement of n complex curves in $\mathbb{C}^2$, the braid monodromy of the curve can be facilitated by use of a braided wiring diagram $\mathcal{W}$. The wires encode information about the position and arrangement of the curves while the braids illustrate how the curves behave before they come to intersect. A braided wiring diagram associated to a curve is not unique and can change depending on the way we construct our braid monodromy for a given line arrangement. These changes are known as Markov moves and braided wiring diagrams can be simplified using these moves. These simplifications with the use of braid relations can make the braid monodromy generators much easier to understand. We describe how to calculate the fundamental group $\pi_1(\mathbb{C}^2 \setminus \mathcal{W})$ and look at several examples.

A fundamental problem in group theory is determining whether two groups are isomorphic. Given two finite presentations $\pi_1(\mathbb{C}^2 \setminus \mathcal{W}_1)$ and $\pi_1(\mathbb{C}^2 \setminus \mathcal{W}_2)$, it is not at all obvious if there exists a method telling us if these groups are isomorphic. This is known as the group isomorphism problem, first mentioned by Tietze [13] in 1908. Around fifty years later, Adian and Rabin independently proved the group isomorphism problem is unsolvable, meaning there does not exist an algorithm solving every instance of this problem. Despite this result, there are various invariants that can help us determine whether two groups are not isomorphic. We give a survey of one of these invariants having close connections with the fundamental group called the Alexander polynomial $\Delta(t)$, a knot invariant introduced by Alexander in the 1920s. It is a Laurent polynomial with integer coefficients that can be computed using the fundamental group of the knot complement. We apply this concept from knot theory to calculate the Alexander polynomial of algebraic curves. We describe how to calculate it using Fox calculus, a tool used for the study of groups defined by generators and relators. This idea was developed through a series of five papers by Fox during the 1950s. Moreover, we show the Alexander polynomial is unchanged under Tietze transformations.

Given our five singularities, we look at a few types of braided wiring diagrams. We first calculate the fundamental group of an n-stranded wiring diagram with n-tuple points. It turns out that the group presentation can be expressed solely in terms of the number of strands.

Theorem 1.1. If $\mathcal{W}$ is an n-stranded wiring diagram with any number of n-tuple points and braiding, the fundamental group is given by

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{W}) = \langle t_1, \ldots, t_n | [t_1, \ldots, t_n] \rangle .$$

We then compute the fundamental group of three-stranded wiring diagrams with two double points. By finding the Alexander polynomials using the fundamental groups, we conclude that these wiring diagrams produce an infinite family of curves.

Theorem 1.2. Let $\mathcal{W}$ be a three-stranded wiring diagram with two double points and a braid σ_1^n between the singularities where $n \geq 0$. Then, the fundamental group $\pi_1(\mathbb{C}^2 \setminus \mathcal{W})$ is different for every n.

Using this calculation, we compare fundamental groups of three-stranded wiring diagrams with three double points by switching the braiding between the singularities. Finally, we look at several four-stranded wiring diagrams having a certain number of branch points, cusp points, tangent points, and cusp points intersecting tangent lines. The examples in this category satisfy the property that the product of the braid monodromy generators is equal to the full twist on four strands, implying these curves can be extended to closed curves in the complex projective plane $\mathbb{CP}^2$.

This thesis is organized as follows. In section 2, we provide background information on the fundamental group. In section 3, we define braid monodromy of algebraic curves and compute the local braid monodromy associated to our five singularities. In section 4, we formally define braided wiring diagrams and describe the group presentation of $\pi_1(\mathbb{C}^2 \setminus \mathcal{W})$. In section 5, we discuss the Alexander polynomial and prove some properties about it. In section 6, we compute the fundamental groups and Alexander polynomials of the braided wiring diagrams mentioned above.

2. Homotopy and the fundamental group

We first start with an example to provide some motivation. Consider a sphere and torus. Intuition tells us these two spaces should not be homeomorphic because the torus has a hole in the center whereas the sphere does not. The fundamental group allows us to see this difference topologically.

Definition 2.1. Let X be a topological space. A loop is a continuous map $f : [0,1] \to X$ such that $f(0) = p = f(1)$ where p is called the basepoint of the loop.

Definition 2.2. Two loops $f, g : [0,1] \to X$ are homotopic, denoted $f \simeq g$, if there exists a continuous function $H : [0,1] \times [0,1] \to X$ such that the following hold.

$$\begin{aligned} H(s,0) &= f(s) \qquad H(0,t) = p = H(1,t) \\ H(s,1) &= g(s) \end{aligned}$$

Example 2.1. Let $X = \mathbb{R}^2$ and $p = (0,0)$. Suppose f is the red loop and g is the green loop.

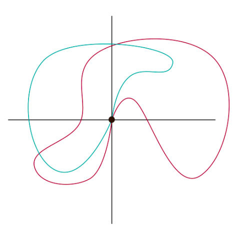

Figure 1. $f, g : [0,1] \to \mathbb{R}^2$

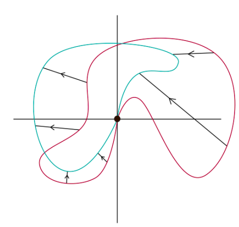

Figure 2. Straight-line homotopy

The homotopy is given by $H(s,t) = f(s)(1-t) + g(s) \cdot t$. This map is called the straight-line homotopy because H traverses a line segment from f to g as t ranges between 0 and 1.

Proposition 2.1. Homotopy is an equivalence relation.

Proof.
$f \simeq f$: Define $H(s,t) = f(s)$ for all $t \in [0,1]$.
$f \simeq g$ implies $g \simeq f$: If we are given $f \simeq g$ with the map $H(s,t)$, then define $\overline{H}(s,t) = H(s, 1-t)$. This new map satisfies the required conditions.

$$\begin{aligned} \overline{H}(s,1-0) &= H(s,1) = g(s) \qquad \overline{H}(0,1-t) = p = \overline{H}(1,1-t) \\ \overline{H}(s,1-1) &= H(s,0) = f(s) \end{aligned}$$

$f \simeq g$ and $g \simeq h$ imply $f \simeq h$: Let Φ be the homotopy from f to g and Λ be the homotopy from g to h. We show $f \simeq h$ using the function

$$\Omega(s,t) = \begin{cases} \Phi(s,2t) & 0 \le t \le \frac{1}{2} \\ \Lambda(s,2t-1) & \frac{1}{2} \le t \le 1 \end{cases}.$$

$\square$

We write $[f]$ to represent all loops homotopic to f. Intuitively, two loops are homotopic if one can be continuously deformed into the other.

Definition 2.3. The fundamental group $\pi_1(X)$ based at p is the set consisting of classes of loops under the equivalence relation of homotopy. The group operation is path multiplication defined by

$$f \cdot g(t) = \begin{cases} f(2t) & 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

In other words, two loops in this group are equivalent if they are homotopic.

Let us go back to our example in the beginning of this section and think about $\pi_1(S^2)$ and $\pi_1(T^2)$. We give an informal explanation using Figure 3.

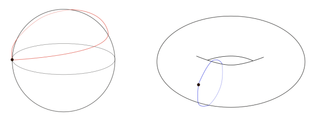

FIGURE 3. Sphere and torus

Suppose the basepoint of each space is the black dot. If we take the red loop and move it around the sphere, we see that it is homotopic to the basepoint. In fact, any loop on the sphere is homotopic to the basepoint by this exact same reasoning. Thus, $\pi_1(S^2) = 0$, the trivial group. However, $\pi_1(T^2) \neq 0$ since the blue loop is not contractible. For more information about the fundamental group, consult Hatcher [4].

3. Braid monodromy of algebraic curves

In this section, we define the braid monodromy of an algebraic curve in $\mathbb{C}^2$ following [2], [7], [8], and [9]. Before doing so, we first review some knot theory.

3.1. Braid group.

A braid is a collection of n intertwined strings with fixed endpoints.

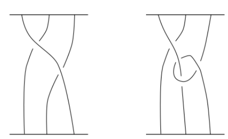

FIGURE 4. $n = 3$

Given any horizontal plane between the two bars, each strand must intersect the horizontal plane exactly once. Therefore, the figure on the right is not a braid because the rightmost strand loops around.

A braid word, represented by σ_i's, is associated to each braid. We use the following convention in this thesis.

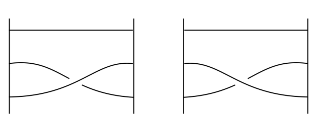

FIGURE 5. σ_1 and σ_1^{-1}

Every n-stranded braid can be expressed using a combination of these σ_i's where $1 \leq i \leq n-1$. Braid words can be simplified using the relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \leq i \leq n-2$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geq 2$.

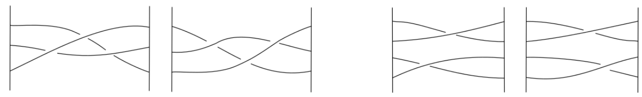

FIGURE 6. $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ and $\sigma_1\sigma_3 = \sigma_3\sigma_1$

The braid group on n strands, denoted B_n, is generated by $\sigma_1, \ldots, \sigma_{n-1}$ with the relations given above, and the presentation is

$$B_n = \big\langle \sigma_1, \ldots, \sigma_{n-1} | \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \le i \le n-2,\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \ge 2 \big\rangle.$$

3.2. **Braid monodromy.** Let S be a degree n algebraic curve in $\mathbb{C}^2$. Let $\pi : \mathbb{C}^2 \to \mathbb{C}$ be a projection onto the first coordinate and $\mathscr{X} = \{x_1, \ldots, x_k\}$ be the set of points in $\mathbb{C}$ for which the fibers of π contain singular points or tangencies. Assume $\pi^{-1}(x_i)$ contains at most one singular point of S and does not belong to the tangent cone of S. Define π to be a generic projection if it satisfies these properties. Choose a point $x_0 \in \mathbb{C} \setminus \mathscr{X}$ such that $\Re(x_0) > \Re(x_i)$ for all $x_i \in \mathscr{X}$. Choose loops $\xi_i : I \to \mathbb{C} \setminus \mathscr{X}$ based at x_0 going below the points $x_1, \ldots, x_{i-1}$ such that it passes around x_i in a counterclockwise direction and comes back the same way. Let D_i denote the small disk around x_i and let s_i be paths connecting x_0 and a point $q_i \in \partial D_i$. Set $a_i = [\xi_i]$. For example, $a_1 = s_1 \cup \partial D_1 \cup s_1^{-1}$ if we look at Figure 7. This gives us a set of generators for $\pi_1(\mathbb{C} \setminus \mathscr{X})$.

Define $L_x = \pi^{-1}(x)$. Fix a basepoint $b \in L_{x_0}$ and consider a collection of loops in $L_{x_0} \setminus (L_{x_0} \cap S)$. These loops represent the generators of $\pi_1(L_{x_0} \setminus (L_{x_0} \cap S))$ based at b. The restriction of the projection map

$$\mathbb{C}^2 \setminus (S \cup L_{\mathscr{X}}) \to \mathbb{C} \setminus \mathscr{X} \tag{1}$$

defines a locally trivial bundle.

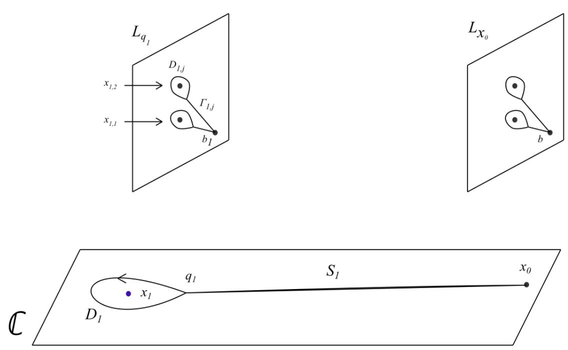

FIGURE 7

The intersection L_{q_i} and S consists of n points, m_i of which have as limit the singular points of S over x_i as q_i approaches x_i. For example, $m_i = 2$ if the singularity is a double point because two points approach the singularity. Label these m_i points in L_{q_i} as $x_{i,1}, \ldots, x_{i,m_i}$. Let $\Gamma_{i,j}$ where $1 \le j \le n$ be paths in L_{q_i} connecting a basepoint $b_i \in L_{q_i}$ with a point on the boundary of a small disk $D_{i,j}$ about the points $L_{q_i} \cap S$. These paths define the generators $\Gamma_{i,j} \cup \partial D_{i,j} \cup \Gamma_{i,j}^{-1}$ of $\pi_1(L_{q_i} \setminus (L_{q_i} \cap S))$ based at b_i.

Now fix a system of non-intersecting segments in L_{q_i} (resp. L_{x_0}) connecting the points $L_{q_i} \cap S$ (resp. $L_{x_0} \cap S$). These segments represent the generators of the braid group $B_n(L_{q_i}, L_{q_i} \cap S)$ (resp. $B_n(L_{x_0}, L_{x_0} \cap S)$), which is interpreted as the group of diffeomorphisms of L_{q_i} (resp. L_{x_0}) fixing $L_{q_i} \cap S$ (resp. $L_{x_0} \cap S$). Each segment defines a diffeomorphism that is a half twist about this segment. A trivialization of the bundle (1) restricted to $\partial D_i \setminus \{q_i\}$ defines a braid $\mu_i^2 \in B_n(L_{q_i}, L_{q_i} \cap S)$, which we call the local braid monodromy at the point x_i. Furthermore, a trivialization of the same bundle restricted to s_i defines diffeomorphisms $\beta_i : (L_{q_i}, L_{q_i} \cap S) \to (L_{x_0}, L_{x_0} \cap S)$.

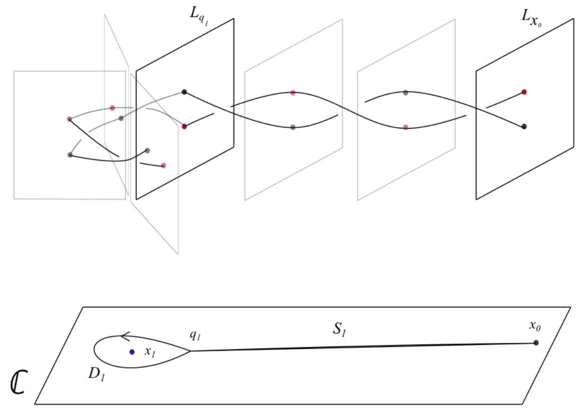

FIGURE 8

Definition 3.1. The homomorphism $\Psi : \pi_1(\mathbb{C} \setminus \mathscr{X}) \to B_n(L_{x_0}, L_{x_0} \cap S)$ sending the ith generator to $\beta_i^{-1}\mu_i^2\beta_i$ is called the braid monodromy.

The braid monodromy of an algebraic curve is not unique and depends on the choices made in defining the generators of $\pi_1(\mathbb{C} \setminus \mathscr{X})$. For instance, we can define our loops ξ_i based at x_0 to go above the points $x_1, \ldots, x_{i-1}$ and pass around x_i in a counterclockwise direction. To understand this indeterminancy, we define braid-equivalence.

Definition 3.2. Two homomorphisms $\Psi : F(k) \to B_n$ and $\Psi' : F(k) \to B_n$, where $F(k)$ is the free group generated by k elements, are equivalent if there exist automorphisms $\psi \in \operatorname{Aut}(F(k))$ and $\phi \in \operatorname{Aut}(F(n))$ with $\phi(B_n) \subset B_n$ such that

$$\Psi' \circ \psi(g) = \phi^{-1} \cdot \Psi(g) \cdot \phi$$

for all $g \in F(k)$. Equivalently, the following diagram commutes.

$$\begin{tikzcd} F(k) \arrow[r, "\Psi"] \arrow[d, "\psi"] & B_n \arrow[d, "\mathrm{conj}_\phi"] \\ F(k) \arrow[r, "\Psi'"] & B_n \end{tikzcd}$$

Moreover, if $\psi \in B_k$ and $\phi \in B_n$, then the homomorphisms are said to be braid-equivalent.

Theorem 3.1 (Theorem 3.7 from [2])**.** The braid monodromy of an algebraic curve S is well-defined up to braid-equivalence.

Proof. Based on our earlier discussion, we know the identification $\pi_1(\mathbb{C} \setminus \mathscr{X}) = F(k)$ depends on the way we define our generators of $\pi_1(\mathbb{C} \setminus \mathscr{X})$. These two choices yield monodromies that differ by a braid automorphism of $F(k)$. Furthermore, the choice of basepoints yields monodromies differing by a cojugation in B_n. Thus, these two cases have braid-equivalent braid monodromy generators. We now examine the effects of a change in the choice of generic projection. Let π and π' be two projections with critical sets $\mathscr{X}$ and $\mathscr{X}'$, respectively. Let the braid monodromies be Ψ and Ψ'. Libgober [9] showed there is a homeomorphism $j : \mathbb{C} \setminus \mathscr{X} \to \mathbb{C} \setminus \mathscr{X}'$ such that the induced isomorphism $j_* : \pi_1(\mathbb{C} \setminus \mathscr{X}) \to \pi_1(\mathbb{C} \setminus \mathscr{X}')$ satisfies $\Psi' \circ j_* = \Psi$. By construction, we observe that j can be taken to be the identity outside a ball of large radius containing $\mathscr{X} \cup \mathscr{X}'$. Then, j_* can be written as the composition of an inner automorphism of $F(k)$ with a braid automorphism of $F(k)$: $j_* = \mathrm{conj}_g \circ \phi$. Trade the inner automorphism of $F(k)$ with an inner automorphism of B_n to obtain $\Psi' \circ \psi = \mathrm{conj}_{\Psi'(g)} \circ \Psi$. $\square$

3.3. **Computation of local braid monodromy.** We consider curves with the following types of singularities. For a more comprehensive list, look at [6].

a – Branch point, topologically equivalent to $y^2 - x = 0$ or $y^2 + x = 0$
b – Tangent point, topologically equivalent to $y(y - x^2) = 0$
c – n-tuple point, intersection of n non-singular branches that are all transversal to each other

d – Cusp, topologically equivalent to $x^3 - y^2 = 0$ or $x^3 + y^2 = 0$
f_1 – Cusp point intersecting a tangent line, topologically equivalent to $y(y^2 - x^3) = 0$

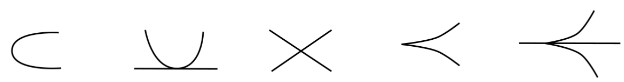

FIGURE 9. Singularities

In the following propositions, we compute the local braid monodromy associated to each singularity.

Proposition 3.2. Let S be the curve $y^2 - x$. The local braid monodromy at the origin is generated by σ_1.

We track the roots of S by taking $x = e^{2\pi i t}$ with $t \in [0, 1]$. The cases plotted below are $t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

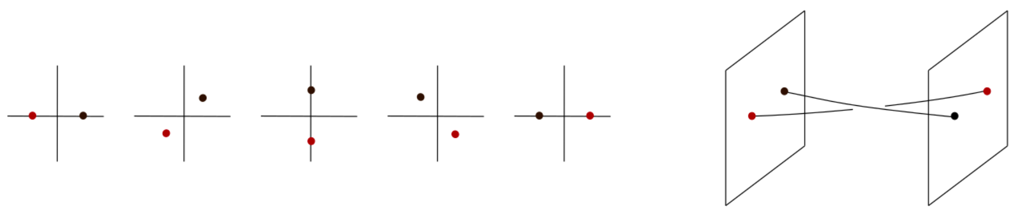

FIGURE 10

Proposition 3.3. Let S be the curve $y(y - x^2)$. The local braid monodromy at the origin is generated by σ_1^4.

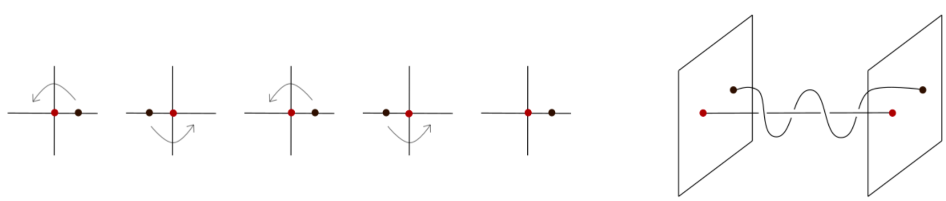

FIGURE 11

Proposition 3.4. Let S be the curve $(y - x)(y + x)$. The local braid monodromy at the origin is generated by σ_1^2.

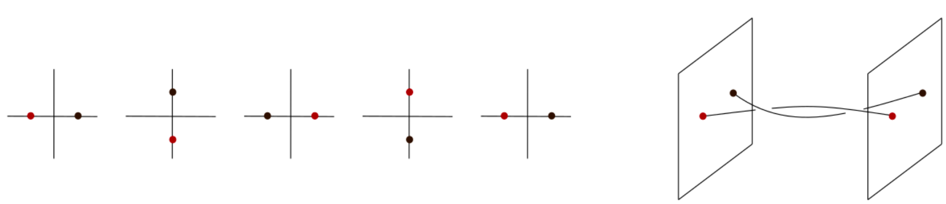

FIGURE 12

Note the local braid monodromy for an arbitrary type c singularity is generated by the full twist on n strands.

$$((\sigma_1 \cdots \sigma_{n-1}) \cdot (\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1\sigma_2) \cdot \sigma_1)^2$$

Proposition 3.5. Let S be the curve $x^3 - y^2$. The local braid monodromy at the origin is generated by σ_1^3.

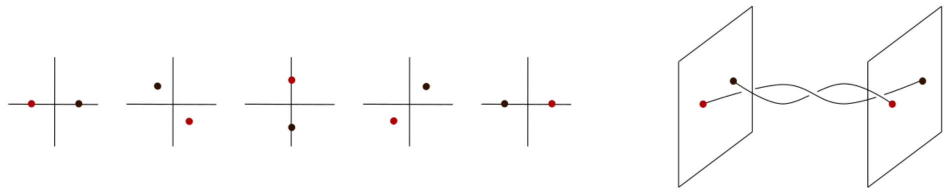

FIGURE 13

Proposition 3.6. Let S be the curve $y(y^2 - x^3)$. The local braid monodromy at the origin is generated by $(\sigma_2\sigma_1\sigma_2)^3$.

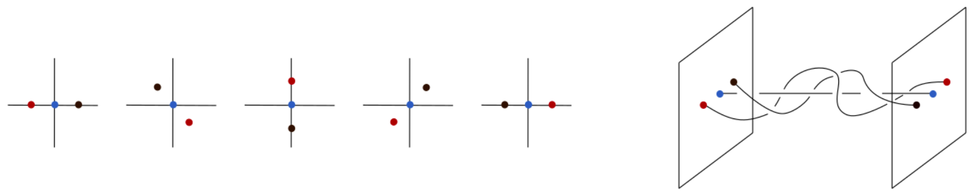

FIGURE 14

3.4. **Fundamental group of the complement of algebraic curves.** The homotopy exact sequence of the bundle (1) is

$$0 \to \pi_1(L_{x_0} \setminus (L_{x_0} \cap S)) \to \pi_1(\mathbb{C}^2 \setminus (S \cup L_{\mathscr{X}})) \to \pi_1(\mathbb{C} \setminus \mathscr{X}) \to 0.$$

This sequence is split exact with the action being the braid monodromy homomorphism Ψ. To obtain a presentation for the group $\pi_1(\mathbb{C}^2 \setminus (S \cup L_{\mathscr{X}}))$, consider $x_0 \in \mathbb{C} \setminus \mathscr{X}$ defined earlier. Based on our discussion in Section 3.2, we can identify $\pi_1(\mathbb{C} \setminus \mathscr{X})$ with $F(k) = \langle a_1, \ldots, a_k \rangle$. Similarly identify $\pi_1(L_{x_0} \setminus (L_{x_0} \cap S))$ based at b with $F(n) = \langle t_1, \ldots, t_n \rangle$. Since we have a split exact sequence, $\pi_1(\mathbb{C}^2 \setminus (S \cup L_{\mathscr{X}}))$ is equal to the semidirect product $F(n) \rtimes_\Psi F(k)$. Therefore, the group presentation is

$$\pi_1(\mathbb{C}^2 \setminus (S \cup L_{\mathscr{X}})) = \left\langle t_1, \ldots, t_n, a_1, \ldots, a_k | a_i^{-1} \cdot t_j \cdot a_i = \Psi(a_i)(t_j) \right\rangle.$$

The fundamental group of the complement of S is the quotient of $\pi_1(\mathbb{C}^2 \setminus (S \cup L_{\mathscr{X}}))$ by the normal closure of $F(k)$.

$$\pi_1(\mathbb{C}^2 \setminus S) = \langle t_1, \ldots, t_n | t_j = \Psi(a_i)(t_j) \rangle$$

4. BRAIDED WIRING DIAGRAMS

Let $\mathcal{A}$ be an arrangement of n complex curves in $\mathbb{C}^2$ represented by a polynomial $f(x, z)$. Recall the definitions of $\pi : \mathbb{C}^2 \to \mathbb{C}$, $\mathscr{X}$, and x_0. Let $\phi : I \to \mathbb{C}$ be a smooth path beginning from x_0 that goes through $x_1, \ldots, x_k$ in order by decreasing real part. The braided wiring diagram $\mathcal{W}$ associated to $\mathcal{A}$ is defined to be

$$\mathcal{W} = \{(x, z) \in \phi \times \mathbb{C} | f(x, z) = 0\}.$$

As we pass between the points in $\mathscr{X}$, the lines of $\mathcal{A}$ may braid. We associate a braid $\beta_{i,i+1}$ to the portion of ϕ from $x_i + \varepsilon$ to $x_{i+1} - \varepsilon$ for some $\varepsilon > 0$. Braided wiring diagrams can be represented using a sequence of these braids and vertices.

$$V_j \xleftarrow{\beta_{j-1,j}} V_{j-1} \leftarrow \cdots \leftarrow V_2 \xleftarrow{\beta_{1,2}} V_1 \xleftarrow{\beta_{0,1}} V_0$$

Let V_0 represent the initial ordering of the strands recorded from the bottom. The elements of the set V_k for $k > 0$ are indices of the wires to the left of vertex k in terms of the order given by V_0 and $\beta_{k,k+1}$ is the braid between vertices k and $k + 1$.

Remark 4.1. The vertices are read from right to left but the braids are read from left to right.

Now suppose there are m strands below V_k and $|V_k| = n$. Let $I_k = \{m + 1, \ldots, m + n\}$ be the local index of V_k.

Example 4.1. For the wiring diagram below, $V_3 = \{1, 2\}$ and $I_3 = \{2, 3\}$.

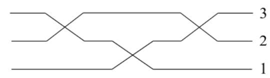

FIGURE 15

To apply the local braid monodromy calculations in Section 3.3 to braided wiring diagrams, we need to use the local indices I_k to determine the appropriate braid. We won't rewrite the propositions but we give one example for the type a singularity. Suppose the kth vertex of a braided wiring diagram is a branch point and $I_k = \{m + 1, m + 2\}$. The local braid monodromy is then generated by σ_{m+1}.

We now provide a formula for the braid monodromy generator of an arbitrary vertex, which is obtained by slightly rewriting the formula presented in Section 3.2.

Theorem 4.1. The braid monodromy generator of the kth vertex of a braided wiring diagram is given by

$$\Psi(a_k) = \Psi_k = \beta_k^{-1} \mu_{I_k}^2 \beta_k$$

where $\mu_{I_k}^2$ is the local braid monodromy of the singularity at vertex k and

$$\beta_k = \beta_{k-1,k} \cdot \mu_{I_{k-1}} \cdot \beta_{k-1}.$$

Example 4.2. Consider a 5-stranded wiring diagram with one branch point and one double point.

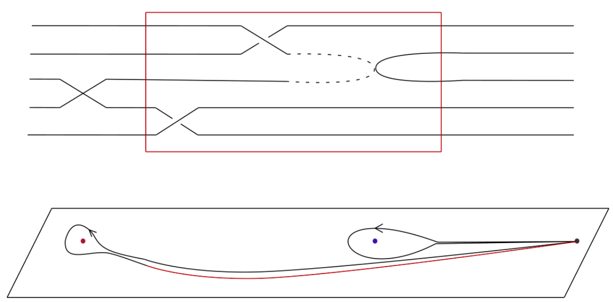

FIGURE 16

Note the dotted lines after the branch point indicate how the wires moved to the imaginary plane. Therefore, to bring the wires back to the real plane, we rotate them counterclockwise by $\frac{\pi}{2}$. The braid monodromy generators are

$$\Psi_1 = \sigma_3, \quad \Psi_2 = \beta_2^{-1} \sigma_2^2 \beta_2$$

where

$$\begin{aligned} \beta_2 &= \beta_{1,2} \cdot \mu_{I_1} \cdot \beta_1 \\ &= \sigma_1 \sigma_4^{-1}. \end{aligned}$$

We explain the calculation for β_2, which represents how the fiber changes as we traverse the red part of the second loop. We first read $\beta_{1,2} = \sigma_1 \sigma_4^{-1}$ from the diagram. We now follow the wires with local indices 3 and 4 because the other wires remain fixed. After the braid σ_4^{-1}, wires 3 and 4 move to the imaginary plane but from the opposite direction so we rotate them counterclockwise by $-\frac{\pi}{2}$. We then apply half of the local braid monodromy contribution from the branch point. The local braid monodromy is σ_3, i.e., rotate wires 3 and 4 by π counterclockwise. Hence, half of this is rotation by $\frac{\pi}{2}$ counterclockwise. Thus, we see that the angles cancel so we get the identity. Finally, there is no braiding afterwards so combining everything gives us $\beta_2 = \sigma_1 \sigma_4^{-1}$.

The fundamental group $\pi_1(\mathbb{C}^2 \setminus \mathcal{W})$ can be calculated using these Ψ_k's. Each generator of the fundamental group, t_i, represents a loop around strand i of the wiring diagram. The presentation is given by

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{W}) = \langle t_1, \ldots, t_n | \Psi_k(t_i) = t_i \text{ for } i \in V_k \setminus \max V_k \text{ and } 1 \leq k \leq j \rangle.$$

We exclude $\max V_k$ because calculating $\Psi_k(t_i) = t_i$ for $i = \max V_k$ gives us a redundant relation. Moreover, if $i \notin V_k$, then we just have the identity relation $t_i = t_i$ so we disregard this case.

The relationship between the braid monodromy generators and t_j's is given by the following.

$$\sigma_i(t_j) = \begin{cases} t_i t_{i+1} t_i^{-1} & i = j \\ t_i & j = i+1 \\ t_j & \text{else} \end{cases} \qquad \text{and} \qquad \sigma_i^{-1}(t_j) = \begin{cases} t_{i+1} & i = j \\ t_{i+1}^{-1} t_i t_{i+1} & j = i+1 \\ t_j & \text{else} \end{cases}$$

Diagrams depicting these relations are shown.

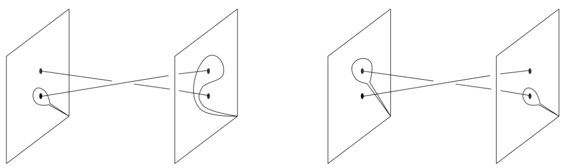

FIGURE 17. $\sigma_1(t_1) = t_1 t_2 t_1^{-1}$, $\sigma_1(t_2) = t_1$

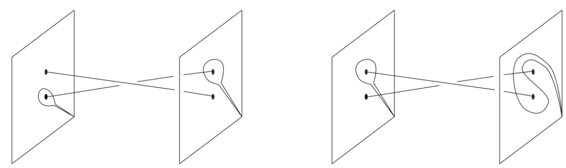

FIGURE 18. $\sigma_1^{-1}(t_1) = t_2$, $\sigma_1^{-1}(t_2) = t_2^{-1} t_1 t_2$

To compute $\sigma_i(t_j^{-1})$ or $\sigma_i^{-1}\left(t_j^{-1}\right)$, take the inverse of $\sigma_i(t_j)$ or $\sigma_i^{-1}(t_j)$, respectively.

Remark 4.2. We apply the σ_i's starting from the left so if we want to calculate $\sigma_1\sigma_2(t_1)$, we do the following:

$$t_1 \xrightarrow{\sigma_1} t_1 t_2 t_1^{-1} \xrightarrow{\sigma_2} t_1 t_2 t_3 t_2^{-1} t_1^{-1}.$$

Example 4.3. We calculate $\pi_1(\mathbb{C}^2 \setminus \mathcal{W})$ for the wiring diagram in Figure 15.

$$\begin{array}{lll} \Psi_1 = \sigma_2^2 & \Psi_2 = \sigma_2^{-1} \cdot \sigma_1^2 \cdot \sigma_2 & \Psi_3 = \sigma_2^{-1}\sigma_1^{-1} \cdot \sigma_2^2 \cdot \sigma_1\sigma_2 = \sigma_1^2 \\ \Psi_1(t_2) = t_2 t_3 t_2 t_3^{-1} t_2^{-1} & \Psi_2(t_1) = t_1 t_2 t_3 t_2^{-1} t_1 t_2 t_3^{-1} t_2^{-1} t_1^{-1} & \Psi_3(t_1) = t_1 t_2 t_1 t_2^{-1} t_1^{-1} \end{array}$$

From the first relation, we know t_2 and t_3 commute so we can simplify the second relation. The fundamental group is then

$$\langle t_1, t_2, t_3 | [t_2, t_3], [t_1, t_3], [t_1, t_2] \rangle$$

where the symbol $[t_1, t_2, \ldots, t_m]$ denotes the family of $m - 1$ relations:

$$t_1 t_2 \cdots t_m = t_2 \cdots t_m t_1 = \cdots = t_m t_1 \cdots t_{m-1}.$$

4.1. **Markov moves.** We consider braided wiring diagrams with type c singularities in this section. For an arbitrary line arrangement $\mathcal{A}$, the way we construct our braid monodromy can affect the wiring diagram associated to $\mathcal{A}$. For example, changing the basepoint x_0 may result in a different braid $\beta_{0,1}$ while changes in the projection may alter the wire ordering. We refer to these different changes as Markov moves, and we describe how these moves affect the braid monodromy. Let $\hat{\Psi}$ be the braid monodromy generator for the altered wiring diagram $\hat{\mathcal{W}}$.

1. Insert a braid $\beta_{0,1}$ at the beginning of the braided wiring diagram.

FIGURE 19. $\Psi_1 = \sigma_1\sigma_2\sigma_1^2\sigma_2\sigma_1$ and $\hat{\Psi}_1 = \sigma_1^{-1} \cdot \sigma_1\sigma_2\sigma_1^2\sigma_2\sigma_1 \cdot \sigma_1$

2. Insert a braid β_{s+1} at the end of the braided wiring diagram.

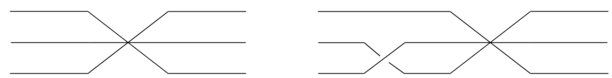

FIGURE 20. $\Psi_1 = \sigma_1\sigma_2\sigma_1^2\sigma_2\sigma_1 = \hat{\Psi}_1$

In the rest of the examples, we assume $\{i, \ldots, j\}$ denotes the local index.
3. Replace vertex $\{i, \ldots, j\}$, then vertex $\{k, \ldots, l\}$ with
a. vertex $\{k, \ldots, l\}$, then vertex $\{i, \ldots, j\}$ if $j < k$ or $i > l$.
Let $i = 1$, $j = 3$, $k = 4$, and $l = 6$.

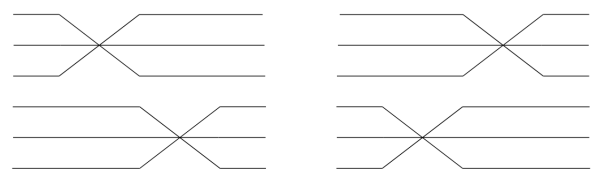

FIGURE 21. $\Psi_1 = \sigma_1\sigma_2\sigma_1^2\sigma_2\sigma_1$, $\Psi_2 = \sigma_4\sigma_5\sigma_4^2\sigma_5\sigma_4$ and $\hat{\Psi}_1 = \sigma_4\sigma_5\sigma_4^2\sigma_5\sigma_4$, $\hat{\Psi}_2 = \sigma_1\sigma_2\sigma_1^2\sigma_2\sigma_1$

b. braid $(\sigma_k \cdots \sigma_{i+1})(\sigma_{k+1} \cdots \sigma_{i+2}) \cdots (\sigma_{l-1} \cdots \sigma_j)$, then vertex $\{i, \ldots, i+l-k\}$, then vertex $\{i+l-k, \ldots, l\}$, then braid $(\sigma_{k-1}^{-1} \cdots \sigma_i^{-1})(\sigma_k^{-1} \cdots \sigma_{i+1}^{-1}) \cdots (\sigma_{l-2}^{-1} \cdots \sigma_{j-1}^{-1})$, if $j = k$
Let $i = 1$, $j = 2 = k$, and $l = 3$.

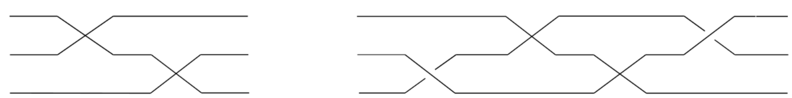

FIGURE 22. $\Psi_1 = \sigma_1^2$, $\Psi_2 = \sigma_1^{-1} \cdot \sigma_2^2 \cdot \sigma_1$ and $\hat{\Psi}_1 = \sigma_2^{-1} \cdot \sigma_1^2 \cdot \sigma_2$, $\hat{\Psi}_2 = \sigma_2^{-1}\sigma_1^{-1} \cdot \sigma_2^2 \cdot \sigma_1\sigma_2$

c. braid $(\sigma_{i-1} \cdots \sigma_k)(\sigma_i \cdots \sigma_{k+1}) \cdots (\sigma_{j-2} \cdots \sigma_{l-1})$, then vertex $\{j+k-l, \ldots, j\}$, then vertex $\{k, \ldots, j+k-l\}$, then braid $(\sigma_i^{-1} \cdots \sigma_{k+1}^{-1})(\sigma_{i+1}^{-1} \cdots \sigma_{k+2}^{-1}) \cdots (\sigma_{j-1}^{-1} \cdots \sigma_l^{-1})$, if $i = l$
Let $i = 2 = l$, $j = 3$, and $k = 1$.

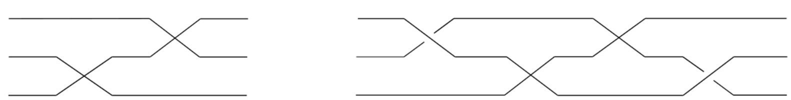

FIGURE 23. $\Psi_1 = \sigma_2^2$, $\Psi_2 = \sigma_2^{-1} \cdot \sigma_1^2 \cdot \sigma_2$ and $\hat{\Psi}_1 = \sigma_1^{-1} \cdot \sigma_2^2 \cdot \sigma_1$, $\hat{\Psi}_2 = \sigma_1^{-1}\sigma_2^{-1} \cdot \sigma_1^2 \cdot \sigma_2\sigma_1$

4. Reduce an intermediate braid $\beta_{k,k+1}$.

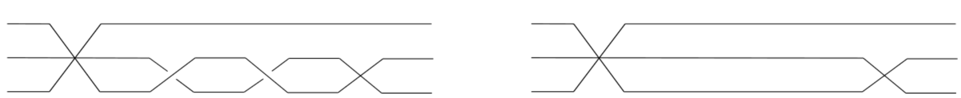

FIGURE 24. $\Psi_1 = \sigma_1^2$, $\Psi_2 = \sigma_1^{-1}\sigma_1\sigma_1^{-1} \cdot \sigma_1\sigma_2\sigma_1^2\sigma_2\sigma_1 \cdot \sigma_1\sigma_1^{-1}\sigma_1$
$\hat{\Psi}_1 = \sigma_1^2$, $\hat{\Psi}_2 = \sigma_1^{-1} \cdot \sigma_1\sigma_2\sigma_1^2\sigma_2\sigma_1 \cdot \sigma_1$

5. Replace braid σ_i, then vertex $\{j, \ldots, k\}$ with
a. vertex $\{j, \ldots, k\}$, then braid σ_i, if $i < j-1$ or $i > k$
Suppose the braid is σ_3 and the local index of the vertex is $\{1, 2\}$.

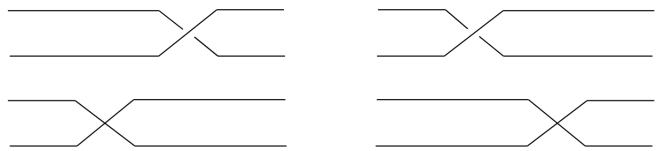

FIGURE 25. $\Psi_1 = \sigma_1^2 = \hat{\Psi}_1$

b. braid $\sigma_j^{-1} \cdots \sigma_{k-1}^{-1}$, then vertex $\{j+1, \dots, k+1\}$, then braid $\sigma_k \cdots \sigma_j$, if $i = k$
Suppose the braid is σ_3 and the local index of the vertex is $\{1, 2, 3\}$.

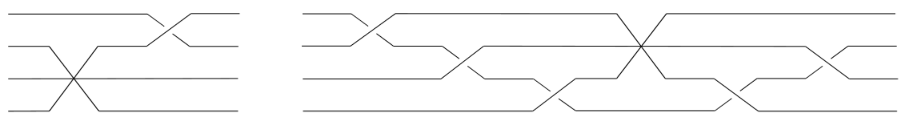

FIGURE 26. $\Psi_1 = \sigma_3^{-1} \cdot \sigma_1\sigma_2\sigma_1^2\sigma_2\sigma_1 \cdot \sigma_3$ and $\hat{\Psi}_1 = \sigma_2\sigma_1 \cdot \sigma_2\sigma_3\sigma_2^2\sigma_3\sigma_2 \cdot \sigma_1^{-1}\sigma_2^{-1}$

c. braid $\sigma_{k-1}^{-1} \cdots \sigma_j^{-1}$, then vertex $\{j-1, \dots, k-1\}$, then braid $\sigma_{j-1} \cdots \sigma_{k-1}$, if $i = j-1$
Suppose the braid is σ_1 and the local index of the vertex is $\{2, 3, 4\}$.

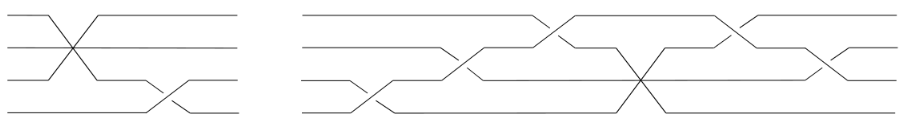

FIGURE 27. $\Psi_1 = \sigma_1^{-1} \cdot \sigma_2\sigma_3\sigma_2^2\sigma_3\sigma_2 \cdot \sigma_1$ and $\hat{\Psi}_1 = \sigma_2\sigma_3 \cdot \sigma_1\sigma_2\sigma_1^2\sigma_2\sigma_1 \cdot \sigma_3^{-1}\sigma_2^{-1}$

d. vertex $\{j, \dots, k\}$, then braid $\sigma_{j+k-i-1}$, if $j \leq i \leq k-1$
Suppose the braid is σ_1 and the local index of the vertex is $\{1, 2, 3\}$.

FIGURE 28. $\Psi_1 = \sigma_1^{-1} \cdot \sigma_1\sigma_2\sigma_1^2\sigma_2\sigma_1 \cdot \sigma_1$ and $\hat{\Psi}_1 = \sigma_1\sigma_2\sigma_1^2\sigma_2\sigma_1$

Note that the braids in moves 3 and 5 can be inverted. For example, we can replace the braid σ_i^{-1}, then vertex $\{j, \dots, k\}$ with vertex $\{j, \dots, k\}$, then σ_i^{-1} if we consider move 5a.

Theorem 4.2 (Theorem 5.7 from [2])**.** The braid monodromy of a braided wiring diagram is invariant under Markov moves. If a braided wiring diagram $\hat{\mathcal{W}}$ is obtained from $\mathcal{W}$ by a finite sequence of Markov moves or their inverses, then the braid monodromy generators are braid-equivalent.

5. Alexander polynomial

In many cases, the fundamental group alone is not sufficient enough to distinguish spaces. For instance, a complicated relation may arise from a braid between two vertices, making it difficult to compare fundamental groups. Thus, we discuss another invariant known as the Alexander polynomial. By assigning a polynomial to each wiring diagram, we may be able to instantly determine whether or not the fundamental groups are isomorphic.

5.1. **Fox calculus.** Suppose, for a topological space X, we are given

$$\pi_1(X) = \langle t_1, \ldots, t_n | r_1, \ldots, r_m \rangle$$

and a surjective homomorphism $\phi : \pi_1(X) \to \mathbb{Z}$. Construct another surjective homomorphism $\psi : F(n) \to \pi_1(X)$. The Fox differential

$$\frac{\partial}{\partial x_j} : \mathbb{C}[F(n)] \to \mathbb{C}[F(n)]$$

is a $\mathbb{C}$-linear map satisfying the following properties.

$$\frac{\partial}{\partial t_i} 1 = 0, \quad \frac{\partial}{\partial t_j} t_i = \delta_{i,j}, \quad \frac{\partial}{\partial t_j}(t_k t_l) = \frac{\partial}{\partial t_j} t_k + t_k \frac{\partial}{\partial t_j} t_l$$

Example 5.1. To calculate the Fox derivative of $t_1 t_2 t_3 t_1$ with respect to t_1, rewrite the word as $t_1(t_2 t_3 t_1)$ and apply the second rule. This means we treat t_1 as t_k and $t_2 t_3 t_1$ as t_l. Continue this method to obtain

$$\frac{\partial}{\partial t_1}(t_1 t_2 t_3 t_1) = 1 + t_1 t_2 t_3.$$

This next formula will be useful when computing Alexander polynomials.

Lemma 5.1. $\frac{\partial}{\partial t_j} t_i^{-1} = \begin{cases} -t_i^{-1} & i = j \\ 0 & i \neq j \end{cases}$

Proof. Using the given properties, we have $0 = \frac{\partial}{\partial t_j} 1 = \frac{\partial}{\partial t_j}(t_i t_i^{-1}) = \delta_{i,j} + t_i \frac{\partial}{\partial t_j} t_i^{-1}$, and the result follows. □

The composition $\phi \circ \psi : F(n) \to \mathbb{Z}$ gives a ring homomorphism $\gamma : \mathbb{C}[F(n)] \to \mathbb{C}[t, t^{-1}]$, and the Alexander matrix A is the $m \times n$ matrix with $a_{i,j} = \gamma\left(\frac{\partial}{t_j} r_i\right)$. To find the Alexander polynomial $\Delta(t)$, we calculate the greatest common divisor of the $(n-1) \times (n-1)$ minors of A. Note the Alexander polynomial is unique up to multiplication by $\pm t^n$ where $n \in \mathbb{Z}$.

Example 5.2. We calculate the Alexander matrix and Alexander polynomial for Example 4.3. The group presentation is

$$\langle t_1, t_2, t_3 | [t_2, t_3], [t_1, t_3], [t_1, t_2] \rangle.$$

We show the calculation for the first row of the Alexander matrix. First, rewrite $[t_2, t_3]$ as $t_2 t_3 t_2^{-1} t_3^{-1} = 1$.

$$\begin{aligned} \frac{\partial}{\partial t_1}(t_2 t_3 t_2^{-1} t_3^{-1}) &= 0 && \mapsto \quad 0 \\ \frac{\partial}{\partial t_2}(t_2 t_3 t_2^{-1} t_3^{-1}) &= 1 - t_2 t_3 t_2^{-1} && \mapsto \quad 1 - t \\ \frac{\partial}{\partial t_3}(t_2 t_3 t_2^{-1} t_3^{-1}) &= t_2 - t_2 t_3 t_2^{-1} t_3^{-1} && \mapsto \quad t - 1 \end{aligned}$$

The Alexander matrix is $A = \begin{pmatrix} 0 & 1-t & t-1 \\ 1-t & 0 & t-1 \\ 1-t & t-1 & 0 \end{pmatrix}$. To find the Alexander polynomial, we calculate the greatest common divisor of the 2×2 minors, which turns out to be $\Delta(t) = (1-t)^2$.

5.2. **Elementary ideals.** In this section, we follow the exposition from [3]. Consider an $m \times n$ matrix A with entries in $\mathbb{C}[t, t^{-1}]$. For $k \in \mathbb{N} \cup \{0\}$, the kth elementary ideal of A is defined by

$$E_k(A) = \begin{cases} 0 & n - k > m \\ \mathbb{C}[t, t^{-1}] & n - k \leq 0 \\ \text{Ideal generated by all } (n-k) \times (n-k) \text{ minors of } A & 0 < n - k \leq m \end{cases},$$

and these ideals form an ascending chain

$$E_0(A) \subset E_1(A) \subset \cdots \subset E_n(A) = E_{n+1}(A) = \cdots = \mathbb{C}[t, t^{-1}].$$

We briefly explain why this ascending chain condition holds. Observe that in $\mathbb{C}[t, t^{-1}]$, the ideal $E_k(A)$ is principal and the element that generates $E_k(A)$ is the greatest common divisor of the $(n-k) \times (n-k)$ minors. Therefore, it suffices to show the greatest common divisor of the $(n-1) \times (n-1)$ minors divides the greatest common divisor of the $n \times n$ minors. This statement holds due to the fact that the determinant of an $n \times n$ matrix can be written as a combination of the cofactors of the elements of a row or column.

In the following lemmas, we assume $0 < n - 1 \leq m$.

Lemma 5.2. Let A be an $m \times n$ matrix and suppose we add a row of zeros after the last row. Denote this $(m+1) \times n$ matrix

$$\left(\begin{array}{ccc} & A & \\ \hdashline 0 & \cdots & 0 \end{array}\right)$$

as $\mathscr{A}$. Then, $E_1(A) = E_1(\mathscr{A})$.

Proof. Consider any $(n-1) \times (n-1)$ minor of A with submatrix C. Let R represent the set of rows and columns we deleted from A to obtain C. Then, $\det(C)$ is equal to the determinant of the $(n-1) \times (n-1)$ submatrix of $\mathscr{A}$ with R and the row of zeros removed. Hence, $E_1(A) \subset E_1(\mathscr{A})$. Now suppose we have an $(n-1) \times (n-1)$ minor of $\mathscr{A}$ with submatrix $\mathscr{C}$. We just covered the case where $\mathscr{C}$ does not include the row of zeros so let us keep that row. Then, the determinant of $\mathscr{C}$ is zero, which does not affect $E_1(\mathscr{A})$. Therefore, $E_1(\mathscr{A}) \subset E_1(A)$. $\square$

Lemma 5.3. Let A be an $m \times n$ matrix and define an $(m+1) \times (n+1)$ matrix $\mathscr{A}$:

$$\left(\begin{array}{ccc:c} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ \hdashline 0 & \cdots & 0 & 1 \end{array}\right).$$

Then, $E_1(A) = E_1(\mathscr{A})$.

Proof. Let us choose any $(n-1) \times (n-1)$ minor of A with submatrix C. Using C, we can construct an $n \times n$ submatrix of $\mathscr{A}$ such that the determinant is the same as $\det(C)$.

$$\left(\begin{array}{ccc:c} & & & 0 \\ & C & & \vdots \\ & & & 0 \\ \hdashline 0 & \cdots & 0 & 1 \end{array}\right)$$

Thus, $E_1(A) \subset E_1(\mathscr{A})$.
Now consider an $n \times n$ minor of $\mathscr{A}$ with submatrix $\mathscr{C}$.
i) $\mathscr{C}$ is one of the following matrices.

$$\left(\begin{array}{ccc} & C & \\ \hdashline 0 & \cdots & 0 \end{array}\right) \qquad \left(\begin{array}{c:c} & 0 \\ C & \vdots \\ & 0 \end{array}\right)$$

Then, $\det \mathscr{C} = 0$ so we can disregard this case.
ii) If $\mathscr{C}$ contains the last row and last column of $\mathscr{A}$, then $\det(\mathscr{C})$ is equal to an $(n-1) \times (n-1)$ minor of A.
iii) If we delete the last row and last column of $\mathscr{A}$, then $\det(\mathscr{C})$ is the determinant of an $n \times n$ submatrix of A. We then use the ascending chain condition, $E_0(A) \subset E_1(A)$, to conclude $E_1(\mathscr{A}) \subset E_1(A)$. $\square$

By properties of determinants, we know adding a linear combination of existing rows to a row of a matrix leaves the determinant unchanged. Therefore, $\mathscr{A}$ in Lemma 5.2 can be replaced with

$$\left(\begin{array}{ccc} & A & \\ \hdashline b_{m+1,1} & \cdots & b_{m+1,n} \end{array}\right)$$

where $b_{m+1,j}$ is a linear combination of the previous rows. Similarly, adding a linear combination of existing columns to a column of a matrix doesn't affect the determinant, so $\mathscr{A}$ in Lemma 5.3 can be replaced with

$$\left(\begin{array}{ccc:c} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ \hdashline c_{m+1,1} & \cdots & c_{m+1,n} & 1 \end{array}\right)$$

where $c_{m+1,j}$ is arbitrary.

Note Lemma 5.2 is equivalent to the following statement: the greatest common divisor of the $(n-1)\times(n-1)$ minors of A is equal to the greatest common divisor of the $(n-1)\times(n-1)$ minors of $\mathscr{A}$. A similar translation follows for Lemma 5.3.

Theorem 5.4. The Alexander polynomial is the same after modifying a group presentation by Tietze transformations.

Proof. Suppose the presentation for the fundamental group is

$$\langle t_1, \ldots, t_n | r_1, \ldots, r_m \rangle$$

so the Alexander matrix is

$$A = \begin{pmatrix} \gamma\left(\frac{\partial}{\partial t_1} r_1\right) & \gamma\left(\frac{\partial}{\partial t_2} r_1\right) & \cdots & \gamma\left(\frac{\partial}{\partial t_n} r_1\right) \\ \gamma\left(\frac{\partial}{\partial t_1} r_2\right) & \gamma\left(\frac{\partial}{\partial t_2} r_2\right) & \cdots & \gamma\left(\frac{\partial}{\partial t_n} r_2\right) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma\left(\frac{\partial}{\partial t_1} r_m\right) & \gamma\left(\frac{\partial}{\partial t_2} r_m\right) & \cdots & \gamma\left(\frac{\partial}{\partial t_n} r_m\right) \end{pmatrix}.$$

i) Let us add a new relation s, which, by definition, is an element of the smallest normal subgroup of $F(n)$ that contains $r_1, \ldots, r_m$. The new presentation is then

$$\langle t_1, \ldots, t_n | r_1, \ldots, r_m, s \rangle$$

with

$$s = \prod_{k=1}^{d} g_k r_{i_k}^{\alpha_k} g_k^{-1}$$

where $\alpha_k = \pm 1$, $1 \leq i_k \leq m$, and g_k is a word with t_β's. If we add a relation, then the number of rows of the Alexander matrix increases by one. Each entry in the new row is given by

$$\begin{aligned} a_{m+1,j} &= \gamma\left(\frac{\partial}{\partial t_j} \prod_{k=1}^{d} g_k r_{i_k}^{\alpha_k} g_k^{-1}\right) \\ &= \gamma\left(\frac{\partial}{\partial t_j} g_1 r_{i_1}^{\alpha_1} g_1^{-1}\right) + \gamma\left(g_1 r_{i_1}^{\alpha_1} g_1^{-1}\right)\gamma\left(\frac{\partial}{\partial t_j} g_2 r_{i_2}^{\alpha_2} g_2^{-1}\right) + \cdots + \gamma\left(\prod_{k=1}^{d-1} g_k r_{i_k}^{\alpha_k} g_k^{-1}\right)\gamma\left(\frac{\partial}{\partial t_j} g_d r_{i_d}^{\alpha_d} g_d^{-1}\right) \\ &= \sum_{k=1}^{d} \gamma\left(\frac{\partial}{\partial t_j} g_k r_{i_k}^{\alpha_k} g_k^{-1}\right) \end{aligned}$$

where the last simplification follows because $\gamma(r_{i_k}) = 1$. Furthermore,

$$\begin{aligned} \frac{\partial}{\partial t_j} g_k r_{i_k}^{\alpha_k} g_k^{-1} &= \frac{\partial}{\partial t_j} g_k + g_k \frac{\partial}{\partial t_j} r_{i_k}^{\alpha_k} + g_k r_{i_k}^{\alpha_k} \frac{\partial}{\partial t_j} g_k^{-1} \\ &= \begin{cases} \frac{\partial}{\partial t_j} g_k + g_k \frac{\partial}{\partial t_j} r_{i_k} - g_k r_{i_k} g_k^{-1} \frac{\partial}{\partial t_j} g_k & \text{if } \alpha_k = 1 \\ \frac{\partial}{\partial t_j} g_k - g_k r_{i_k}^{-1} \frac{\partial}{\partial t_j} r_{i_k} - g_k r_{i_k}^{-1} g_k^{-1} \frac{\partial}{\partial t_j} g_k & \text{if } \alpha_k = -1 \end{cases}. \end{aligned}$$

We substitute the second expression into the first and simplify:

$$\begin{aligned} a_{m+1,j} &= \sum_{k=1}^{d} \gamma\left(\iota_k \frac{\partial}{\partial t_j} r_{i_k} + (1 - g_k r_{i_k}^{\alpha_k} g_k^{-1}) \frac{\partial}{\partial t_j} g_k\right) \\ &= \sum_{k=1}^{d} \gamma\left(\iota_k \frac{\partial}{\partial t_j} r_{i_k}\right) \\ &= \sum_{k=1}^{d} \gamma(\iota_k)\gamma\left(\frac{\partial}{\partial t_j} r_{i_k}\right) \end{aligned}$$

where

$$\iota_k = \begin{cases} g_k & \text{if } \alpha_k = 1 \\ -g_k r_{i_k}^{-1} & \text{if } \alpha_k = -1 \end{cases}.$$

We have shown that each entry of the last row is a linear combination of the previous rows. The Alexander matrix is

$$\begin{pmatrix} \gamma\left(\frac{\partial}{\partial t_1} r_1\right) & \cdots & \gamma\left(\frac{\partial}{\partial t_{n-1}} r_1\right) & \gamma\left(\frac{\partial}{\partial t_n} r_1\right) \\ \vdots & \ddots & \vdots & \vdots \\ \gamma\left(\frac{\partial}{\partial t_1} r_m\right) & \cdots & \gamma\left(\frac{\partial}{\partial t_{n-1}} r_m\right) & \gamma\left(\frac{\partial}{\partial t_n} r_m\right) \\ \sum_{k=1}^{d} \gamma(\iota_k)\gamma\left(\frac{\partial}{\partial t_1} r_{i_k}\right) & \cdots & \sum_{k=1}^{d} \gamma(\iota_k)\gamma\left(\frac{\partial}{\partial t_{n-1}} r_{i_k}\right) & \sum_{k=1}^{d} \gamma(\iota_k)\gamma\left(\frac{\partial}{\partial t_n} r_{i_k}\right) \end{pmatrix},$$

and by Lemma 5.2, the Alexander polynomial is the same.

ii) Let y be a new generator with relation given by $y = w$ where w is a word consisting of t_β's. The new presentation is

$$\langle t_1, \ldots, t_n, y | r_1, \ldots, r_m, yw^{-1} \rangle.$$

If we insert a new generator and relation, we must add one column and row to the Alexander matrix. We have

$$a_{m+1,n+1} = \gamma\left(\frac{\partial}{\partial y} yw^{-1}\right) = 1.$$

The remaining entries in column $n+1$ are

$$\gamma\left(\frac{\partial}{\partial y} r_i\right) = 0$$

since the original relations don't contain the new generator y. The first n entries of row $m+1$ are

$$\gamma\left(\frac{\partial}{\partial t_j} yw^{-1}\right).$$

Therefore, the new Alexander matrix is

$$\begin{pmatrix} \gamma\left(\frac{\partial}{\partial t_1} r_1\right) & \gamma\left(\frac{\partial}{\partial t_2} r_1\right) & \cdots & \gamma\left(\frac{\partial}{\partial t_n} r_1\right) & 0 \\ \gamma\left(\frac{\partial}{\partial t_1} r_2\right) & \gamma\left(\frac{\partial}{\partial t_2} r_2\right) & \cdots & \gamma\left(\frac{\partial}{\partial t_n} r_2\right) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \gamma\left(\frac{\partial}{\partial t_1} r_m\right) & \gamma\left(\frac{\partial}{\partial t_2} r_m\right) & \cdots & \gamma\left(\frac{\partial}{\partial t_n} r_m\right) & 0 \\ \gamma\left(\frac{\partial}{\partial t_1} yw^{-1}\right) & \gamma\left(\frac{\partial}{\partial t_2} yw^{-1}\right) & \cdots & \gamma\left(\frac{\partial}{\partial t_n} yw^{-1}\right) & 1 \end{pmatrix},$$

and by Lemma 5.3, the Alexander polynomial is the same. □

6. Applications and examples

Let $\mathcal{W}$ denote the wiring diagram in each section.

6.1. *n*-stranded wiring diagram with *n*-tuple points.

Let $n \geq 3$. Consider an n-stranded wiring diagram that has two n-tuple points with a braid $\beta_{1,2}$ in between.

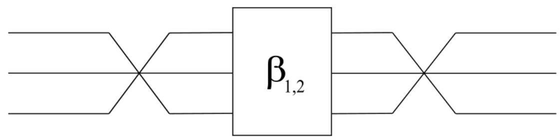

Figure 29. $n = 3$

By Theorem 4.1, the braid monodromy generators are

$$\Psi_1 = \mu_{I_1}^2 \quad \text{and} \quad \Psi_2 = \mu_{I_1}^{-1}\beta_{1,2}^{-1} \cdot \mu_{I_2}^2 \cdot \beta_{1,2}\mu_{I_1}$$

where

$$\mu_{I_1} = (\sigma_1 \cdots \sigma_{n-1}) \cdot (\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1\sigma_2) \cdot \sigma_1 = \mu_{I_2}.$$

Theorem 6.1. $\pi_1(\mathbb{C}^2 \setminus \mathcal{W}) = \langle t_1, \ldots, t_n | [t_1, \ldots, t_n] \rangle$

To prove this theorem, it suffices to show the first relation is $[t_1, \ldots, t_n]$ and $\Psi_2 = \Psi_1$ for any braid $\beta_{1,2}$. We first define some terms and prove a lemma.

Let Σ be a surface with fixed marked points $z_1, \ldots, z_n \in \Sigma$. Then, the mapping class group is defined to be the following:

$$\text{MCG}(\Sigma, (z_1, \ldots, z_n)) = \text{Homeo}(\Sigma, (z_1, \ldots, z_n))/\sim$$

where $\text{Homeo}(\Sigma, (z_1, \ldots, z_n))$ is the set of homeomorphisms $f : \Sigma \to \Sigma$ satisfying $f(z_i) = z_j$ and the equivalence relation is isotopy of maps. If we consider the surface $\mathbb{R}^2$, then $\text{MCG}(\mathbb{R}^2, (z_1, \ldots, z_n)) = B_n$. Now suppose we have a plane with n holes removed and a disk surrounding those n points. We know the braid $\mu_{I_1}^2$ is a full twist on n strands, meaning this action rotates the disk by 2π clockwise.

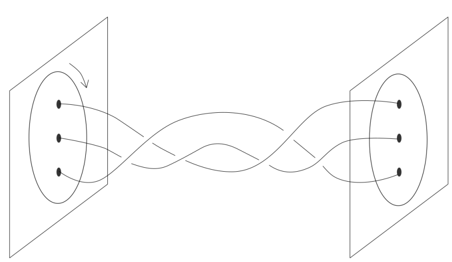

Figure 30. Full twist on 3 strands

Let $\psi \in B_n$ be this full twist and η be a loop representing the boundary of the disk. Then, $\eta = t_1 \cdots t_n$. For any loop γ around a point involved in η, we get $\psi(\gamma) = \eta\gamma\eta^{-1}$.

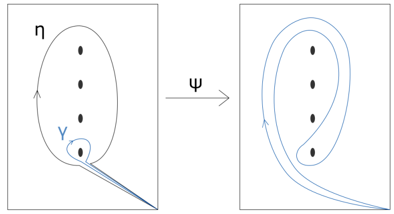

FIGURE 31. $\gamma \mapsto \psi(\gamma)$

Lemma 6.2. The full twist $\big((\sigma_1 \cdots \sigma_{n-1}) \cdot (\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1\sigma_2) \cdot \sigma_1\big)^2$ is an element of the center of the braid group B_n for $n \geq 3$.

Proof. Let $\Delta_n = (\sigma_1 \cdots \sigma_{n-1}) \cdot (\sigma_1 \cdots \sigma_{n-2}) \cdots (\sigma_1\sigma_2) \cdot \sigma_1$. We show $\sigma_i\Delta_n = \Delta_n\sigma_{n-i}$ where $1 \leq i \leq n-1$. This equality can be easily seen for small values of n. Suppose $n = 3$. Then,

$$\begin{aligned} \sigma_1\Delta_3 &= \sigma_1 \cdot \sigma_1\sigma_2\sigma_1 \\ &= \sigma_1 \cdot \underbrace{\sigma_2\sigma_1\sigma_2}_{\sigma_1\sigma_2\sigma_1} \\ &= \Delta_3\sigma_2. \end{aligned}$$

Now assume $\sigma_i\Delta_n = \Delta_n\sigma_{n-i}$ is true and $i \neq 1$. By definition, $\Delta_{n+1} = (\sigma_1 \cdots \sigma_n)\Delta_n$ so

$$\begin{aligned} \sigma_i\Delta_{n+1} &= \sigma_i(\sigma_1 \cdots \sigma_n)\Delta_n \\ &= \sigma_1 \cdots \sigma_{i-2}\sigma_i\sigma_{i-1}\sigma_i\sigma_{i+1} \cdots \sigma_n\Delta_n \\ &= \sigma_1 \cdots \sigma_{i-2}\underbrace{\sigma_{i-1}\sigma_i\sigma_{i-1}}_{\sigma_i\sigma_{i-1}\sigma_i} \cdots \sigma_n\Delta_n \\ &= \sigma_1 \cdots \sigma_{i-2}\sigma_{i-1}\sigma_i\sigma_{i+1} \cdots \sigma_n\sigma_{i-1}\Delta_n \\ &= \sigma_1 \cdots \sigma_n\Delta_n\sigma_{n+1-i} \\ &= \Delta_{n+1}\sigma_{n+1-i}. \end{aligned}$$

We consider the case when $i = 1$.

$$\begin{aligned} \sigma_1\Delta_{n+1} &= \sigma_1(\sigma_1 \cdots \sigma_n)(\sigma_1 \cdots \sigma_{n-1})\Delta_{n-1} && (1) \\ &= \sigma_1(\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2 \cdots \sigma_n\sigma_{n-1})\Delta_{n-1} && (2) \\ &= \sigma_1(\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3 \cdots \sigma_n\sigma_{n-1}\sigma_n)\Delta_{n-1} && (3) \\ &= \sigma_1(\sigma_2 \cdots \sigma_n)(\sigma_1 \cdots \sigma_n)\Delta_{n-1} && (4) \\ &= (\sigma_1 \cdots \sigma_n)(\sigma_1 \cdots \sigma_{n-1})\Delta_{n-1}\sigma_n && (5) \\ &= \Delta_{n+1}\sigma_n \end{aligned}$$

Each step for this case is explained below.
(1) $\to$ (2) We use the relation $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| \geq 2$.

$$\begin{aligned} \sigma_1(\sigma_1 \cdots \sigma_n)(\sigma_1\sigma_2\sigma_3 \cdots \sigma_{n-1})\Delta_{n-1} &= \sigma_1(\sigma_1\sigma_2\sigma_1\sigma_3 \cdots \sigma_n)(\sigma_2\sigma_3 \cdots \sigma_{n-1})\Delta_{n-1} \\ &= \sigma_1(\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4 \cdots \sigma_n)(\sigma_3 \cdots \sigma_{n-1})\Delta_{n-1} \\ &\;\;\vdots \\ &= \sigma_1(\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3 \cdots \sigma_n\sigma_{n-1})\Delta_{n-1} \end{aligned}$$

(2) → (3) We repeatedly apply the relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$.

$$\begin{aligned}\sigma_1(\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\cdots\sigma_n\sigma_{n-1})\Delta_{n-1} &= \sigma_1(\underbrace{\sigma_2\sigma_1\sigma_2}_{\sigma_1\sigma_2\sigma_1}\sigma_3\sigma_2\cdots\sigma_n\sigma_{n-1})\Delta_{n-1}\\ &= \sigma_1(\sigma_2\sigma_1\underbrace{\sigma_3\sigma_2\sigma_3}_{\sigma_2\sigma_3\sigma_2}\cdots\sigma_n\sigma_{n-1})\Delta_{n-1}\\ &\ \vdots\\ &= \sigma_1(\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\cdots\sigma_n\sigma_{n-1}\sigma_n)\Delta_{n-1}\end{aligned}$$

(3) → (4) We use the relation $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j|\geq 2$.

$$\begin{aligned}\sigma_1(\sigma_2\sigma_1\sigma_3\sigma_2\sigma_4\sigma_3\cdots\sigma_n\sigma_{n-1}\sigma_n)\Delta_{n-1} &= \sigma_1(\sigma_2\sigma_3\sigma_1\sigma_2\sigma_4\sigma_3\cdots\sigma_n\sigma_{n-1}\sigma_n)\Delta_{n-1}\\ &= \sigma_1(\sigma_2\sigma_3\sigma_4\sigma_1\sigma_2\sigma_3\cdots\sigma_n\sigma_{n-1}\sigma_n)\Delta_{n-1}\\ &\ \vdots\\ &= \sigma_1(\sigma_2\sigma_3\sigma_4\cdots\sigma_n)(\sigma_1\cdots\sigma_n)\Delta_{n-1}\end{aligned}$$

(4) → (5) Since every element in Δ_{n-1} is of the form σ_k where $1\leq k\leq n-2$, we conclude $|k-n|\geq 2$ and $\sigma_n\Delta_{n-1} = \Delta_{n-1}\sigma_n$.

Thus, $\sigma_i\Delta_n\Delta_n = \Delta_n\sigma_{n-i}\Delta_n = \Delta_n\Delta_n\sigma_i$. □

We now have the tools to prove the theorem.

Proof of Theorem 6.1. From earlier, we know $\psi(\gamma) = \eta\gamma\eta^{-1}$ where $\eta = t_1\cdots t_n$ and $\gamma = t_i$. Therefore, we conclude $\Psi_1(t_i) = t_1\cdots t_n t_i t_n^{-1}\cdots t_1^{-1}$. We derive the first relation in the following way.

$$\begin{aligned}&i=1: t_1\cdots t_n = t_2\cdots t_n t_1\\ &i=2: t_2t_1\cdots t_n = t_1\cdots t_n t_2 \quad\Rightarrow\quad t_1\cdots t_n = t_3\cdots t_n t_1 t_2\\ &i=3: t_3t_1\cdots t_n = t_1\cdots t_n t_3 \quad\Rightarrow\quad t_1\cdots t_n = t_4\cdots t_n t_1 t_2 t_3\\ &\quad\vdots\\ &i=n-1: t_{n-1}t_1\cdots t_n = t_1\cdots t_n t_{n-1} \quad\Rightarrow\quad t_1\cdots t_n = t_n t_1\cdots t_{n-1}\end{aligned}$$

Thus, the relation obtained from the first braid monodromy generator is $[t_1,\ldots,t_n]$. We now want to show $\Psi_2 = \Psi_1$ for any braid $\beta_{1,2}$, and this result follows immediately from Lemma 6.2 because $\mu_{I_2}^2$ commutes with every σ_i where $1\leq i\leq n-1$. □

Observe that adding more arbitrary braids and n-tuple points to $\mathcal{W}$ won't affect the group presentation. To see why, suppose $\Psi_k = \Psi_1$ for some $k>1$ and consider $\beta_{k,k+1}$. We can just apply Theorem 6.1 to conclude $\Psi_{k+1} = \Psi_k$. Hence, we have Theorem 1.1.

The relation $[t_1,\ldots,t_n]$ yields $n-1$ separate equations so we have an $(n-1)\times n$ Alexander matrix. The Alexander matrix is

$$A = \begin{pmatrix} 1-t^{n-1} & t-1 & t^2-t & \cdots & t^{n-1}-t^{n-2}\\ 1-t^{n-2} & t-t^{n-1} & t^2-1 & \cdots & t^{n-1}-t^{n-3}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1-t^2 & t-t^3 & \cdots & t^{n-2}-t^n & t^{n-1}-t\\ 1-t & t-t^2 & \cdots & t^{n-2}-t^{n-1} & t^{n-1}-1\end{pmatrix}.$$

Example 6.1. The Alexander matrix for Figure 29 is

$$\begin{pmatrix} 1-t^2 & t-1 & t^2-t\\ 1-t & t-t^2 & t^2-1\end{pmatrix}.$$

The Alexander matrix A obtained from the relation $[t_1,\ldots,t_n]$ has the property that if we take any column of A, then that column equals the sum of the other columns. Then, by properties of the determinant, we conclude the Alexander polynomial is the determinant of any $(n-1)\times(n-1)$ submatrix of A. Since the number of terms in the polynomial increases as n becomes larger, we conclude each fundamental group of an n-stranded wiring diagram with n-tuple points is different.

6.2. **Three-stranded wiring diagrams.**

6.2.1. *Three-stranded wiring diagram with two double points.*
Consider the braided wiring diagram given below.

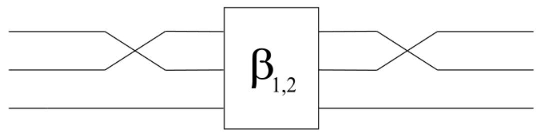

FIGURE 32. $\beta_{1,2} = \sigma_1^n$ for $n \in \mathbb{Z}$

The first braid monodromy generator $\Psi_1 = \sigma_2^2$ gives the relation $[t_2, t_3]$. The second braid monodromy generator is

$$\Psi_2 = \sigma_2^{-1}\sigma_1^{-n} \cdot \sigma_2^2 \cdot \sigma_1^n \sigma_2.$$

We use the following result to help us calculate the second relation.

Lemma 6.3. Let $k \geq 0$.

$$\begin{aligned}
\sigma_i^{2k}(t_{i+1}) &= (t_i t_{i+1})^{k-1} t_i t_{i+1} t_i^{-1} (t_{i+1}^{-1} t_i^{-1})^{k-1} & \sigma_i^{2k+1}(t_{i+1}) &= (t_i t_{i+1})^k t_i (t_{i+1}^{-1} t_i^{-1})^k \\
\sigma_i^{2k}(t_{i+1}^{-1}) &= (t_i t_{i+1})^{k-1} t_i t_{i+1}^{-1} t_i^{-1} (t_{i+1}^{-1} t_i^{-1})^{k-1} & \sigma_i^{2k+1}(t_{i+1}^{-1}) &= (t_i t_{i+1})^k t_i^{-1} (t_{i+1}^{-1} t_i^{-1})^k
\end{aligned}$$

$$\begin{aligned}
\sigma_i^{-2k}(t_{i+1}) &= (t_{i+1}^{-1} t_i^{-1})^k t_{i+1} (t_i t_{i+1})^k & \sigma_i^{-2k-1}(t_{i+1}) &= (t_{i+1}^{-1} t_i^{-1})^k t_{i+1}^{-1} t_i t_{i+1} (t_i t_{i+1})^k \\
\sigma_i^{-2k}(t_{i+1}^{-1}) &= (t_{i+1}^{-1} t_i^{-1})^k t_{i+1}^{-1} (t_i t_{i+1})^k & \sigma_i^{-2k-1}(t_{i+1}^{-1}) &= (t_{i+1}^{-1} t_i^{-1})^k t_{i+1}^{-1} t_i^{-1} t_{i+1} (t_i t_{i+1})^k
\end{aligned}$$

Proof. We only need to consider cases of the form $\sigma_i^n(t_{i+1})$ because $\sigma_i^n(t_{i+1}^{-1}) = \big(\sigma_i^n(t_{i+1})\big)^{-1}$ by definition. We use induction to prove this lemma.
Base case for $\sigma_i^{2k}(t_{i+1})$: $\sigma_i^2(t_{i+1}) = t_i t_{i+1} t_i^{-1}$
Inductive step: We assume the expression $\sigma_i^{2k}(t_{i+1}) = (t_i t_{i+1})^{k-1} t_i t_{i+1} t_i^{-1} (t_{i+1}^{-1} t_i^{-1})^{k-1}$ holds until some positive k. Then,

$$\begin{aligned}
\sigma_i^{2(k+1)}(t_{i+1}) &= \sigma_i^{2k} \sigma_i^2(t_{i+1}) \\
&= \sigma_i^2 \bigg((t_i t_{i+1})^{k-1} t_i t_{i+1} t_i^{-1} (t_{i+1}^{-1} t_i^{-1})^{k-1} \bigg) \\
&= (t_i t_{i+1})^{k-1} t_i t_{i+1} t_i t_{i+1} t_i^{-1} t_{i+1}^{-1} t_i^{-1} (t_{i+1}^{-1} t_i^{-1})^{k-1} \\
&= (t_i t_{i+1})^k t_i t_{i+1} t_i^{-1} (t_{i+1}^{-1} t_i^{-1})^k.
\end{aligned}$$

The other cases are omitted. □

Proposition 6.4. Let $k \geq 0$. If $\beta_{1,2} = \sigma_1^n$, then the second relation is one of the following.

$$\begin{aligned}
&[t_2, (t_1 t_3)^{k-1} t_1 t_3 t_1^{-1} (t_3^{-1} t_1^{-1})^{k-1}] && n = 2k \\
&[t_2, (t_1^{-1} t_3^{-1})^{k-1} t_1^{-1} t_3 t_1 (t_3 t_1)^{k-1}] && n = -2k \\
&[t_2, (t_1 t_3)^k t_1 (t_3^{-1} t_1^{-1})^k] && n = 2k+1 \\
&[t_2, (t_1^{-1} t_3^{-1})^k t_1 (t_3 t_1)^k] && n = -2k-1
\end{aligned}$$

Proof. Suppose $n = 2k$ so

$$\Psi_2 = \sigma_2^{-1}\sigma_1^{-2k} \cdot \sigma_2^2 \cdot \sigma_1^{2k} \sigma_2.$$

We use Lemma 6.3 to apply everything except σ_2 to t_2.

$$(t_1 t_2)^{k-1} t_1 t_2 t_1^{-1} (t_2^{-1} t_1^{-1})^{k-1} t_3 (t_1 t_2)^{k-1} t_1 t_2^{-1} t_1^{-1} (t_2^{-1} t_1^{-1})^{k-1}$$

From the first generator, we know t_2 and t_3 commute so applying the last σ_2 gives the relation

$$\Psi_2(t_2) = (t_1t_3)^{k-1}t_1t_3t_1^{-1}(t_3^{-1}t_1^{-1})^{k-1}t_2(t_1t_3)^{k-1}t_1t_3^{-1}t_1^{-1}(t_3^{-1}t_1^{-1})^{k-1}.$$

The other cases are proved in a similar manner. □

If $\beta_{1,2}$ is the trivial braid, then $\pi_1(\mathbb{C}^2 \setminus \mathcal{W}) = \langle t_1, t_2, t_3 | [t_2, t_3] \rangle$. Since we have one relation and three generators, this group gives a 1×3 Alexander matrix. However, we want to find the 2×2 minors for the Alexander polynomial. To fix this issue, let $k > 0$ if n is even and $k \geq 0$ if n is odd. We list the Alexander matrix and Alexander polynomial for different values of n.

$$\begin{pmatrix} 0 & 1-t & t-1 \\ (1-t)^2\left(\sum_{i=0}^{k-1} t^{2i}\right) & t-1 & t(1-t)\left(\sum_{i=0}^{2k-2}(-1)^i t^i\right) \end{pmatrix} \qquad \Delta_{2k}(t) = (1-t)^3\left(\sum_{i=0}^{k-1} t^{2i}\right)$$

$$\begin{pmatrix} 0 & 1-t & t-1 \\ -\frac{1}{t}(1-t)^2\left(\sum_{i=0}^{k-1} t^{-2i}\right) & t-1 & \frac{1}{t}(1-t)\left(\sum_{i=0}^{2k-2}(-1)^i t^{-i}\right) \end{pmatrix} \qquad \Delta_{-2k}(t) = \frac{1}{t}(1-t)^3\left(\sum_{i=0}^{k-1} t^{-2i}\right)$$

$$\begin{pmatrix} 0 & 1-t & t-1 \\ (1-t)\left(\sum_{i=0}^{2k}(-1)^i t^i\right) & t-1 & t(1-t)\left(\sum_{i=0}^{2k-1}(-1)^i t^i\right) \end{pmatrix} \qquad \Delta_{2k+1}(t) = (1-t)^2\left(\sum_{i=0}^{2k}(-1)^i t^i\right)$$

$$\begin{pmatrix} 0 & 1-t & t-1 \\ (1-t)\left(\sum_{i=0}^{2k}(-1)^i t^{-i}\right) & t-1 & \frac{1}{t}(1-t)\left(\sum_{i=0}^{2k-1}(-1)^i t^{-i}\right) \end{pmatrix} \qquad \Delta_{-2k-1}(t) = (1-t)^2\left(\sum_{i=0}^{2k}(-1)^i t^{-i}\right)$$

By observing this list, we conclude this leads to Theorem 1.2. Note this theorem also holds if we consider negative values of n.

6.2.2. *Three-stranded wiring diagram with three double points.*
We add another braid and vertex to our diagram from the previous section.

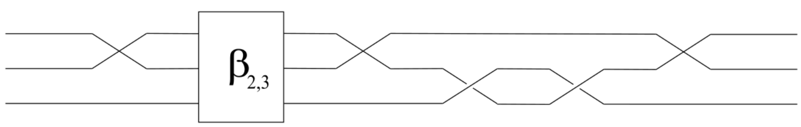

FIGURE 33. $\beta_{1,2} = \sigma_1^2$ and $\beta_{2,3} = \sigma_1^n$

The first two relations are $[t_2, t_3]$ and $[t_2, t_1t_3t_1^{-1}]$. The third braid monodromy generator is

$$\Psi_3 = \sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}\sigma_1^{-n} \cdot \sigma_2^2 \cdot \sigma_1^n\sigma_2\sigma_1^2\sigma_2.$$

Lemma 6.5. If $\beta_{1,2} = \sigma_1^2$ and $\beta_{2,3} = \sigma_1^k$ where $k \geq 2$, then

$$\Psi_3 = \sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}\sigma_1^{-k} \cdot \sigma_2^2 \cdot \sigma_1^k\sigma_2\sigma_1^2\sigma_2 = \sigma_1^{-k+2}\sigma_2^2\sigma_1^{k-2}.$$

Proof. We prove this claim using induction. When $k = 2$, the third braid monodromy generator is

$$\Psi_3 = \sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}\sigma_1^{-2} \cdot \sigma_2^2 \cdot \sigma_1^2\sigma_2\sigma_1^2\sigma_2 = \sigma_2^2.$$

Now suppose the statement holds for all natural numbers until k. Let $\beta_{2,3} = \sigma_1^{k+1}$. Then,

$$\begin{aligned} \Psi_3 &= \sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}\sigma_1^{-k-1} \cdot \sigma_2^2 \cdot \sigma_1^{k+1}\sigma_2\sigma_1^2\sigma_2 \\ &= \sigma_1^{-1}\left(\sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}\sigma_1^{-k} \cdot \sigma_2^2 \cdot \sigma_1^k\sigma_2\sigma_1^2\sigma_2\right)\sigma_1 \\ &= \sigma_1^{-1} \cdot \sigma_1^{-k+2}\sigma_2^2\sigma_1^{k-2} \cdot \sigma_1 \\ &= \sigma_1^{-k+1}\sigma_2^2\sigma_1^{k-1}. \end{aligned}$$

□

Proposition 6.6. Let $\beta_{1,2} = \sigma_1^2$ and $\beta_{2,3} = \sigma_1^n$. Suppose $k \geq 1$ if n is even and $k \geq 0$ if n is odd. The third relation is one of the following.

$$\begin{array}{ll} [t_3, (t_1t_2)^{k-2}t_1t_2t_1^{-1}(t_2^{-1}t_1^{-1})^{k-2}] & n = 2k \\ [t_3, (t_1^{-1}t_2^{-1})^k t_1^{-1}t_2t_1(t_2t_1)^k] & n = -2k \\ [t_3, (t_1t_2)^{k-1}t_1(t_2^{-1}t_1^{-1})^{k-1}] & n = 2k+1 \\ [t_3, (t_1^{-1}t_2^{-1})^{k+1}t_1(t_2t_1)^{k+1}] & n = 2k-1 \end{array}$$

Proof. Let $n = 2k$. By Lemma 6.5, we conclude $\Psi_3 = \sigma_1^{-2k+2}\sigma_2^2\sigma_1^{2k-2}$. To obtain the third relation, we use Lemma 6.3.

$$\Psi_3(t_3) = (t_1t_2)^{k-2}t_1t_2t_1^{-1}(t_2^{-1}t_1^{-1})^{k-2}t_3(t_1t_2)^{k-2}t_1t_2^{-1}t_1^{-1}(t_2^{-1}t_1^{-1})^{k-2}$$

The proof is the same when $n = 2k+1$, but we check what happens when $n = 1$ because Lemma 6.5 only applies to σ_1^n for $n \geq 2$. If $\beta_{2,3} = \sigma_1$, then

$$\Psi_3 = \sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}\sigma_1^{-1} \cdot \sigma_2^2 \cdot \sigma_1\sigma_2\sigma_1^2\sigma_2 = \sigma_2^{-1}\sigma_1^2\sigma_2$$

and the relation is $[t_3, t_1]$, which agrees with our formula.

Now suppose $n = -2k$. The third braid monodromy generator is given by

$$\Psi_3 = \sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}\sigma_1^{2k} \cdot \sigma_2^2 \cdot \sigma_1^{-2k}\sigma_2\sigma_1^2\sigma_2.$$

We calculate $\Psi_3(t_3)$ using Lemma 6.3 followed by some simplifications:

$$\Psi_3(t_3) = (t_1^{-1}t_2^{-1})^k t_1^{-1}t_2t_1(t_2t_1)^k t_3(t_1^{-1}t_2^{-1})^k t_1^{-1}t_2^{-1}t_1(t_2t_1)^k.$$

The same method applies when $n = -2k-1$. □

If $\beta_{2,3}$ is the trivial braid or $\beta_{2,3} = \sigma_1^2$, then both fundamental groups are given by

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{W}) = \langle t_1, t_2, t_3 | [t_2, t_3], [t_2, t_1t_3t_1^{-1}] \rangle.$$

We already calculated the Alexander matrix and Alexander polynomial for this group in the previous section so we only consider $k \geq 2$ if $n = 2k$. If $n = -2k$, then let $k \geq 1$. The braids $\beta_{2,3} = \sigma_1^{\pm 1}$ and $\beta_{2,3} = \sigma_1^3$ give the same relation so let $k \geq 1$ for odd values of n.

$$\begin{pmatrix} 0 & 1-t & t-1 \\ 1-2t+t^2 & t-1 & t-t^2 \\ (1-t)^2\left(\sum_{i=0}^{k-2} t^{2i}\right) & t(1-t)\left(\sum_{i=0}^{2k-4}(-1)^i t^i\right) & t-1 \end{pmatrix} \qquad \Delta_{2k}(t) = (t-1)^3$$

$$\begin{pmatrix} 0 & 1-t & t-1 \\ 1-2t+t^2 & t-1 & t-t^2 \\ -\frac{1}{t}(1-t)^2\left(\sum_{i=0}^{k} t^{2i}\right) & \frac{1}{t}(1-t)\left(\sum_{i=0}^{2k}(-1)^i t^{-i}\right) & t-1 \end{pmatrix} \qquad \Delta_{-2k}(t) = (t-1)^3$$

$$\begin{pmatrix} 0 & 1-t & t-1 \\ 1-2t+t^2 & t-1 & t-t^2 \\ (1-t)\left(\sum_{i=0}^{2k-2}(-1)^i t^i\right) & t(1-t)\left(\sum_{i=0}^{2k-3}(-1)^i t^i\right) & t-1 \end{pmatrix} \qquad \Delta_{2k+1}(t) = (t-1)^2$$

$$\begin{pmatrix} 0 & 1-t & t-1 \\ 1-2t+t^2 & t-1 & t-t^2 \\ (1-t)\left(\sum_{i=0}^{2k+2}(-1)^i t^{-i}\right) & \frac{1}{t}(1-t)\left(\sum_{i=0}^{2k+1}(-1)^i t^{-i}\right) & t-1 \end{pmatrix} \qquad \Delta_{-2k-1}(t) = (t-1)^2$$

Let us now switch the braids.

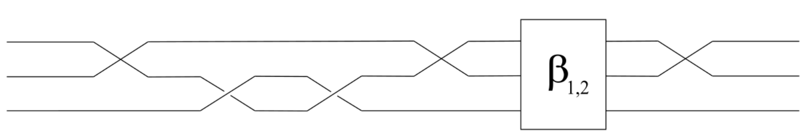

FIGURE 34. $\beta_{1,2} = \sigma_1^n$ and $\beta_{2,3} = \sigma_1^2$

We know the possible values for the second relation from Section 6.2.1. The third braid monodromy generator is

$$\Psi_3 = \sigma_2^{-1}\sigma_1^{-n}\sigma_2^{-1}\sigma_1^{-2} \cdot \sigma_2^2 \cdot \sigma_1^2\sigma_2\sigma_1^n\sigma_2.$$

If $\beta_{1,2} = 1$, then $\Psi_3 = \sigma_2^{-2}\sigma_1^{-2}\sigma_2^2\sigma_1^2\sigma_2^2$ and the third relation is $[t_2, t_1^{-1}t_3t_1]$.

Lemma 6.7. If $\beta_{1,2} = \sigma_1^n$ for $n \geq 2$ and $\beta_{2,3} = \sigma_1^2$, then

$$\Psi_3 = \sigma_2^{-1}\sigma_1^{-n}\sigma_2^{-1}\sigma_1^{-2} \cdot \sigma_2^2 \cdot \sigma_1^2\sigma_2\sigma_1^n\sigma_2 = \sigma_2^{-1}\sigma_1^{-n+2}\sigma_2^2\sigma_1^{n-2}\sigma_2.$$

Proof. Simplify using braid relations. □

Proposition 6.8. Let $k \geq 1$ if n is even and $k \geq 0$ if n is odd. If $\beta_{1,2} = \sigma_1^n$ and $\beta_{2,3} = \sigma_1^2$, then the third relation is one of the following.

$$\begin{array}{ll} [t_2, (t_1t_3)^{k-2}t_1t_3t_1^{-1}(t_3^{-1}t_1^{-1})^{k-2}] & \quad n = 2k \\ [t_2, (t_1^{-1}t_3^{-1})^{k}t_1^{-1}t_3t_1(t_3t_1)^{k}] & \quad n = -2k \\ [t_2, (t_1t_3)^{k-1}t_1(t_3^{-1}t_1^{-1})^{k-1}] & \quad n = 2k+1 \\ [t_2, (t_1^{-1}t_3^{-1})^{k+1}t_1(t_3t_1)^{k+1}] & \quad n = -2k-1 \end{array}$$

Proof. It is similar to the proof of Proposition 6.6. □

The Alexander polynomials for this group of wiring diagrams are $\Delta_n(t) = (t-1)^3$ for even values of n and $\Delta_n(t) = (t-1)^2$ for odd values of n. These are the exact same polynomials as the wiring diagrams with $\beta_{1,2} = \sigma_1^2$ and $\beta_{2,3} = \sigma_1^n$. Therefore, they do not help us differentiate the fundamental groups. Moreover, the relations are difficult to compare so some future work involving these wiring diagrams would be to try to find a way that gives us a more concrete relationship between the fundamental groups. In other words, how exactly does switching braids in our three-stranded wiring diagram affect the fundamental group?

6.2.3. *Three-stranded wiring diagram with one double point and triple point.* Let ψ be an arbitrary three-stranded wiring diagram that is placed between a double point and triple point.

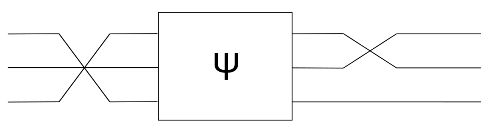

FIGURE 35. ψ can contain type c singularities or braids

Theorem 6.9. Suppose the triple point in the figure is vertex k. The kth relation is given by $[t_1, t_2, t_3]$. In particular, the fundamental group is abelian.

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{W}) = \langle t_1, t_2, t_3 | [t_1, t_2], [t_1, t_3], [t_2, t_3] \rangle$$

Proof. The kth braid monodromy generator is

$$\Psi_k = \beta_k^{-1}\mu_{I_k}^2\beta_k.$$

We know $\mu_{I_k}^2 = \sigma_1\sigma_2\sigma_1^2\sigma_2\sigma_1 \in Z(B_3)$ so we conclude $\Psi_3 = \mu_{I_k}^2$, giving our desired relation. The fundamental group is abelian because we can use the first relation $[t_2, t_3]$ and third relation $[t_1, t_2, t_3]$ to derive the two remaining commutators: $[t_1, t_2]$, $[t_1, t_3]$. □

Since every generator of the fundamental group commutes with each other, the remaining relations coming from $\Psi_2, \ldots, \Psi_{k-1}$ can always be simplified. Furthermore, we can add more vertices or braids after the last triple point and it will not change the presentation. The Alexander matrix is then

$$\begin{pmatrix} 1-t & t-1 & 0 \\ 1-t & 0 & t-1 \\ 0 & 1-t & t-1 \end{pmatrix}$$

and the Alexander polynomial is $\Delta(t) = (1-t)^2$.

6.3. **Four-stranded wiring diagrams.** We now consider some four-stranded wiring diagrams that satisfy the property that the product of the braid monodromy generators is equal to the full twist on four strands:

$$\prod_{i=1}^{k} \Psi_i = \Delta_4^2, \quad k = \# \text{ of singularities.}$$

6.3.1. *Four-stranded wiring diagram with three cusps and three branch points.*

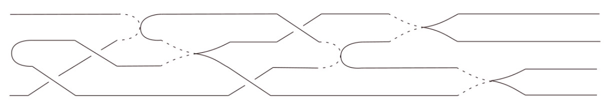

Figure 36

The braid monodromy generators are

$$\begin{aligned}
&\Psi_1 = \sigma_1^3,\\
&\Psi_2 = \sigma_1^{-1}\sigma_3^3\sigma_1 = \sigma_3^3,\\
&\Psi_3 = \sigma_1^{-1}\sigma_3^{-1}\sigma_2\sigma_3\sigma_1,\\
&\Psi_4 = \sigma_1^{-1}\sigma_3^{-1}\sigma_3\sigma_1\sigma_2^3\sigma_1^{-1}\sigma_3^{-1}\sigma_3\sigma_1 = \sigma_2^3,\\
&\Psi_5 = \sigma_1^{-1}\sigma_3^{-1}\sigma_3\sigma_1\sigma_2^{-1}\sigma_3\sigma_2\sigma_1^{-1}\sigma_3^{-1}\sigma_3\sigma_1 = \sigma_2^{-1}\sigma_3\sigma_2,\\
&\Psi_6 = \sigma_1^{-1}\sigma_3^{-1}\sigma_3\sigma_1\sigma_2^{-1}\sigma_2\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_2\sigma_3^{-1}\sigma_1^{-1}\sigma_3\sigma_1 = \sigma_1\sigma_2\sigma_1^{-1}.
\end{aligned}$$

The relation associated to each singularity is the following.

$$\begin{aligned}
\Psi_1(t_1) &= t_1t_2t_1t_2t_1^{-1}t_2^{-1}t_1^{-1}\\
\Psi_2(t_3) &= t_3t_4t_3t_4t_3^{-1}t_4^{-1}t_3^{-1}\\
\Psi_3(t_1) &= t_1t_3t_4t_3^{-1}t_1^{-1}\\
\Psi_4(t_2) &= t_2t_3t_2t_3t_2^{-1}t_3^{-1}t_2^{-1}\\
\Psi_5(t_2) &= t_2t_4t_2^{-1}\\
\Psi_6(t_3) &= t_2^{-1}t_1t_2
\end{aligned}$$

Then,

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{W}) = \left\langle t_1, t_2, t_3, t_4 | t_1t_2t_1 = t_2t_1t_2, t_3t_4t_3 = t_4t_3t_4, t_1 = t_3t_4t_3^{-1}, t_2t_3t_2 = t_3t_2t_3, t_2 = t_4, t_3 = t_2^{-1}t_1t_2 \right\rangle.$$

Note we can simplify this presentation further.

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{W}) = \left\langle t_1, t_2, t_3 | t_1t_2t_1 = t_2t_1t_2, t_2t_3t_2 = t_3t_2t_3, t_1 = t_3t_2t_3^{-1}, t_1 = t_2t_3t_2^{-1} \right\rangle$$

The Alexander matrix is

$$A = \begin{pmatrix} 1+t^2-t & t-t^2+1 & 0 \\ 0 & 1+t^2-t & t-t^2+1 \\ 1 & -t & t-1 \\ 1 & t-1 & -t \end{pmatrix}$$

and the Alexander polynomial is $\Delta(t) = 1$.

6.3.2. *Four-stranded wiring diagram with two type b singularities and four branch points.*

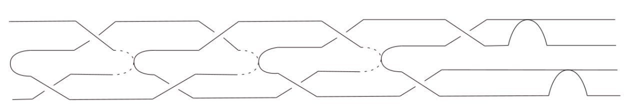

Figure 37

The braid monodromy generators are

$$\begin{aligned}
\Psi_1 &= \sigma_1^4,\\
\Psi_2 &= \sigma_1^{-2}\sigma_3^4\sigma_1^2 = \sigma_3^4,\\
\Psi_3 &= \sigma_1^{-2}\sigma_3^{-2}\sigma_3\sigma_1\sigma_2\sigma_1^{-1}\sigma_3^{-1}\sigma_3^2\sigma_1^2 = \sigma_1^{-1}\sigma_3^{-1}\sigma_2\sigma_3\sigma_1,\\
\Psi_4 &= \sigma_1^{-2}\sigma_3^{-2}\sigma_3\sigma_1\sigma_3\sigma_1\sigma_2\sigma_1^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_3^2\sigma_1^2 = \sigma_2,\\
\Psi_5 &= \sigma_1^{-2}\sigma_3^{-2}\sigma_3\sigma_1\sigma_3\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3\sigma_1^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_3^2\sigma_1^2 = \sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3,\\
\Psi_6 &= \sigma_1^{-2}\sigma_3^{-2}\sigma_3\sigma_1\sigma_3\sigma_1\sigma_3^{-1}\sigma_1^{-1}\sigma_3\sigma_1\sigma_2\sigma_1^{-1}\sigma_3^{-1}\sigma_1\sigma_3\sigma_1^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_3^2\sigma_1^2 = \sigma_2.
\end{aligned}$$

Since $\Psi_3 = \Psi_5$ and $\Psi_4 = \Psi_6$ and the strands involved in these singularities are the same, we only need to consider the first four braid monodromy generators to obtain the relations for the fundamental group.

$$\begin{aligned}
\Psi_1(t_1) &= t_1t_2t_1t_2t_1t_2^{-1}t_1^{-1}t_2^{-1}t_1^{-1},\\
\Psi_2(t_3) &= t_3t_4t_3t_4t_3t_4^{-1}t_3^{-1}t_4^{-1}t_3^{-1},\\
\Psi_3(t_1) &= t_1t_3t_4t_3^{-1}t_1^{-1},\\
\Psi_4(t_2) &= t_2t_3t_2^{-1}.
\end{aligned}$$

After some simplifications, we get

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{W}) = \langle t_2, t_4 | t_2t_4t_2t_4 = t_4t_2t_4t_2\rangle .$$

We compute the Alexander matrix and polynomial.

$$A = \begin{pmatrix}1+t^2-t^3-t & t+t^3-t^2-1\end{pmatrix}, \quad \Delta(t) = 1+t^2-t^3-t.$$

6.3.3. *Four-stranded wiring diagram with three branch points and one type f_1 singularity.*

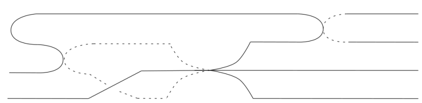

FIGURE 38

The braid monodromy generators are

$$\begin{aligned}
\Psi_1 &= \sigma_3,\\
\Psi_2 &= (\sigma_1\sigma_2\sigma_1)^3,\\
\Psi_3 &= \sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1 = \sigma_2^{-1}\sigma_1\sigma_2,\\
\Psi_4 &= \sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2 = \sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}\sigma_3\sigma_2\sigma_1^2\sigma_2,
\end{aligned}$$

and the relations are

$$\begin{aligned}
\Psi_1(t_3) &= t_3t_4t_3^{-1},\\
\Psi_2(t_1) &= t_1t_2t_3t_1t_2t_3t_2^{-1}t_1^{-1}t_3^{-1}t_2^{-1}t_1^{-1},\\
\Psi_2(t_2) &= t_1t_2t_3t_1t_2t_1^{-1}t_3^{-1}t_2^{-1}t_1^{-1},\\
\Psi_3(t_1) &= t_1t_2t_3t_2^{-1}t_1^{-1}\\
\Psi_4(t_3) &= t_1t_2t_3t_2^{-1}t_1^{-1}t_4^{-1}t_2^{-1}t_1^{-1}t_4t_1t_2t_4t_1t_2t_3^{-1}t_2^{-1}t_1^{-1}
\end{aligned}$$

Therefore, the fundamental group is

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{W}) = \langle t_2, t_3 | [t_2, t_3]\rangle .$$

Like the previous example, we have a 1×2 Alexander matrix.

$$A = \begin{pmatrix}1-t & t-1\end{pmatrix}$$

The Alexander polynomial is $\Delta(t) = 1 - t$.

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