REMARKS ON COMBINATORIAL WALL-CROSSING

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1. Introduction

The symmetric group. Let $n \geq 1$. Recall that the *symmetric group* S_n is the set of all permutations of n, that is, bijections on the set $\{1, 2, \ldots, n\}$. This is naturally a group, with the group structure given by function composition.

We will use cycle notation, for example, $(a_1, \ldots, a_k)$ denotes the permutation sending $a_1 \mapsto a_2 \mapsto \cdots \mapsto a_k \mapsto a_1$ and fixing every other element of $\{1, \ldots, n\}$. In this case, we will call k the *length* of the cycle. Note that for an element $a_1 \in \{1, \ldots, n\}$, the 1-cycle (a_1) is the identity.

We say that two cycles $a = (a_1, \ldots, a_k)$ and $b = (b_1, \ldots, b_j)$ are *disjoint* if $\{a_1, \ldots, a_k\} \cap \{b_1, \ldots, b_j\} = \emptyset$. Note that under this assumption, we have $ab = ba$. The following result is well-known.

Lemma 1.1 ([1])**.** Let $\sigma \in S_n$. Then, σ admits a decomposition as a product of pairwise disjoint cycles. Moreover, this decomposition is unique up to a reordering of the cycles.

Example 1.2. Consider the element

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 6 & 2 & 7 & 1 & 3 & 9 & 8 & 5 \end{pmatrix} \in S_9$$

Then, we can decompose it in cycles as $(14795)(263)(8)$.

Let $\sigma \in S_n$ and let $\sigma = c_1 \ldots c_k$ be its decomposition as a product of disjoint cycles. For each $i = 1, \ldots, k$, we let $\lambda_i > 0$ be the length of the cycle c_i. We may reorder the cycles so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. Note that $\lambda_1 + \cdots + \lambda_k = n$. In this way, to each element $\sigma \in S_n$, we may uniquely associate a *partition* λ of n.

Definition 1.3 (Partition)**.** Let $n > 0$. A *partition* of n is a non-increasing sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ of numbers such that $n = \lambda_1 + \cdots + \lambda_k$.

For $\sigma \in S_n$, we call its associated partition λ the *cycle type* of σ.

Recall that two elements $\sigma_1, \sigma_2 \in S_n$ are said to be *conjugates* if there exists $\sigma \in S_n$ such that $\sigma_1 = \sigma \sigma_2 \sigma^{-1}$. It is easy to see that this defines an equivalence relation on S_n. Its equivalence classes are known as *conjugacy classes*.

Lemma 1.4. Two elements $\sigma_1, \sigma_2 \in S_n$ are conjugates if and only if they have the same cycle type. It follows that conjugacy classes in S_n are parametrized by partitions of n.

Example 1.5. Let $\sigma_1 = (135)(24)$ and $\sigma_2 = (235)(14)$ be in S_3. We want to find σ such that $\sigma_1 = \sigma \sigma_2 \sigma^{-1}$. In this case, $(135)(24) = \sigma(235)(14)\sigma^{-1}$. It's not very difficult to find that $\sigma = (12)$. Thus we can say that σ_1, σ_2 are conjugates.

Representation theory of S_n in characteristic zero. For basics in representation theory, we refer to [1]. Recall that the set of irreducible representations of S_n (in fact, of any group) is in bijective correspondence with the set of conjugacy classes in S_n. From Lemma 1.4 we get.

Proposition 1.6. The set of irreducible representations of S_n is in bijective correspondence with the set of partitions of n.

An explicit bijection $\lambda \mapsto V_\lambda$ has been constructed in [7]. We will not review the construction. We will just comment that there is a special 1-dimensional representation, the *sign* representation, such that for any other representation V_λ we have sign $\otimes V_\lambda = V_{\lambda^T}$. Here λ^T is the *transpose* of the partition λ, a construction we will review in Section 2 below.

Representation theory of S_n in positive characteristic. The representation theory of S_n in positive characteristic is much more subtle than in characteristic zero. Let p be a prime and $\mathbb{F}$ an algebraically closed field of characteristic zero. We will assume that $p > 2$. Note that we still get a trivial and a sign representation and, because $p > 2$, triv $\ncong$ sign.

The irreducible representations of S_n over $\mathbb{F}$ are parametrized by a special subset of partitions.

Definition 1.7 (p-restricted partition)**.** Let p be a prime number. A partition λ of n is called *p-restricted* if no part of λ repeats p times.

If λ is p-restricted, we denote by V_λ the irreducible representation of S_n over $\mathbb{F}$ labeled by λ. Note that sign $\otimes_{\mathbb{F}} V_\lambda$ is another irreducible representation of S_n over $\mathbb{F}$. We call the unique p-restricted partition λ^M such that $V_{\lambda^M} \cong$ sign $\otimes V_\lambda$ the *Mullineux dual* of λ. Note that, by definition, $(\lambda^M)^M = \lambda$. For this reason, we call the map $\lambda \mapsto \lambda^M$ the *Mullineux involution.*

To compute λ^M is complicated and it involves a lot of steps. We will review this in Section 3.

Extensions of the Mullineux involution. Note that, to define a p-restricted partition, we do not need the number p to be prime.

Definition 1.8 (p-restricted partition)**.** Let p be any number and λ a partition of n. We say that λ is p-restricted if no part of λ repeats p-times.

An interesting question is whether there exists an analogous of the Mullineux involution when p is not prime. The answer is yes. From the representation-theoric point of view, this is motivated by the representation theory of Hecke algebras at a p-th root of unity, see [2, 5]. We will review this construction in Section 3.

Wall-crossing. The main goal of this work is to study a further extension of the Mullineux involution, called *wall-crossing.* One advantage of wall-crossing is that it is defined on *all* partitions, rather than just the p-restricted ones. A minor disadvantage is that it is no longer an involution (however, see Lemma 4.7). We will review the definition and main properties of wall-crossing in Section 4, after we have studied the Mullineux involution.

2. Partitions

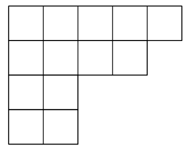
Definition 2.1. A partition λ is a finite, ordered non-increasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ where $\lambda_i \geq \lambda_{i+1}$. The largest value of k such that $\lambda_k > 0$ is called the *length* of λ. The *size* of λ is $|\lambda| := \sum_i \lambda_i$.If $|\lambda| = n$, we usually say that it is a partition of n, and write $\lambda \vdash n$.

For example, the sequence $\lambda = (5, 4, 2, 2)$ is a partition of 13. The length of λ is 4.

Definition 2.2. Let λ be a partition. The *Young diagram* of λ (sometimes also known as the Ferrers diagram) is a collection of upper-left aligned boxes with λ_1 boxes in the first row, λ_2 boxes in the second row and, in general, λ_i boxes in the i-th row.

Remark 2.3. Sometimes, it will be convenient to write a partition in a different way. We write $\lambda = [1^{a_1}, 2^{a_2}, \dots, n^{a_n}]$ for the partition that has a_1 parts equal to 1; a_2 parts equal to 2 and so on.

Example 2.4. The Young diagram of $\lambda = (5, 4, 2, 2) = [1^0, 2^2, 3^0, 4^1, 5^1]$ is



The coordinates of the vertices of the boxes of the diagram are given by: $(i, j) \in \{\mathbb{N} \times \mathbb{N} \mid 1 \leq i, 1 \leq j \leq \lambda_i\}$. Label by (i, j) the box whose southeast vertex has coordinates (i, j). We remark that, here, our x- and y-axes are not oriented as usual, but rather follow the following pattern.

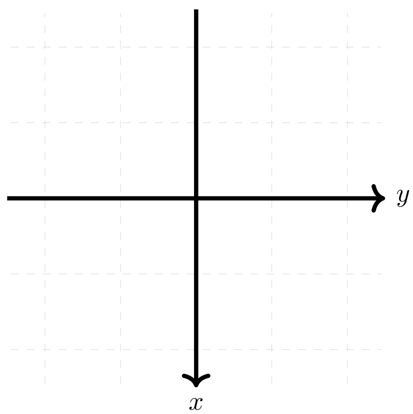


Figure 1. The orientation of the plane that we will be using for the rest of our work

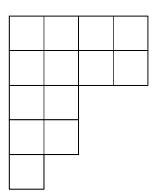
We do not distinguish between a partition and its Young diagram. In particular, we sometimes abuse notation and say that a box belongs to λ to express that the box belongs to the Young diagram of λ.

Definition 2.5. Denote by P_n the set of partitions of n. Let $\lambda \in P_n$. A box $R \in \lambda$ is called a *removable* box of λ if $\lambda \setminus R \in P_{n-1}$. A box B is called an *addable* box of λ, if $\lambda \cup B \in P_{n+1}$.

Definition 2.6. Given a partition λ, its *transpose* is the partition whose diagram is that of λ flipped over the diagonal line $y = x$. We denote the transpose of λ by λ^T.

Example 2.7. The transpose of $\lambda = (5, 4, 2, 2)$ is $\lambda^T = (4, 4, 2, 2, 1)$,

$\lambda^T =$



The following result gives us a way to compute the transpose λ^T without resorting to the diagram of λ.

Lemma 2.8. Let λ be a partition. Then, $\lambda^T = (\lambda_1^T, \lambda_2^T, \cdots)$ where $\lambda_i^T = |\{j : \lambda_j \geq i\}|$.

Proof. From Definition 2.1, we know λ can be written as the form $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ where $\lambda_i \geq \lambda_{i+1}$. Graphically, λ_i is the number of boxes in the i-th row of λ, and λ_i^T is the number of boxes in the i-th column of λ. If there exists j such that $\lambda_j \geq i$, the box in the j-th row will contribute to the box in the i-th column.

$\square$

The following result, that we state without a proof, gives a representation-theoretic interpretation of the transpose operation.

Proposition 2.9. Let λ be a partition of n and V_λ the corresponding irreducible representation of S_n over $\mathbb{C}$. Then, $\mathrm{sign} \otimes V_\lambda$ is an irreducible representation isomorphic to V_{λ^T}.

To finish this chapter, for future reference, let us define the *content* of a box (i, j) as $\mathrm{ct}(i, j) = j - i$.

3. The Mullineux involution

In this section, we study an operation that mimics transposition on a certain subset of partitions that depends on a positive integer p. Let us first define this subset.

Definition 3.1. Let $p > 0$ be an integer, and let λ be a partition. We say that λ is *p-regular* if no nonzero part of λ repeats p times.

Example 3.2. Consider the partition $\lambda = (4, 3, 2, 2)$. Then λ is 3-regular, but not 2-regular. Note that $\lambda^T = (4, 4, 2, 1)$ is not 2-regular either. On the other hand, consider the partition $\mu = (4, 4, 4)$. This partition is 4-regular. Note, however, that $\mu^T = (3, 3, 3, 3)$ is not 4-regular. This shows that the transpose of a p-regular partition does not need to be p-regular itself.

The previous example tells us that we should be careful if we want to define an operation on p-regular partitions that has the properties of the transpose. This operation is called the *Mullineux involution*. Our next goal will be to define this operation. First, we need a "positive characteristic" analogue of the content of a box.

Definition 3.3. The *p-content* of a box is defined as $\mathrm{ct}_p(i, j) := i - j \mod p \in \mathbb{Z}/(p)$.

The Mullineux involution is based on the notion of a good box of residue i, for $i \in \mathbb{Z}/(p)$. This will be a removable box $\square$ of λ with $\mathrm{ct}_p(\square) = i$. We will define it using an algorithmic procedure.

Definition 3.4. A good box of residue i where $i \in \{0, 1, ..., p-1\}$ of a partition λ is defined as:

1. Label the boxes of λ by their p-residue.
2. From southwest to northeast, write R for the removable boxes and A for the addable boxes of residue i. This way we get a sequence that is called an RA-sequence.

(3) Inductively cancel the RA-sequence until there is no RA appearing.

Then, the removable box of residue i corresponding to the first R from the very left is called a *good box* of residue i. If no such R exists, then we say that λ has no good removable box of residue i.

Similarly, from the reduced RA-sequence we define the good addable box of residue i to be the box corresponding to the first A from the left. If no such A exists, we say that λ has no good addable box of residue i.

Note that, if λ has a good removable box of residue i, then this box is unique. This allows us to define the following function.

Definition 3.5. Let λ be a p-regular partition and $i \in \mathbb{Z}/(p)$. We define $f_i(\lambda)$ to be the partition obtained by removing the good removable box of λ of residue i, if such box exists. Otherwise, we define $f_i(\lambda) = 0$.

Similarly, e_i is the operator that adds the good removable box of residue i, and we define $e_i(\lambda) = 0$ if no such box exists. Note that if $f_i\lambda \neq 0$, then $e_i f_i(\lambda) = \lambda$. Likewise, if $e_i(\lambda) \neq 0$ then $f_i e_i(\lambda) = \lambda$.

Definition 3.6. An f-sequence of μ is a sequence $(i_1, i_2, \ldots, i_n) \in (\mathbb{Z}/p\mathbb{Z})^n$ if $f_{i_1} f_{i_2} \cdots f_{i_{n-1}} f_{i_n} \mu = \emptyset$. Then, the Mullineux transpose is defined as

$$\mu^M = e_{-i_n} e_{-i_{n-1}} \cdots e_{-i_2} e_{-i_1} \emptyset,$$

where $(i_1, i_2, \ldots, i_n)$ is an f-sequence of μ .

Note that, in principle, there may be many f-sequences for a partition μ, but μ^M does not depend on these sequences. More specifically, we have the following important result.

Theorem 3.7. The partition μ^M does not depend on the f-sequence.

Proof. See [2, 5]. $\square$

Example 3.8. Consider the partition $\mu = (7, 5, 2, 1)$. Note that this is 3-regular, and let us compute λ^{M_3}. The 3-contents of boxes of μ are given as following.

$$\mu = \begin{array}{|c|c|c|c|c|c|c|}\hline 0 & 1 & 2 & 0 & 1 & 2 & 0 \\ \hline 2 & 0 & 1 & 2 & 0 \\ \cline{1-5} 1 & 2 \\ \cline{1-2} 0 \\ \cline{1-1}\end{array}$$

Let us start with content 0. The RA sequence for this content is $RARR$. Cross-cancelling the left-most RA, we get $f_0\mu = (7, 4, 2, 1)$.

$$f_0\mu = \begin{array}{|c|c|c|c|c|c|c|}\hline 0 & 1 & 2 & 0 & 1 & 2 & 0 \\ \hline 2 & 0 & 1 & 2 \\ \cline{1-4} 1 & 2 \\ \cline{1-2} 0 \\ \cline{1-1}\end{array}$$

Doing content 0 again, the RA sequence is $RAAR$. Cross-cancelling the left-most RA, we get $f_0 f_0\mu = (6, 4, 2, 1)$.

$$f_0 f_0 \mu = \begin{array}{|c|c|c|c|c|c|} \hline 0 & 1 & 2 & 0 & 1 & 2 \\ \hline 2 & 0 & 1 & 2 & & \\ \cline{1-4} 1 & \boxed{2} & & & & \\ \cline{1-2} 0 & & & & & \\ \cline{1-1} \end{array}$$

Note that the RA-sequence for 0 now is $RAAA$. After cancelling the left-most RA we do not get R's. So there is no good removable box of content 0. cannot do f_0 because there is no R after cancellation. Note that the RA-sequence for 1 is AA, so there is no good removable box of content 1.

Thus, there must be a good removable box of content 2. Let us find it. The RA-sequence if $ARRR$, so the good removable box corresponds to the left-most R.

$$f_2 f_0^2 \mu = (6,4,1,1) = \begin{array}{|c|c|c|c|c|c|} \hline 0 & 1 & 2 & 0 & 1 & 2 \\ \hline 2 & 0 & 1 & \boxed{2} & & \\ \cline{1-4} 1 & & & & & \\ \cline{1-1} 0 & & & & & \\ \cline{1-1} \end{array}$$

Now the RA-sequence for 2 is $AARR$. Thus,

$$f_2^2 f_0^2 \mu = (6,3,1,1) = \begin{array}{|c|c|c|c|c|c|} \hline 0 & 1 & 2 & 0 & 1 & \boxed{2} \\ \hline 2 & 0 & 1 & & & \\ \cline{1-3} 1 & & & & & \\ \cline{1-1} 0 & & & & & \\ \cline{1-1} \end{array}, \qquad f_2^3 f_0^2 \mu = (5,3,1,1) = \begin{array}{|c|c|c|c|c|} \hline 0 & 1 & 2 & 0 & 1 \\ \hline 2 & 0 & 1 & & \\ \cline{1-3} 1 & & & & \\ \cline{1-1} \boxed{0} & & & & \\ \cline{1-1} \end{array}$$

We cannot do f_2 anymore because there is no R in the RA-sequence. Let us go back to 0 again. We have that the RA-sequence is $RAAA$ so

$$f_0 f_2^3 f_0^2 \mu = (5,3,1) = \begin{array}{|c|c|c|c|c|} \hline 0 & 1 & 2 & 0 & 1 \\ \hline 2 & 0 & 1 & & \\ \cline{1-3} 1 & & & & \\ \cline{1-1} \end{array}, \qquad f_1^3 f_0 f_2^3 f_0^2 \mu = (4,2) = \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 0 \\ \hline 2 & 0 & & \\ \cline{1-2} \end{array}.$$

$$f_0^2 f_1^3 f_0 f_2^3 f_0^2 \mu = (3,1) = \begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline 2 & & \\ \cline{1-1} \end{array}, \qquad f_0 f_1 f_2^2 f_0^2 f_1^3 f_0 f_2^3 f_0^2 \mu = \emptyset$$

By the definition of M_3, we have that μ^{M_3} is $e_{-0}^2 e_{-2}^3 e_{-0} e_{-1}^3 e_{-0}^2 e_{-2}^2 e_{-1} e_{-0} \emptyset$. Since we are working modulo 3, this is

$$\mu^{M_3} = e_0^2 e_1^3 e_0 e_2^3 e_0^2 e_1^2 e_2 e_0 \emptyset = (5,3,3,2,2) = \begin{array}{|c|c|c|c|c|} \hline 0 & 1 & 2 & 0 & 1 \\ \hline 2 & 0 & 1 & & \\ \cline{1-3} 1 & 2 & 0 & & \\ \cline{1-3} 0 & 1 & & & \\ \cline{1-2} 2 & 0 & & & \\ \cline{1-2} \end{array}.$$

Remark 3.9. If $p \gg |\lambda|$, then $\lambda^{M_p} = \lambda^T$.

Proof. When p is very large, then all the contents of all the addable and removable boxes of λ are distinct, so we can choose in which order to remove the boxes. We can fill in a series of consecutive numbers into the partition, and let them be the order as follows: "1" for the first row first column box, "2" for the first row second column... when finishing the first row, continue to the second row and do the same thing.

1	2	3	4
5	6	7	
8	9		
10	11		

We remove the boxes in order, starting with the box labeled by n, then $n-1$ and so on. We get an f-sequence and the Mullineux transpose is using the opposite sign of f-sequence. The series of numbers are going downward, in other words, it is the transpose partition.

1	5	8	10
2	6	9	11
3	7		
4			

$\square$

4. Wall-Crossing

We would like to extend the definition of the Mullineux transpose to include all partitions, not just the p-regular ones. This is the process of *wall-crossing*. We follow [6], see also [3]. First, we remark that we can decompose every partition into a p-regular partition and a partition where each part repeats a multiple of p-times.

Proposition 4.1. Given a partition λ and a positive integer p, λ can be written as a union of multi-sets $\lambda = \mu \cup \nu$ where ν means that every part repeats a multiple of p times and μ is p-regular, i.e. no part of the partition repeats p-parts.

Proof. Let $\lambda = (i_1^{e_1}, i_2^{e_2}, \ldots, i_n^{e_n})$. Since p is an arbitrary number, by division algorithm, we have $e_j = c_j p + b_j$ where $c_j \geq 0$ and $0 \leq b_j < p$. Define $\nu = (i_1^{pc_1}, i_2^{pc_2}, \ldots, i_n^{pc_n})$ and $\mu = (i_1^{b_1}, i_2^{b_2}, \ldots, i_n^{b_n})$. It is clear that μ and ν satisfy the requirements of the proposition. $\square$

Example 4.2. Consider the partition $\lambda = (1^{10})$ and $p = 4$. Since $8 = 2 \times 4$, $\nu = (1^8)$, $\mu = (1^2)$.

Definition 4.3. Let ν be a partition where each part repeats a multiple of p-times, say $\nu = (i_1^{c_1 p}, \ldots, i_n^{c_n p})$. We define the partition $\underline{\nu} := (i_1^{c_1}, \ldots, i_n^{c_n})$.

Similarly, if $\nu = (i_1^{e_1}, \ldots, i_n^{e_n})$ is any partition, then we define $\nu^p := (i_1^{pe_1}, \ldots, i_n^{pe_n})$.

The following lemma will be useful.

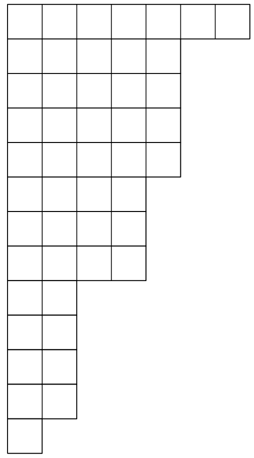
Lemma 4.4. Let ν be a partition where each part repeats a multiple of p-times. Then, $\underline{\nu}^p = \nu$.

Now we are ready to define the extension of the Mullineux involution.

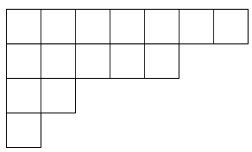
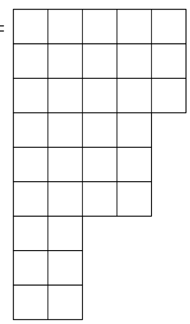
Definition 4.5. Let λ be a partition and $p > 0$. Decompose $\lambda = \mu \cup \nu$, where μ is b-regular and every part of ν repeats a multiple of p-times as in Proposition 4.1. We define the wall-crossing transformation of λ to be

$$\lambda^{W_p} := (\mu^{M_p} \cup (\underline{\nu}^T)^p)^T.$$

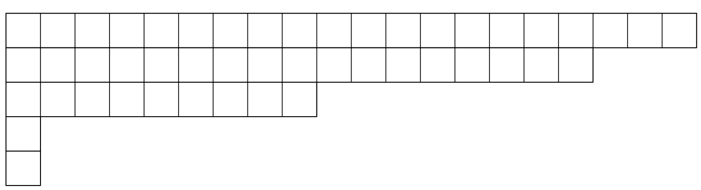
Example 4.6. Consider the partition $\lambda = (7,5,5,5,5,4,4,4,2,2,2,2,1)$, and $p = 3$.

$\lambda =$ 

The p-regular part is $\mu = (7,5,2,1)$ and $\nu = (5,5,5,4,4,4,2,2,2)$.

$\mu =$  $\nu =$ 

In Example 3.8 we computed that $\mu^{W_3} = (5,3,3,2,2)$ and $\underline{\nu} = (5,4,2)$, so $\underline{\nu}^T = (3,3,2,2,1)$ and $(\underline{\nu}^T)^3 = (3,3,3,3,3,3,2,2,2,2,2,2,1,1,1)$. Thus, $\lambda^{W_3} = (5,3^8,2^8,1^3)^T = (20,17,9,1,1)$.

$\lambda^{W_3} =$ 

Unlike the Mullineux involution, wall-crossing is not an involution, that is, $(\lambda^{W_b})^{W_b} \neq \lambda$, in general. However, if we compose wall-crossing with transposition, we do have an involution.

Lemma 4.7. Let λ be a partition and $p > 0$. Then,

$$\lambda^{W_p T W_p T} = \lambda.$$

Proof. First, we notice that $\lambda^{W_p} = (\mu^{M_p} \cup (\underline{\nu}^T)^p)^T$, so our equation becomes $\lambda^{W_pT} = (\mu^{M_p} \cup (\underline{\nu}^T)^p)^{TT}$. Transpose a partition twice does nothing to the partition, so we can just cancel them. The right hand side becomes $\mu^{M_p} \cup (\underline{\nu}^T)^p$. Applying the second wall crossing transpose, we will have $\lambda^{W_pTW_p} = (\mu^{M_p} \cup (\underline{\nu}^T)^p)^{W_p}$. Applying W_p has the same meaning as applying M_p(Mullineux transpose) to the p-regular part, μ, and applying regular transpose and power of p one more time. So, we can get $\lambda^{W_pTW_p} = ((\mu^{M_p})^{M_p} \cup (\underline{\nu}^p))^T$. Since μ is already p-regular, μ^{M_p} is also p-regular. If we apply M_p one more time to the p-regular partition, it does nothing. Thus, we can get $\lambda^{W_pTW_p} = (\mu \cup \nu^p)^T$. Applying one more regular transpose to both sides, since transpose a partition will get back to the original partition, then $\lambda^{W_pTWpT} = \mu \cup \underline{\nu}^p$. Applying Lemma 4.4, we would finally get $\lambda^{W_pTWpT} = \mu \cup \nu = \lambda$. □

Corollary 4.8. The inverse of the wall-crossing transformation is $\lambda^{W_p^{-1}} = \lambda^{TW_pT}$

5. Farey sequences and sliding boxes

Throughout this section, we follow [3].

Definition 5.1. The *Farey sequence* of order n is the sequence of completely reduced fractions between 0 and 1, which when in lowest terms have denominators less than or equal to n, arranged in order of increasing size, i.e.

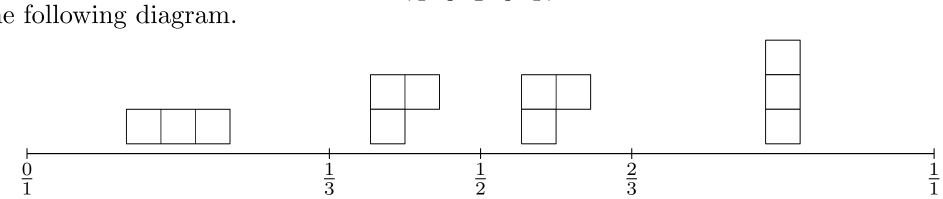
$$F_n = \left\{ \frac{a}{b} \in [0,1] \mid b \leq n \right\}.$$

Definition 5.2. Fix $n > 0$. We define a function $\Phi_n : [0,1] \setminus F_n \to P_n$ as following. For $0 < \epsilon < 1/n$, $\Phi_n(\epsilon) = (n)$ and, if $a/b \in F_n$ and δ is such that $\left(\frac{a}{b} - \delta, \frac{a}{b} + \delta\right) \cap F_n = \frac{a}{b}$ then

$$\Phi_n\left(\frac{a}{b} + \epsilon\right) = \Phi_n\left(\frac{a}{b} - \epsilon\right)^{W_p}$$

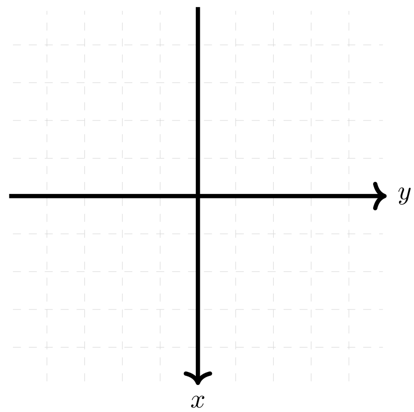
for every $0 < \epsilon < \delta$.

Example 5.3. When $n = 3$, $F_n = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$ and the function Φ_3 is given in the following diagram.



We now describe an algorithm to computing the function Φ_n that we call the "sliding boxes" procedure.

Recall that we are working with the following orientation of the xy-plane.

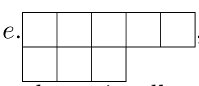


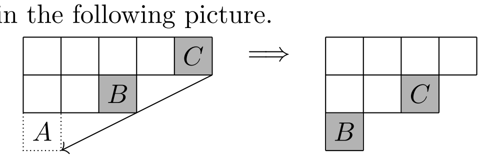
Definition 5.4. For two non-negative integers $0 < a < b$, and $\gcd(a, b) = 1$, we define the "sliding box" process as follows.
For any partition λ (identifying as a set of integer points in the plane), ladders are defined as lines

$$L_c := \left\{ (x, y) \in \mathbb{R}^2 \mid y + \frac{b-a}{a} x = \frac{c}{a} \right\}$$

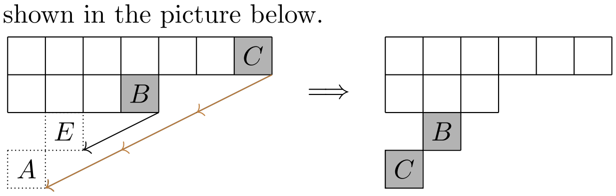
with $c \in \mathbb{Z}$, and identify L_c with the set of integer points on it. If a box of λ is on the line L_c, we slide it down along the same line until we either hit the x-axis or another box of λ. We obtain a collection of boxes which may not be a partition, as we will show in Example 5.5.

Note that, in fact, sliding a box on a ladder with parameter a, b is to slide it a spaces down and $(b - a)$ spaces to the left.

Example 5.5. Given a partition $\lambda = (5, 3)$, *i.e.* , and take $a = 1$, $b = 3$. Then, applying the "sliding boxes" procedure visually, we will get a new partition as shown in the following picture.



However, note that after applying the "sliding box" procedure we may not get a partition sometimes. For example, when applying the "sliding box" procedure to $\lambda = (7, 4)$ it is not a partition any more. The box C slides to where A is, B slides to position E, shown in the picture below.



The "sliding boxes procedure" is important to us for the following reason. For a partition λ, let $\mathcal{S}_{\frac{a}{b}}(\lambda)$ be the resulting shape obtained after applying the "sliding boxes" procedure to λ.

Theorem 5.6. Let $n > 0$ and let $a/b \in F_n$. Then, for sufficiently small $\epsilon > 0$,

$$\Phi_n\left(\frac{a}{b}+\epsilon\right)=\mathcal{S}_{\frac{a}{b}}\left(\frac{a}{b}-\epsilon\right)$$

Proof. See [3]. □

Corollary 5.7. For ϵ sufficiently small, $\Phi_n(1-\epsilon)=(1^n)$.

The following is our first main result.

Theorem 5.8. The function Φ_n is symmetric transpose around $\frac{1}{2}$, meaning that for every $x\in[0,1/2]\setminus F_n$,

$$\Phi_n(x)=\Phi_n(1-x)^T.$$

Proof. We actually prove something stronger.Indeed, the function Φ_n does not need to start with the partition (n). Let us assume that Φ_n starts with any partition of n and we make the following assumption:

$$\Phi_n(1-\epsilon)=\Phi(\epsilon)^T \textit{ for sufficiently small } \epsilon.$$

We will show that the function Φ_n satisfies the conclusions of the Theorem. Note that if $\Phi_n(\epsilon)=(n)$ then our assumption holds thanks to Corollary 5.7.

Let us order the Farey sequence F_n by the order it appears in the real line. We will do an induction on F_n. Let $x_0=0<x_1<\cdots<x_k<1/2$ be the numbers in the Farey sequence to the left of $1/2$, and let $y_i:=1-x_i$, so that we have $1/2<y_k<\cdots<y_1<y_0=1$. We will show by induction on i that the transpose of the partition appearing to the right of x_i coincides with the partition appearing to the left of y_i.

The base of induction is trivial by assumption. Now assume the result is true for x_i, and let us show it for x_{i+1}. Now, $x_{i+1}=a_{i+1}/b_{i+1}$ with $\gcd(a_{i+1},b_{i+1})=1$. Then, $y_{i+1}=(b_{i+1}-a_{i+1})/b_{i+1}$. Assume that the partition appearing to the right of x_i is λ (same as the left of x_{i+1}). Then, by definition, the partition appearing to the right of x_{i+1} is $\lambda^{W_{b_{i+1}}}$ and by assumption, the partition appearing to the left of y_i is λ^T (same as the right of y_{i+1}). Also, by definition, the partition appearing to the left of y_{i+1} is $\lambda^{TW^{-1}_{b_{i+1}}}$. By Lemma 4.7, we have that the transpose of the partition $\lambda^{W_{b_{i+1}}}$ coincides with the partition $\lambda^{TW^{-1}_{b_{i+1}}}$. Thus, the inductive steps are also true. □

Let us conclude this section with an example of the function Φ_n when $n=7$, that we give in Figure 2.

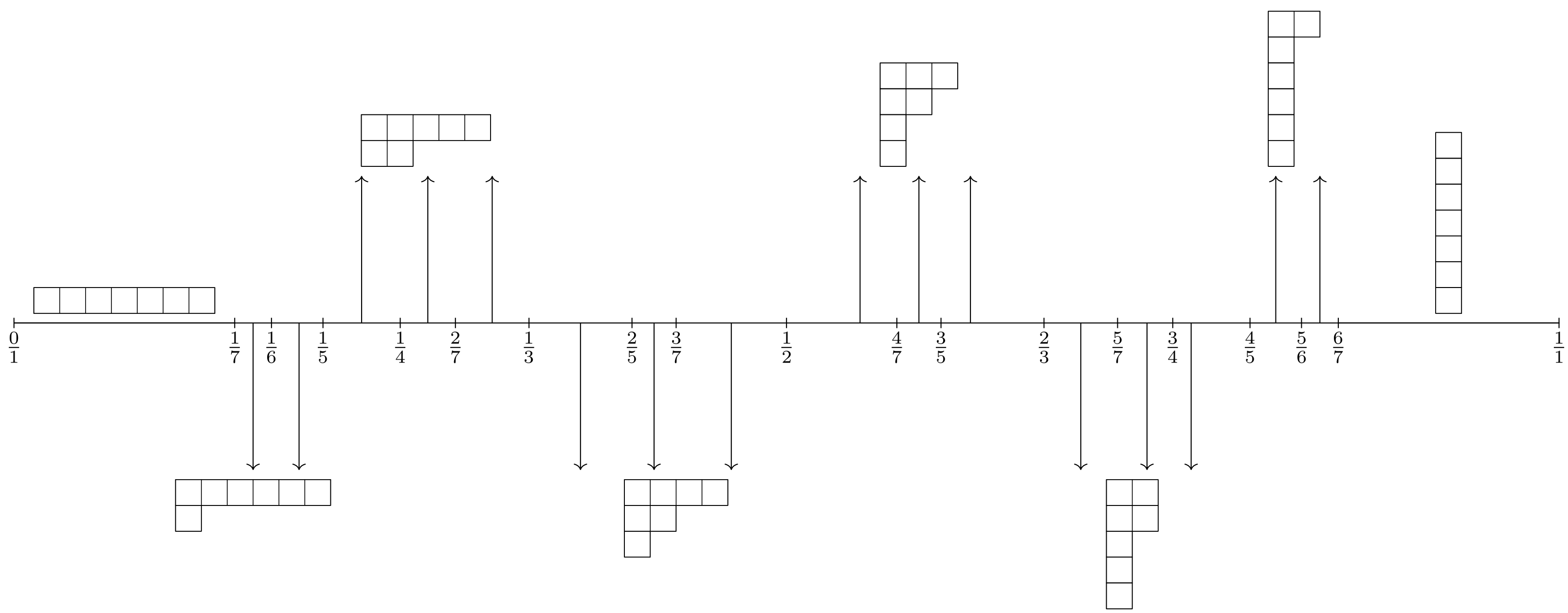


FIGURE 2. The wall-crossing transformations when $n = 7$. Note the transpose symmetry around 1/2.

6. Wall-crossing using partial orders

In this section, we study a procedure to compute Φ_n using a partial order on the set of boxes in the plane. First, let $0 < a < b$ be coprime integers. We define a function $f_{\frac{a}{b}} : \mathbb{Z}^2 \to \mathbb{Z}$ by

$$f_{\frac{a}{b}}(x, y) = (b - a)x + ay$$

Definition 6.1. Consider coprime numbers $a < b$. We define a partial order on the boxes $\{(x, y) : x \geq 1, y \geq 1\}$ as follows:

$$(x, y) \prec_r (x', y') \text{ if } \begin{cases} f_{\frac{a}{b}}(x, y) < f_{\frac{a}{b}}(x', y'), \text{ or} \\ f_{\frac{a}{b}}(x, y) = f_{\frac{a}{b}}(x', y') \text{ and } x < x' \end{cases}$$

Note that $\prec_r$ is in fact a linear order, and the minimal box is $(1, 1)$.

An example of the partial order $\prec_r$ when $a = 1$, $b = 3$ is given in Figure 6.3.

Theorem 6.2. Let $n > 0$ and $a/b \in F_n$. For sufficiently small $\epsilon > 0$, the value of $\Phi_n\left(\frac{a}{b} + \epsilon\right)$ is given by taking the first n boxes with respect to the partial order $\prec_r$ defined in Definition 6.1.

Proof. See [4]. $\square$

Example 6.3. Let the n be 7, and we want to find the partition after $\frac{1}{3}$(for example) in Farey Sequence using the computation above.Here, $b = 3, a = 1, b - a = 2$.

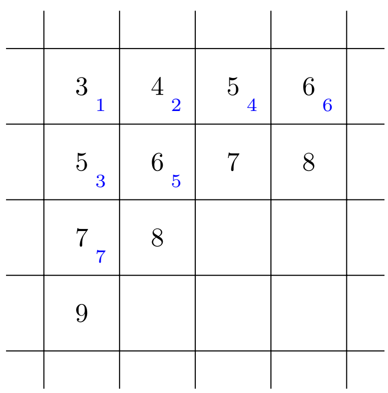
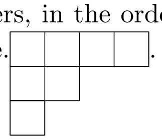


Figure 3. The partial order $\prec_r$ when $a = 1$, $b = 3$. The black numbers in the boxes are the values of the function $(b - a)x + ay$, and the small blue numbers indicate the order of the boxes.

By the theorem above, we are picking the seven boxes marked with blue numbers, in the order of those numbers in Figure 3.We will get the partition $(4, 2, 1)$, i.e. . We can check with Figure 2 that this is the correct answer.

Our second main result gives a similar way to compute the partition *to the left* of a/b. More precisely, given $\epsilon > 0$ sufficiently small, we define a similar partial order $\prec_\ell$ that gives the value of $\Phi\left(\frac{a}{b} - \epsilon\right)$.

Definition 6.4. Let $a < b$ with $\gcd(a, b) = 1$. Define the partial order on the boxes $\{(x, y) \mid x \geq 1, y \geq 1\}$ as follows:

$$(x, y) \prec_\ell (x', y') \text{ if } \begin{cases} f_{\frac{a}{b}}(x, y) < f_{\frac{a}{b}}(x', y'), \text{ or} \\ f_{\frac{a}{b}}(x, y) = f_{\frac{a}{b}}(x', y') \text{ and } x' < x \end{cases}$$

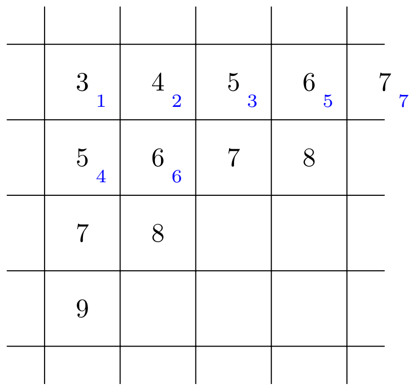


FIGURE 4. The partial order $\prec_\ell$ when $a = 1, b = 3$. The black numbers in the boxes give the values of the function $(b - a)x + ay$ and the small blue numbers indicate the partial order $\prec_\ell$.

Theorem 6.5. Let $n > 0$ and $\frac{a}{b} \in F_n$. For sufficiently small $\epsilon > 0$, the value of $\Phi_n\left(\frac{a}{b} - \epsilon\right)$ is given by taking the first n boxes with respect to the partial order $\prec_\ell$ defined in Definition 6.4.

Proof. We want to find out the partition before $\frac{a}{b}$ by using the total partial order process. As is shown in Theorem 5.8, we have the partition on the left of $\frac{a}{b}$ is the same as the transpose of the partition on the right of $\frac{b-a}{b}$.

Let us examine the order $\prec_\ell$ with respect to $\frac{a}{b}$ and the order $\prec_r$ with respect to $\frac{b-a}{b}$. First, we note that

$$f_{\frac{a}{b}}(x, y) = (b - a)x + ay = (b - (b - a))y + (b - a)x = f_{\frac{b-a}{b}}(y, x)$$

This implies the following claim.

Claim. For two boxes $(x, y), (x', y')$:

$$(x, y) \prec_\ell (x', y') \Leftrightarrow (y, x) \prec_r (y', x')$$

If $f_{\frac{a}{b}}(x, y) < f_{\frac{a}{b}}(x', y')$ then $f_{\frac{b-a}{b}}(y, x) < f_{\frac{b-a}{a}}(y', x')$, since $(b - a)x + ay < (b - a)x' + ay'$ implies $(b - (b - a)y + (b - a)x < (b - (b - a)y' + (b - a)x'$.

If $f_{\frac{a}{b}}(x, y) = f_{\frac{a}{b}}(x', y')$ and $x' < x$, then $f_{\frac{b-a}{b}}(y, x) = f_{\frac{b-a}{b}}(y', x')$ and $y < y'$, since $(b - a)x + ay = (b - a)x' + ay'$ and $x' < x$ implies $(b - (b - a)y + (b - a)x = (b - (b - a)y' + (b - a)x'$ and $y < y'$. Then, by Definition 6.4, the claim is proved. By Definition 6.1, since we've already known how to compute $\prec_r$, we just need to change (y, x) to (x, y) and it will be trivial to find the total partition of $\prec_\ell$.

$\square$

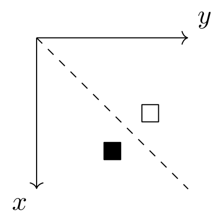


FIGURE 5. Two boxes reflected around the $x = y$ line. Note that $f_{\frac{b}{a}}(\square) = f_{\frac{b-a}{a}}(\blacksquare)$.

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