Euler characteristics of surfaces

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1 Introduction

In this note we study Euler characteristics of surfaces and graphs on them. The Euler characteristic of a graph G is defined as $\chi(G) = V - E + F$ where V, E and F are respectively the numbers of vertices, edges and faces of a graph G.

In section 2, we discuss χ of graphs on a given surface. First, we prove Theorem 2.1 about the Euler characteristic of a planar graph with k connected components. Then, we move to the case where $k = 1$ (i.e when the graph is connected) but the surface is arbitrary and prove in Theorem 2.5 that χ is independent of the choice of a (connected) graph we have on a given surface, and we will provide two methods for doing so. We define the Euler characteristic $\chi(S)$ of a surface S as the Euler characteristic of arbitrary (good) graph on S.

In section 3, we study the Euler Characteristics of various surfaces. We will introduce orientability (and non-orientability), connected sum of two surfaces, and classification of surfaces.

The exposition mainly follows Chapter 4 in [1].

2 Euler characteristic on a given surface

In this section, we want to show that Euler characteristic is the same for any good graph on a given surface (could be any, e.g torus, sphere,etc).

Let G be a good graph (we shall define later), then we say that the Euler characteristic of G is $V - E + F$ where V = number of vertices, E = number of edges, and F = number of faces.

Here the number of faces F includes the unbounded infinite face. For example, a graph with no edges has one face.

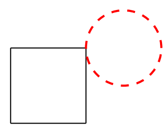
Theorem 2.1. *If G is a planar graph with k connected components, then the Euler characteristic for it is $V - E + F = k + 1$.*

Proof. Induction by the number of edges.
Base case: $E = 0$: We have 0 edges, i.e we only have k vertices, so

$$
\begin{gathered}
V = k \\
E = 0 \\
F = 1 \\
V - E + F = k + 1
\end{gathered}
$$

Step: $E \to E + 1$:
We add an edge, this increases E to $E + 1$. We have three different cases.

Case 1): New edge is a loop.



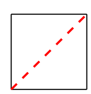
The numbers of vertices, edges and faces change as follows:

$$
\begin{aligned}
V &\to V \\
E &\to E + 1 \\
F &\to F + 1 \\
k &\to k
\end{aligned}
$$

Therefore

$$
V - (E + 1) + (F + 1) = V - E + F = k + 1
$$

Case 2): New edge is connected to another vertex within the same connected component:



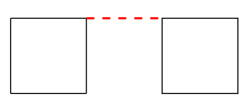
The numbers of vertices, edges and faces change as follows:

$$
\begin{aligned}
V &\to V \\
E &\to E + 1 \\
F &\to F + 1 \\
k &\to k
\end{aligned}
$$

Therefore

$$
V - (E + 1) + (F + 1) = V - E + F = k + 1
$$

Case 3): The new edge connects two connected components:



The numbers of vertices, edges and faces change as follows:

$$\begin{aligned} V &\to V \\ E &\to E+1 \\ F &\to F \\ k &\to k-1 \end{aligned}$$

Therefore

$$V-(E+1)+(F+0)=(k-1)+1=k$$

In this case the number of connected components decreases by 1. Therefore we conclude that in all three cases the equation $V-E+F=k+1$ remains true. ■

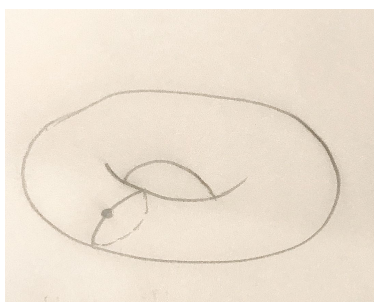
Corollary 2.2. *If G is a connected planar graph. then $v-e+f=2$.*

Remark 2.3. We can also consider planar graphs as graphs on the surface of the sphere by adding a point at infinity.

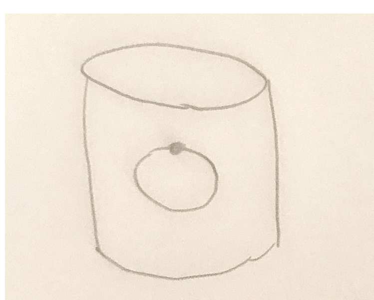
Definition 2.1. Let G be any graph on some surface S, we say G is good if all faces are homeomorphic to disks. Here we define the faces of G as connected components of $S-G$.

If S is a surface with boundary, then we require that each boundary component has at least one vertex of G and the boundary consists of edges of G.

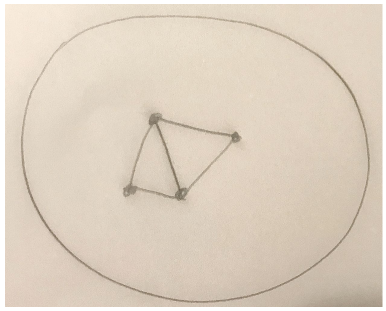
Remark 2.4. 1)The following graphs are not good:
a) A graph on torus:



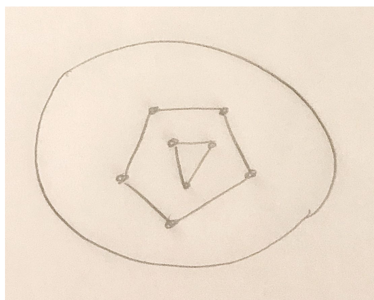
b) A graph on cylinder



2) Connected planar graph is always good.



3) Disconnect planar graph is not good because infinite face is not a disk.



Theorem 2.5 (Euler Characteristic on a given surface remains the same)**.** *If G_1 and G_2 are good graphs on same surface S, then*

$$V_{G_1} - E_{G_1} + F_{G_1} = V_{G_2} - E_{G_2} + F_{G_2} = \chi(S)$$

.

To prove the theorem, we consider the graph G which is the union of G_1 and G_2 (with the intersection points as additional vertices). We are going to prove that

$$V_G - E_G + F_G = V_{G_1} - E_{G_1} + F_{G_1},$$

then similarly

$$V_G - E_G + F_G = V_{G_2} - E_{G_2} + F_{G_2},$$

and the theorem follows. We define I to be the set of intersection points. Define ${G_1}' = G_1 + I$.

Lemma 2.6. *The Euler characteristics of G_1 and G_1' are the same:*

$$V_{G_1} - E_{G_1} + F_{G_1} = V_{G_1'} - E_{G_1'} + F_{G_1'}.$$

Proof. When we add k intersection points,

$$V \to V + k$$
$$E \to E + k$$
$$F \to F$$

So $V - E + F$ does not change. same for G_1 and G_1' (think pf tikz pic here) ∎

Lemma 2.7. *Here we want to prove that $\chi({G_1}'_f) = \chi(G_f)$, where ${G_1}'_f$ is a single face of G_1'; G_f is a single face of G.*

Proof. Let us present three pictures to better present our proof: Figures 2, 1, and 3. Figures 1 and 3 are separated parts of Figure 2.

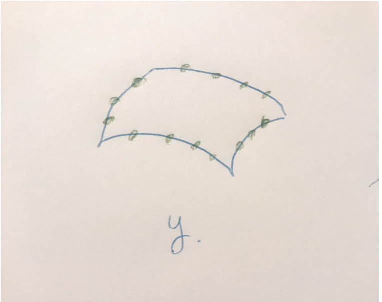


Figure 1:

First, $V_{{G_1}'_f} = E_{{G_1}'_f} = E_{blue}$ (see Figure 1) , here $V_{{G_1}'_f}$ is all the points on the boundary of ${G_1}'_f$ (i.e a single face of G_1'), $E_{{G_1}'_f}$ is all the edges on the boundary of ${G_1}'_f$. (it is just those edges we colored in blue E_{blue})

$E_{green} = E_{inside}$ (see Figure 3 , here E_{green} is all the edges that we marked green, so just all the edges inside the ${G_1}'_f$, which is E_{inside})

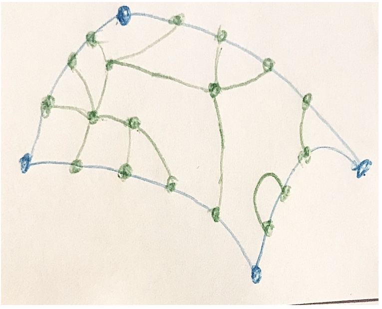


Figure 2:

In this face, $V_{G_f} = V_{{G_1}'_f} + V_{inside}$ (see Figure 2, here V_{G_f} is all the vertices in the picture)

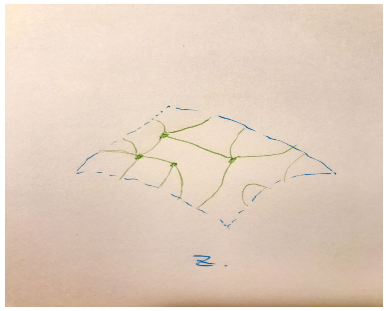


Figure 3:

$E_{G_f} = E_{blue} + E_{green} = E_{{G_1}'_f} + E_{inside}$ (please look at Figure 3, here E_{G_f} is all the edges in the picture)

$$F_{{G_1}'_f} = 2 \tag{1}$$

See Figure 1, we can see the number of faces on the graph ${G_1}'_f$ is 2. Also,

$$F_{G_f} = 1 + F_{inside} \tag{2}$$

(see Figure 2, F_{G_f} is all the faces in the picture, F_{inside} is all the faces inside the face of ${G_1}'_f$. Now $F_{G_f} = F_{{G_1}'_f} - 1 + F_{inside}$ follows from (1) and (2).

Next we calculate the Euler Characteristic for G on the plane:

$$V_{G_f} - E_{G_f} + F_{G_f} = 2,$$

this follows from Corollary 2.2.

Therefore $V_{G_f} - E_{G_f} + (1 + F_{inside}) = 2$, then

$$(V_{{G_1}'_f} + V_{inside}) - (E_{{G_1}'_f} + E_{inside}) + (1 + F_{inside}) = 2.$$

Since $V_{{G_1}'_f} = E_{{G_1}'_f}$ we get $V_{inside} - E_{inside} + F_{inside} = 1$.

Finally, we can see that

$$\begin{aligned}
&V_{G_f} - E_{G_f} + F_{G_f} \\
&\quad = (V_{{G_1}'_f} + V_{inside}) - (E_{{G_1}'_f} + E_{inside}) + (1 + F_{inside}) \\
&\quad = V_{{G_1}'_f} + V_{inside} - E_{{G_1}'_f} - E_{inside}) + 1 + F_{inside} \\
&\quad = V_{{G_1}'_f} + V_{inside} - E_{{G_1}'_f} - E_{inside}) + F_{{G_1}'_f} - 1 + F_{inside} \\
&\quad = V_{{G_1}'_f} + V_{inside} - E_{{G_1}'_f} - E_{inside}) + F_{{G_1}'_f} - 1 + F_{inside} \\
&\quad = V_{{G_1}'_f} - E_{{G_1}'_f} + F_{{G_1}'_f} + (V_{inside} - E_{inside} + F_{inside}) - 1 \\
&\quad = V_{{G_1}'_f} - E_{{G_1}'_f} + F_{{G_1}'_f} + (1 - 1) \\
&\quad = V_{{G_1}'_f} - E_{{G_1}'_f} + F_{{G_1}'_f}
\end{aligned}$$

■

Proof of Theorem 2.5. Our main goal is to prove that $\chi(G_1) = \chi(G_1)$, given (G_1) and (G_2) are arbitrary graphs on a given surface. To do so, we want to prove the equations $\chi(G_1) = \chi(G)$ and $\chi(G_2) = \chi(G)$. Since (G_1) and (G_2) are arbitrary graphs, it suffices to prove $\chi(G_1) = \chi(G)$, and then we can apply same reasoning for (G_2).

Therefore, to prove $\chi(G_1) = \chi(G)$, we want to prove:

$$\chi(G_1') = \chi(G) \tag{3}$$

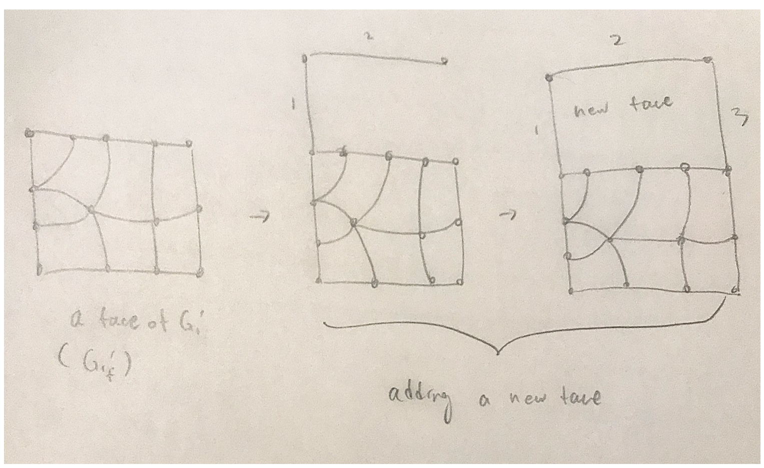
$$\chi(G_1) = \chi(G_1') \tag{4}$$

Since we already proved (4) in Lemma 2.6, it suffices to show (3) which we prove by induction:

Base case: From Lemma 2.7, we already know that $\chi({G_1}'_f) = \chi(G_f)$. (Note: $({G_1}'_f)$ is a face of (G_1'), and (G'_f) is a face of (G))

Step: we want to show that, by first adding a new face of (G_1'), and then adding points and edges from G_2 to that new face, the Euler characteristic χ will be the same. Our proof is as follows:

First, we want to show that χ is the same after adding a new face of (G_1'):



We shall omit the proof , as the proof can be easily illustrated in the picture. In the process of adding a new face, χ remains the same after each step.

Then, from Lemma 2.7, we know that adding points from G_2 to a face of (G_1') will keep Euler characteristic the same.

Therefore we claim that χ keeps the same, when we move from n faces to $(n+1)$ face (i.e. we proved our induction step)

Thus we showed that $\chi(G_1') = \chi(G)$. And we completed our proof, in other words, we've shown that two arbitrary graphs G_1 and G_2 on a given surface will have the same χ.

■

Another proof using the triangulation method. A triangulation of a surface is just dividing a surface into a finite number of triangles. Essentially, a triangulation of a surface is just a graph such that all its faces are triangles.

We have two arbirary triangulation G_1 and G_2 of a given surface (see Figures 2 and 2), and we want to show $\chi(G_1) = \chi(G_2)$.

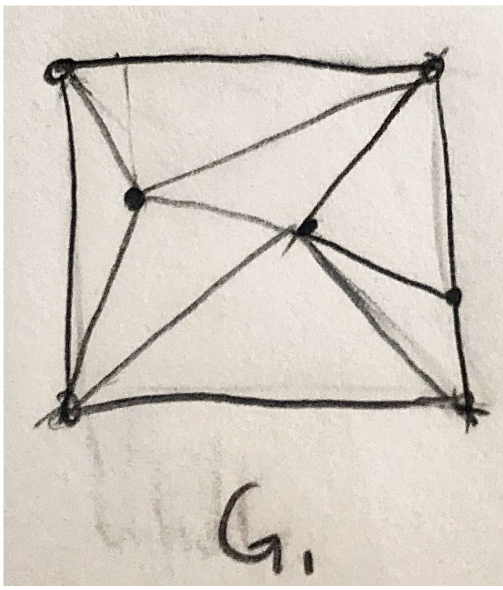


Figure 4: The triangulation G_1

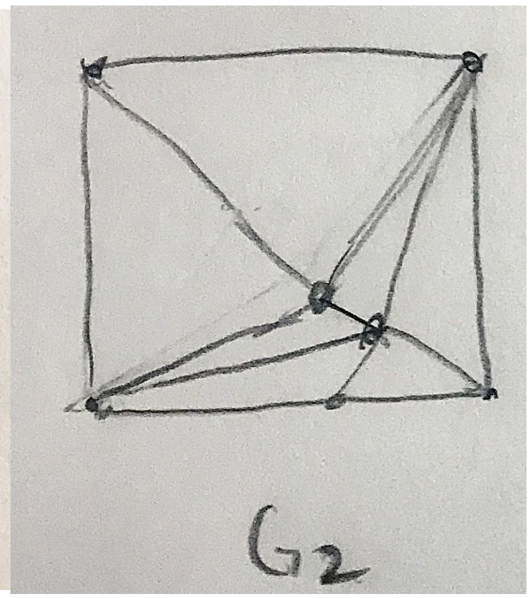
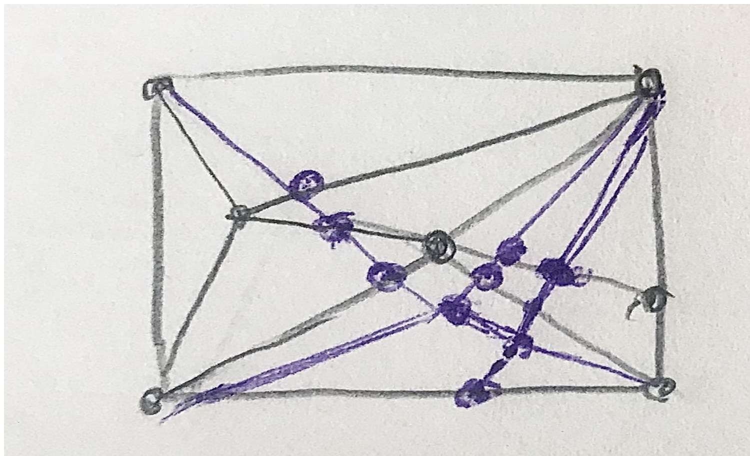


Figure 5: The triangulation G_2

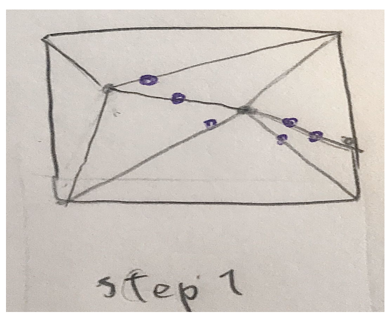
To show this, we want to prove the equations $\chi(G_1) = \chi(G)$ and $\chi(G_2) = \chi(G)$,where $G = G_1 \cup G_2 \cup I$. ($I :=$ the set of intersection points of G_1 and G_2. i.e the purple points in the picture below).



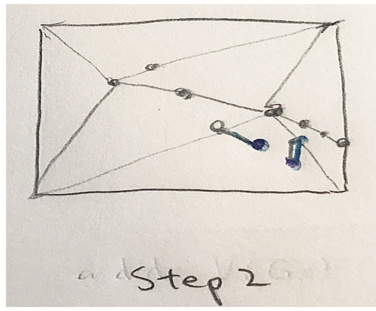
Since G_1 and G_2 are arbitrary graphs, it suffice to prove $\chi(G_1) = \chi(G)$, and apply same reasoning for G_2. Now we prove $\chi(G_1) = \chi(G)$:

The graph G can be constructed from G_1 by three steps: Step 1, adding I; Step 2, add all vertices of G_2, and connect each vertex of G_2 with G' using only one edge from of G_2; Step 3, connect the rest of vertices of G_2 and G' using the rest of edges of G_2. If we can show for each step, χ did not change, then we will prove that $\chi(G)$ is same as $\chi(G')$.

First, we prove that Step 1 preserves χ. This follows from Lemma 2.6, χ won't change by adding more points on the edges of G_1.



Then we prove that Step 2 preserves χ: here we can see, that both V and E increase by the number of new vertices, while F does not change.



Last we prove that Step 3 preserves χ: each of our actions is equivalent to connecting two boundary points in a disk. This is because after triangulation, each triangle face is homeomorphic to a disk.

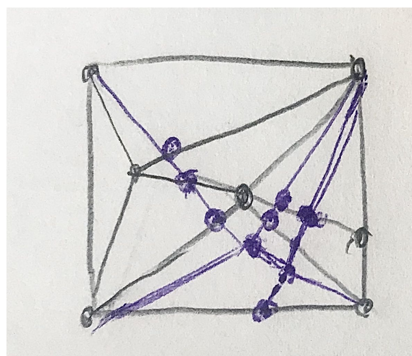


Figure 6: Step 3

When we add an edge, the number of faces goes up by 1 (i.e face cut in half), the number of edges goes up by 1, and the number of vertices does not change, so the Euler characteristic remains the same.

Here we proved $\chi(G) = \chi(G_1)$ and completed the proof. ■

Remark 2.8. Triangulation is not necessary, dividing the surface to polygons is also fine.

3 Examples of surfaces

In this section, we want to discuss the relation of Euler Characteristics and different surfaces. We will introduce orientability (non-orientability), connected sum of two surfaces, and classification of surfaces.

Our main goal here is for better understanding of these concepts, therefore, the examples and proofs we give will be less rigorous and more intuitive.

3.1 Orientable surfaces

Example 3.1. *The Euler characteristic of S^2 equals 2.*

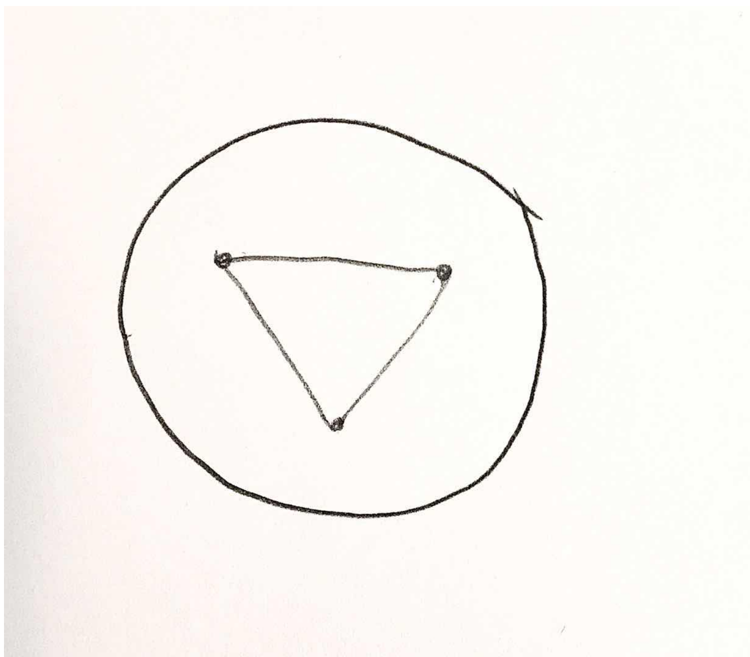


Figure 7: Graph on a sphere

In the graph in Figure 7, we see $V = 3, E = 3, F = 2$, so $V - E + F = 2$. Since we show that χ independent of graph on a given surface, so we know $\chi(S^2)$ is 2 (we can also use Corollary 2.2).

Example 3.2. *The Euler characteristic of the the torus equals* $\chi(T^2) = 0$

Indeed, we can obtain the torus by gluing the opposite sides of a rectangle. This gives a good graph with $V = 1, E = 2, F = 1$, therefore

$$\chi(T^2) = 1 + 1 - 2 = 0.$$

Next we want to obtain the new surfaces by gluing them from smaller ones.

Lemma 3.3. *Suppose we have two surfaces S_1 and S_2 with boundary. If we combine them by their boundaries, the Euler characteristic of new surface is the sum of Euler characteristics of S_1 and S_2.*

Proof. Let us choose two good graphs G_1 and G_2 on S_1 and S_2 such that both graphs have 1 vertex and 1 edge on the boundary. Then

$$\chi(S_1) = V_1 - E_1 + F_1,\ \chi(S_2) = V_2 - E_2 + F_2.$$

The union of G_1 and G_2 has $V_1 + V_2 - 1$ vertices, $E_1 + E_2 - 1$ edges and $F_1 + F_2$ faces. Then

$$\chi(S) = (V_1 + V_2 - 1) - (E_1 + E_2 - 1) + (F_1 + F_2) = \chi(S_1) + \chi(S_2).$$

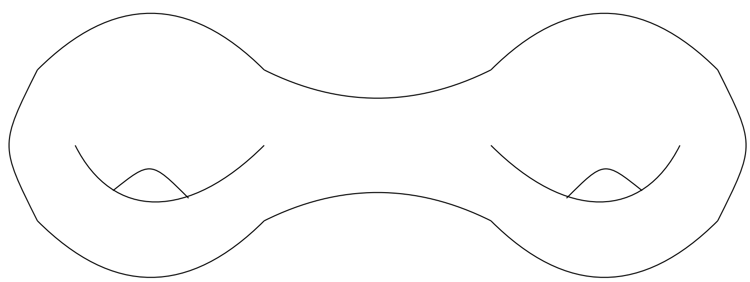
■

Definition 3.1. The connected sum $S_1\#S_2$ of two surfaces S_1 and S_2 is obtained by cutting two small disks out of S_1 and S_2 and gluing the resulting surfaces along the boundary.

Definition 3.2. The **genus** g surface is defined as the connected sum of g tori or, equivalently, as the sphere with g handles added to it.

Remark 3.4. We can think of the genus as the number of holes in an (orientable) surface.

Our previous two examples, sphere and torus, are of genus 0 and genus 1. Here is an example of genus 2 surface:



Theorem 3.5. *If S_1 and S_2 are two surfaces. Then*

$$\chi(S_1\#S_2) = \chi(S_1) + \chi(S_2) - 2.$$

Proof. Let us only look at the two boundaries combined at first, and not look at the remaining triangulations on the two surfaces as they are unchanged. Then we will notice that we lose two faces as we remove the interiors. Also, the number of vertices of a surface reduce by 3, the edge also reduce by 3 when combined. In other words,

$$\chi(S_1) = V_1 - E_1 + F_1,$$

$$\chi(S_2) = V_2 - E_2 + F_2.$$

Therefore,

$$\chi(S_1\#S_2) = V_1 - 3 - (E_1 - 3) + F_1 - 1 + V_2 - E_2 + F_2 - 1 =$$

$$\chi(S_1) + \chi(S_2) - 2.$$

■

Corollary 3.6. *If we have a genus g orientable surface, the χ of it $2-2g$.*

Proof. Here we use proof by induction. Note that a genus g surface is just the connected sum of g tori.
Base case: For $g=0$ we get $2-2\cdot 0 = 2 = \chi(S^2)$ by Example 3.1. For $g=1$ we get $2-2\cdot 1 = 0 = \chi(torus)$ by Example 3.2.
Step: Assume $2-2\cdot g$ true for genus g, we want to show that $\chi(genus)(g+1) = 2-2\cdot(g+1)$

Let genus g surface be S_1, and torus be S_2. By assumption , $\chi(S_1) = 2-2g$. From Example 3.2, we get $\chi(S_2) = 0$. Then by Theorem 3.5 the Euler characteristic of genus $(g+1)$ surface is

$$(S1\#S2) = 2-2g+0-2 = 2-2(g+1)$$

which proves the step. And here we complete the proof.

■

3.2 Unorientable surfaces

Definition 3.3. We define the projective plane RP^2 as the following: suppose we have a sphere, and we identify all $(x,y,z) \sim (-x,-y,-z)$. Then, RP^2 is the quotient set of all equivalent classes. See Figure 8.

Lemma 3.7. $\chi(RP^2) = 1$

Proof. To prove $\chi(RP^2) = 1$, we start from a special construction on a sphere.

Let us we pick two opposite points (x,y,z) and $(-x,-y,-z)$ and connect them by a big circle, as in Figure 8. We have a graph with two vertices, two edges and two faces:

$$V = 2, E = 2, F = 2.$$

But on RP^2, the two points we pick are equivalent, from the definition of RP^2, so the two edges and two vertices are also equivalent.

Therefore now $V' = 2/2 = 1$,$E' = 1$ and $F' = 1$ (for the same reason as above.) We conclude that $\chi(RP^2) = V' - E' + F' = 1$ ■

Remark 3.8. Also it is interesting to note that RP^2 is not half sphere, although similar in some places.

Remark 3.9. We can also calculate $\chi(RP^2)$ by observing that it is composed of Mobius strip and disk combined by their boundaries, see Figure 9. We have

$$\chi(mobius\ strip) = 0,\ \chi(disk) = 1,$$

so by Lemma 3.3 we get

$$\chi(RP^2) = \chi(mobius\ strip) + \chi(disk) = 0 + 1 = 1.$$

please see the following picture as an illustration.

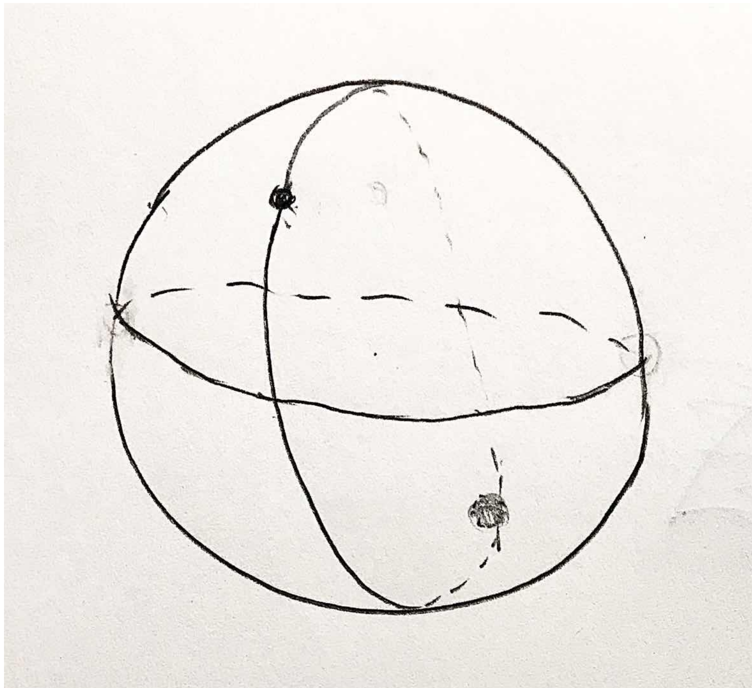


Figure 8: Our construction of RP^2 from the sphere

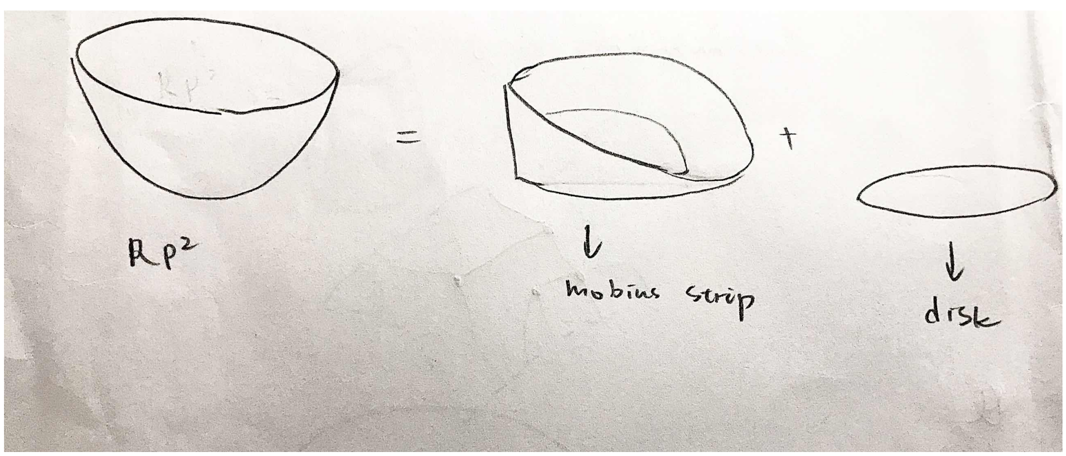


Figure 9: How RP^2 is composed of a Mobius strip and a disk

Definition 3.4. We define Klein bottle by gluing two Mobius strips along the boundary, see Figure 10.

Example 3.10. *Now we want to find* χ*(klein bottle).*

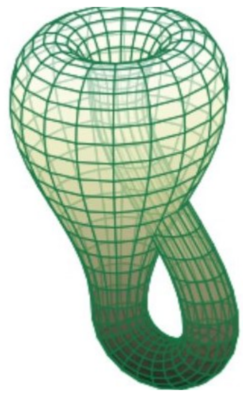


Figure 10: Klein Bottle (figure from [2])

By definition, we know Klein bottle is two mobius strips combined by their boundaries. Therefore
$\chi(\textit{klein bottle}) = \chi(mob) + \chi(mob) = 0.$

We give three definitions for **orientability**.

Definition 3.5. A surface is orientable if it has two sides which can be painted with two different colors. If not, the surface is non orientable.

For the other two definitions we need a notion of local orientation.

Definition 3.6. We say that a pair of vectors (v_1, v_2) at the same point p is positively oriented, if the shortest direction from v_1 to v_2 is counter-clockwise.

Given a pair of vectors (v_1, v_2) at a point p and a path connecting p with q, we can continuously move v_1 and v_2 along the path.

Definition 3.7. A surface is orientable if one can start with a pair of vectors (v_1, v_2) at some point p, and transport them by a path to other point q, will end up with the same orientation, no matter what the path is.
Similarly, if not, the surface is non orientable.

Definition 3.8. A surface is orientable if, whichever path it takes, a pair of vectors (v_1, v_2) remains the same orientation when back to its origin, after navigating around the surface.
Similarly, if not, the surface is non orientable.

Remark 3.11. It would be an interesting exercise to show the three definitions are equivalent to each other, but we shall omit it here.

Example 3.12. *The sphere and torus are orientable.*

Example 3.13. *Connected sum of two orientable surfaces is orientable, so any genus g surface is orientable.*

Lemma 3.14. *RP^2 is non orientable.*

Proof. RP^2 is non-orientable because it contains a Mobius strip.

Then we use proof by contradiction, assume RP^2 is orientable. Since if a surface contains a mobius strip, that means there exist a path that will reverse orientation after navigating around the surface back to the origin. Therefore, this contradicts our Definition 3.8. And thus RP^2 is non-orientable. ∎

Similarly, the Klein bottle is not orientable.

3.3 Classification of surfaces

In this subsection we review the classification of two-dimensional surfaces following [1].

Theorem 3.15. *Let S be a two-dimensional surface without boundary.*

1) If S is orientable then it is homeomorphic to the connected sum of k tori. (the exact value of k depends on specific orientable surface).

2) If S is non-orientable then it is homeomorphic to the connected sum of k RP^2's (the exact value of k depends on specific non-orientable surface)

3) All the surfaces listed above are pairwise not homeomorphic.

Remark 3.16. Here we assume that S^2 is the connected sum of $k = 0$ tori, or a genus 0 surface.

We do not prove parts (1) and (2) here, but prove part (3) using the Euler characteristic.

First, observe that orientable surface cannot be homeomorphic to a non-orientable one. By corollary 3.6 the Euler characteristic of the connected sum of k tori (that is, genus k surface) equals $2 - 2k$. Therefore for different k these surfaces are not homeomorphic, as they have different Euler characteristic.

Corollary 3.17. *If we have a connected sum of n RP^2 , the Euler characteristic of it is $2 - n$.*

Proof. Here we use proof by induction.
Base case: $n = 1$ $2 - 1 = 1 = \chi(RP^2)$ by Lemma 3.7 therefore base case right.
Step: Assume $2 - n$ is true for n, we want to show that

$$\chi(\text{connected sum of } (n+1)\ RP^2\text{s}) = 2 - (n+1)$$

for $(n + 1)$.

Let S_1 be the connected sum of n RP^2s, by assumption $\chi(S_1) = 2 - n$. Let S_2 be RP^2, Euler characteristic is 1. From Theorem 3.5, the Euler characteristic of $(n + 1)$ RP^2s equals

$$(S1\#S2) = 2 - n + 1 - 2 = 2 - (n + 1),$$

which proves the step. ∎

As a corollary, we conclude that for different k the connect sums of k RP^2s are not homeomorphic, as they have different Euler characteristic. We also get the following corollary.

Corollary 3.18. *1) If S is an orientable surface then it is homeomorphic to the connected sum of g tori where*

$$g = \frac{2 - \chi(S)}{2}.$$

2) If S is a non-orientable surface then it is homeomorphic to the connected sum of k RP^2s where

$$k = 2 - \chi(S).$$

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References

[1] C. Adams. The knot book. An elementary introduction to the mathematical theory of knots. American Mathematical Society, Providence, RI, 2004.

[2] Klein Bottle, Wikipedia. `https://en.wikipedia.org/wiki/Klein_bottle`