Relations on the Mapping Class Monoids of Planar Surfaces

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1 Background Information

1.1 Introduction

The mapping class group of a surface is the set of isotopy classes of homeomorphisms from the set to itself, with its binary operation being function composition. Notably, the entire mapping class group of any surface is generated by finitely many Dehn twists, a particular kind of homeomorphism that is performed relative to a chosen curve on a surface. In their article *Geometric Presentations for the Pure Braid Group*, Margalit and McCammond presented a full set of relations on the mapping class group of a genus 0 surface with punctures.

The focus of this thesis is on relations that hold on the mapping class monoid of certain planar surfaces. A monoid is a set with a binary operation that has all of the same axioms as a group, except for closure under inverses. This means that some elements of a monoid may not have an inverse element also in the monoid. The mapping class monoid of a surface consists of all products of positive Dehn twists along any curve on the surface. In this thesis, we will consider relations that exist in the mapping class monoids of particular surfaces, and we will occasionally be working in the submonoid that consists only of positive Dehn twists generated by positive twists on the generators of the mapping class group.

The structure of this thesis is as follows: We will first discuss surfaces and their classification, the mapping class group of a surface, and methods for better understanding the elements of the mapping class group. The majority of this information comes from Farb and Margalit's *A Primer on Mapping Class Groups* [2]. Then we will discuss the generators of the mapping class group and use braids as a method of comparison between mapping classes, and finally, we will present new relations and prove them using the tools described.

1.2 Surfaces

We will begin by defining surfaces and their classification so that we may define the mapping class group of a surface.

Definition 1.1. *A **surface** is a 2-dimensional manifold.*

In this thesis, we will be focusing on the mapping class monoids of bounded, connected, orientable surfaces. Therefore, we can use the following theorem to classify the surfaces we consider.

Theorem 1.2. (The Classification of Surfaces) *Every closed, connected, orientable surface is homeomorphic to the connect sum of a 2-dimensional sphere sphere and $g \geq 0$ tori. Every compact, connected, orientable surface can be obtained by removing $b \geq 0$ disjoint open disks from the interior of a closed surface. There exists a bijective correspondence between the set of pairs $\{(g, b) : g, b \geq 0\}$ and the set of homeomorphism types of compact surfaces [4].*

In particular, the value $g \geq 0$ is called the genus of a surface, and $b \geq 0$ is the number of boundary components. Furthermore, removing $n \geq 0$ points from the interior of a compact surface yields a noncompact surface. In this case, the noncompact surface is said to have n punctures. Occasionally it will be convenient to instead consider punctures to be marked points on the surface. In this case, we distinguish them as individual points. The topological properties of the surface in either case are the same, so we can refer to them as marked points or punctures. Additionally, boundary components are also called holes, and we will use this term when it is convenient.

For the remainder of this thesis, we will use 'surface" to refer to a compact, closed, oriented surface or a noncompact surface resulting from removing $n \geq 0$ points from the interior of a compact, closed, oriented surface. Therefore, all surfaces considered can be represented by the triple (g, n, b). We will use the notation $S^b_{g,n}$ to denote a surface of genus g with n punctures and b boundary components.

1.3 The Mapping Class Group of a Surface

The following terms will be used to describe the elements of the mapping class group of a surface.

Definition 1.3. *A* ***homeomorphism*** *between topological spaces X and Y is a function $f : X \to Y$ such that f is bijective and continuous and has a continuous inverse.*

Definition 1.4. *A* ***homotopy*** *between functions f and g is a map $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.*

Furthermore, an ***isotopy*** *is a homotopy $H : X \times [0, 1] \to Y$ such that, for all $t \in [0, 1]$, $H(x, t) : X \to Y$ is an embedding. Equivalently, $H(x, t)$ is a homeomorphism onto its image.*

We are now equipped to define the mapping class group of a surface.

Definition 1.5. *The* ***mapping class group*** *of a surface S, denoted $\text{Mod}(S)$, is the quotient*

$$\text{Homeo}^+(S, \partial S)/\text{Homeo}_0(S, \partial S),$$

where $\text{Homeo}^+(S, \partial S)$ is the group of orientation-preserving homeomorphisms from S to itself that restrict to the identity on its boundary, with function composition as the binary operation. $\text{Homeo}_0(S, \partial S)$ denotes the connected component of the identity within this group.

Equivalently, $\text{Mod}(S)$ is the group of isotopy classes of homeomorphisms from S to itself. In other words, two elements of $\text{Mod}(S)$ are equivalent if they are isotopic to one another.

In order to better understand the elements of the mapping class group of a surface, called mapping classes, we can use a proposition called the Alexander method to narrow our focus to the action of mapping class on finitely many curves and arcs on a surface.

First, we will define some terms.

Definition 1.6. *A* ***closed curve*** *on a surface S is a continuous map $f : S^1 \to S$. Furthermore, closed curve is* ***simple*** *if f is an embedding, or equivalently, if $f : S^1 \to f(S)$ is a homeomorphism, and a closed curve is* ***essential*** *if it is not homotopic to a point or a boundary component.*

Definition 1.7. *A* ***proper arc*** *on a surface S is a map $\alpha : [0, 1] \to S$ such that $\alpha^{-1}(\mathcal{P} \cup \partial S) = \{0, 1\}$, where $\mathcal{P}$ is the set of marked points on S. A proper arc is* ***simple*** *if its restriction to $(0, 1)$ is an embedding.*

Occasionally we will consider curves and arcs up to homotopy. In particular, the homotopy class of a curve or arc α contains all curves β such that there exists a homotopy $H : \alpha \times [0, 1] \to \beta$.

Definition 1.8. *The* ***geometric intersection number*** *between the homotopy classes a and b of simple closed curves in a surface S is the minimal number of intersection points between representatives of each class. Furthermore, two curves α and β on a surface are in* ***minimal position*** *if $|\alpha \cap \beta|$ is equal to the geometric intersection number of their respective homotopy classes.*

Definition 1.9. *A collection of simple closed curves $\{\gamma_i\}$* ***fills*** *a surface S if the complement of $\bigcup_i \gamma_i$ in S is a disjoint union of disks and once-punctured disks.*

The following result will allow us to classify elements of a mapping class group by considering its action on a finite collection of curves and arcs.

Proposition 1.10. (Alexander method, [2]) *Let S be a compact surface, with or without marked points, and let $\phi \in Homeo^+(S, \partial S)$. Let $\gamma_1, \ldots, \gamma_n$ be a collection of essential simple closed curves and simple proper arcs in S such that*

1. For all $i \neq j$, γ_i and γ_j are in minimal position.

2. For all $i \neq j$, γ_i and γ_j are not isotopic.

3. For all distinct i, j, k, one of $\gamma_i \cap \gamma_j$, $\gamma_i \cap \gamma_k$, and $\gamma_j \cap \gamma_k$ is empty.

Then the following two statements are true:

i. If there is a permutation σ of $\{1, \ldots, n\}$ such that, for all i, $\phi(\gamma_i)$ is isotopic to $\gamma_{\sigma(i)}$ relative to ∂S, then $\phi(\cup\gamma_i)$ is isotopic to $\cup\gamma_i$ relative to ∂S.

ii. Assume $\{\gamma_i\}$ fills S. If ϕ_* fixes each vertex and each edge of Λ while preserving orientation, then ϕ is isotopic to the identity. Otherwise, there exists a nontrivial power of ϕ that is isotopic to identity.

This result will allow us to compare homeomorphisms from a surface S to tself. In particular, we can use statement (ii) by choosing a set of curves and arcs on S that satisfy the conditions in the theorem, find the image of the collection of curves under each homeomorphism. Then the two homeomorphisms are isotopic to each other if and only if the images of the collection of curves and arcs are homotopic to each other.

1.4 The Finite Generation of the Mapping Class Group

As mentioned in the introduction, the mapping class group of any surface is generated by a finite number of Dehn twists, a specific kind of homeomorphism. We will now define and introduce this idea precisely.

Definition 1.11. *Given an oriented surface S, and α a simple closed curve in S, let N be a regular neighborhood of α. Choose an orientation preserving homeomorphism $\phi : A \to N$, where A denotes the annulus.*

Then the ***Dehn twist about*** *α is the homeomorphism $T_\alpha : S \to S$ defined by*

$$T_\alpha(x) = \begin{cases} \phi \circ T \circ \phi^{-1}(x) & \text{if } x \in N \\ x & \text{if } x \in S/N \end{cases}$$

Figure 1.1 shows an example of a Dehn twist being performed on an annulus. When applied to a curve on a surface, the neighborhood around such a curve forms a sub-annulus, and we perform a Dehn twist on this annulus.

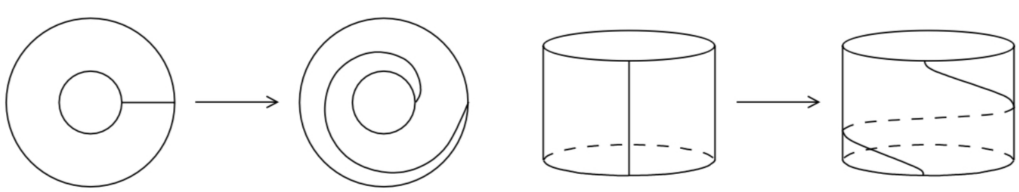


Figure 1.1

The Dehn twist pictured above is a **left-handed** Dehn twist. This is because, if we follow the arc shown from another point on the surface heading toward the sub annulus, once it arrives at the sub annulus, it will turn left to follow the shape of the curve. This is well-defined, even without determining an orientation for the curve [2]. By convention, we will consider left-handed Dehn twists to be **positive Dehn twists**. The inverse of a positive Dehn twist will be a right-handed, or negative Dehn twist.

Dehn twists are also well-defined on isotopy classes of curves [2]. Typically the notation for a Dehn twist would be T_a, where a is the isotopy class of a simple closed curve α. However, we will use T_α to denote the Dehn twist over the isotopy class of α.

Dehn twists are relevant because the mapping class group of a surface is generated by Dehn twists over finitely many simple closed curves on that surface. The following theorems are all related to this result.

Theorem 1.12. (Humphries generators, [3]) Let $g \geq 2$. Then the group $\mathrm{Mod}(S_{g,n})$ is generated by the Dehn twists about the $2g+1$ isotopy classes of nonseparating simple closed curves

$$a_1, \ldots, a_g, c_1, \ldots, c_{g-1}, m_1, m_2$$

as shown in Figure 1.2.

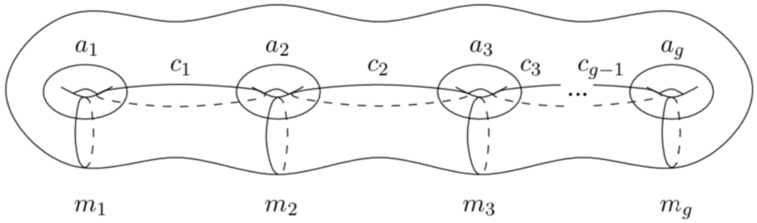


Figure 1.2

In the following theorem, let PMod(S) denote the subgroup of Mod(S) where each element fixes the punctures individually.

Corollary 1.13. *Let S be any surface of genus $g \geq 2$. Then PMod(S) is generated by finitely many Dehn twists about nonseparating simple closed curves in S [2].*

In particular, twists over the following set of curves generate PMod(S), where the small circles can be taken to be punctures or boundary components.

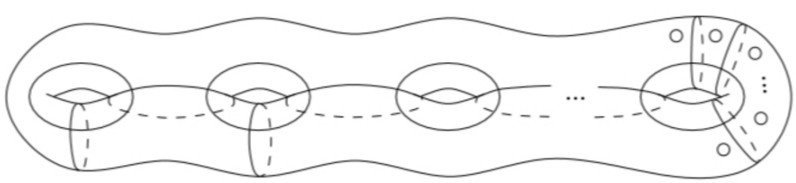


Figure 1.3

Furthermore, for surfaces with at most one boundary component, Mod(S) is generated by the curves shown in the figure above. However, if S has $n \geq 2$ boundary components, then Mod(S) is generated by the curves depicted and $n-1$ boundary parallel curves [2].

2 Comparing Products of Dehn Twists

2.1 Braids Corresponding to Dehn Twists

For the remainder of this thesis, we will restrict our focus to the surface of genus 0 with with $n \geq 0$ boundary components. In particular, we know that any element of the mapping class group of a surface can be written as the product of Dehn twists, so relations between products of Dehn twists will help us determine whether two mapping classes are equivalent.

We will first develop techniques for comparing mapping classes of the sphere with 1 boundary component and $n-1$ punctures, which we will apply to the sphere with no punctures and n boundary components.

Most importantly, we will be able to represent Dehn twists on such a surface using braids.

Definition 2.1. *A **braid** is a set of strings that each have fixed endpoints on a bar on the left and on the right, as in the figure below, where any vertical plane between the bars intersects each strand exactly once.*

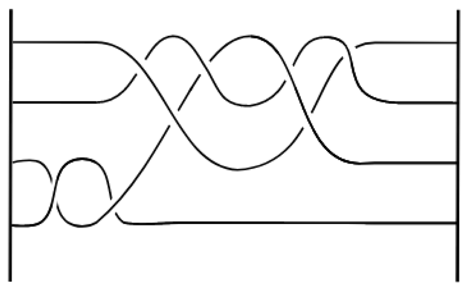


Figure 2.1

We can define an isotopy between braids if we equate a braid with its embedding into $\mathbb{R}^3$. Then a braid is isotopic to another braid if there exists an isotopy between their embedding maps. Two braids are considered equivalent if they are isotopic. In other words, two braids are equivalent if one can be converted to another by stretching and moving its strands without moving the endpoints of any strands or breaking a strand.

Any vertical plane between the plates of a braid intersects each strand once, and we can perform an isotopy to ensure that there is a clear ordering on the relative vertical positions of the strands everywhere except directly at their crossings. We call the highest strand the first strand, the one below it the second strand, and so on. After each crossing, this order will change, but we will apply this terminology based on current relative position.

We will use the following notation for braids:

Let σ_i denote the crossing where the ith strand passes in front of the $(i+1)$th strand. Then σ_i^{-1} represents the crossing where the $(i+1)$th strand passes in front of the ith. Then a braid can be represented by a product of σ_i's and σ_i^{-1}'s, called a **word** [1].

For example, we can denote the braid in Figure 2.1 by the word

$$\sigma_3^{-1}\sigma_3^{-1}\sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1$$

Because the strands can be ordered everywhere except at the crossings themselves, this notation is well-defined. Furthermore, strands must cross adjacent strands before reaching others, so the collection of σ_i and σ_i^{-1} for $1 \leq i-1 \leq n$ describe all possible crossings in a braid.

We can use this notation to determine whether two braids are isotopic. In particular, we have the following theorem:

Theorem 2.2. ([1]) Two braids are isotopic if and only if their associated words are equivalent using the following relations:

1. $\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i$, and both are equivalent to no crossing at all.

2. $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

3. $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| > 1$

We can also use braids to represent disks with punctures. A disk with punctures is equivalent to the surface of genus 0 with one boundary component and n punctures, $S^1_{0,n-1}$. We begin with such a disk, and we perform a homeomorphism that arranges the punctures into a semicircle. Then we can turn the disk to the side, as in the example below with $S^5_{0,1}$. We will assume that $S^1_{0,n-1}$ has a metric, and that the punctures are arranged so that they form the vertices of a convex polygon under this metric.

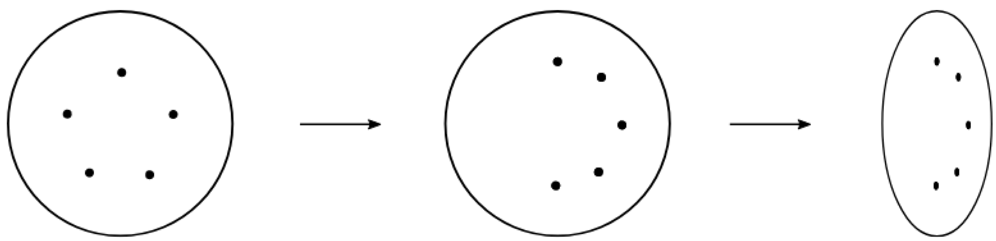


Figure 2.2

Additionally, we can perform a type of homeomorphism called a convex swing on such a disk.

A convex swing on a convexly punctured disk is the movement of a collection of n punctures contained in a convex curve, where the punctures permute themselves n times, each time sending the puncture in the first position to the second position, the second goes to the third, and so on, until each puncture returns to its original position. By convention, we will consider convex swings to go in a clockwise direction.

This motion can be represented by a braid. We can treat the disk the bar on one side of a braid and each puncture as the basepoint of a strand. Then the strands can represent the movement of each puncture under a convex swing.

For example, if the first and second puncture twist around each other in a clockwise fashion, we can track their movement in the following way:

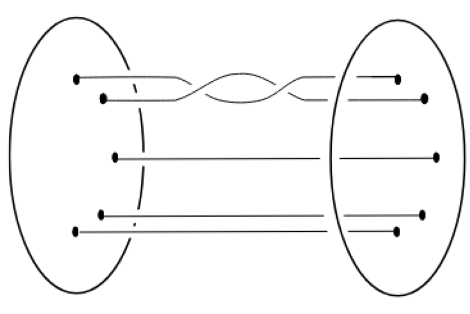


Figure 2.3

Using the notation for crossings in braids, we can say that this corresponds to the braid denoted by $\sigma_1^{-1} \sigma_1^{-1}$.

For simplicity, when we use this technique later in the thesis, we will arrange the punctures so that they sit on a vertical line. In such a case, some curves containing various punctures will not be convex, but we can always skew the punctures so that they are in the shape of a half circle.

Definition 2.3. *A **pure braid** is a braid such that each strand has the same position on both endpoints.*

We can also think of braids as the motion of points in space. With this conceptualization we can say that, in a pure braid, the points return to their original positions at the end of the braid.

We can compose two braids by connecting the endpoints of one with the starting points of the other. Under the composition of braids, the set of pure braids on a set of points A, denoted $\text{PB}\textsc{raid}_A$, forms a group.

We can relate pure braids to Dehn twists with the following result:

Let D_A be a disk with n punctures, where A is the set of punctures. Then $\text{PB}\textsc{raid}_A$ is isomorphic to the mapping class group of D_A [5].

In particular, Lemma 2.4 tells us how to relate Dehn twists to pure braids.

Lemma 2.4. *Let D_A be a convexly punctured disk. In the mapping class group, every convex swing over a set of punctures B is equivalent to the Dehn twist over the convex curve whose interior contains exactly the punctures in B. Additionally, every Dehn twist over a convex curve b is equivalent to the convex swing over the points B contained in the interior of b [(1)].*

By the isomorphism between the pure braid group and the mapping class group, any product of convex swings on a disk D_A is exactly equivalent to an element of the pure braid group. Furthermore, a product of Dehn twists is equivalent to a product of convex swings. Therefore, given two products of Dehn twists on D_A, they are equivalent in the mapping class group if and only if their associated braids are isotopic.

Because the focus of this thesis is on spheres with boundary components and no punctures, we will apply this result to such surfaces. In particular, given a surface $S^n_{0,0}$, we can perform a homeomorphism that stretches one boundary component around the others so that it lies flat. Then it looks like a disk with one boundary component on the outside and $n-1$ interior boundary components. We can perform an action called capping to the interior boundary components, in which we take $n-1$ once-punctured disks and glue the boundary of each one to an interior boundary component. Then we are left with the disk with $n-1$ punctures, as before. In particular, we can use braids to represent products of Dehn twists on spheres with boundary by capping its interior boundary components and finding the braids associated with the Dehn twists over the same set of curves.

However, the condition that the braids for two product of Dehn twists are isotopic is no longer sufficient for determining equivalency in the mapping class group. Therefore, we must find a stronger condition. We will use a value called multiplicity to provide a necessary and sufficient condition.

Definition 2.5. *Under a product of Dehn twists over a collection of convex simple closed curves $\{\gamma_1, \ldots, \gamma_n\}$ on a surface with boundary, the **multiplicity** of a boundary component b_i is the number of curves γ_j such that b_i is contained in the interior of γ_j.*

Using this term, we make the following claim:

Theorem 2.6. *Two products of Dehn twists over convex curves in $S^n_{0,0}$ are equivalent if and only if their associated braids are isotopic and each interior boundary components have the same multiplicity under each product of Dehn twists.*

If we have two products of Dehn twists on the sphere with boundary that are equivalent, then capping each of the interior boundary components with a punctured disk and performing each product of twists will yield the same braid. Furthermore, the first half of the proof of Lemma 2.3 in [6] shows that multiplicity is well-defined under any element of the mapping class group, so the multiplicities of each boundary component under each product must be equal as well.

The proof of Lemma 2.3 in [6] also shows that, if two products of Dehn twists have isotopic braids and each boundary component has the same multiplicity, then the two products of Dehn twists are equivalent in the mapping class group. This completes the argument.

Finally, we will present one last result that will be useful in determining what relations may exist between products of Dehn twists.

Definition 2.7. *A **link** is a set of knotted loops that are tangled together. Each loop is an embedding of the circle S^1 into three-dimensional space. Two links are equivalent if there exists an isotopy between them.*

Definition 2.8. *The **closure** of a braid is the link that results from connecting the starting and ending points of the braid.*

If there exists an isotopy that relates two braids, then their closures are also isotopic. This is because the conversion from braid to link is an isotopy itself, and the composition of two isotopies is an isotopy.

In particular, this means that link invariants can be applied to braids by taking their closure. We will present one such invariant.

Definition 2.9. *Let M and N be two oriented components of a link. Then the **linking number** between them is calculated in the following way:*

First we assign a value of 1 or -1 to each crossing between M and N, using the following guide:

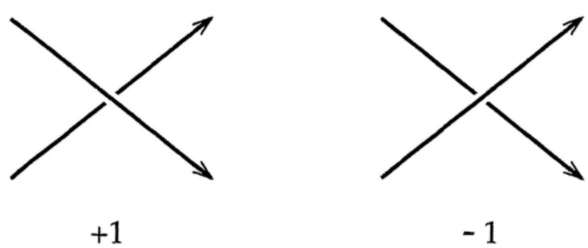


Figure 2.4

Finally, we add up the values assigned to each crossing and divide the sum by 2. The resulting value is the linking number between M and N.

Linking number can also be applied to pure braids. This is because the crossings are preserved when taking a braid to its closure, and each strand in a pure braid corresponds to a distinct component of its closure, so the pairwise linking numbers between components are exactly the pairwise linking numbers between strands. Furthermore, we consider the strands to have an orientation from right to left, and this can be taken to be the orientation of the resulting component of the link.

Additionally, the linking number between any two components of a link is a link invariant [1]. This means that, if two links are isotopic, then the linking numbers of any pair of components in each link are equal. In particular, if we have two braids such that any choice two strands has a different linking number in each braid, this means that their closures are not isotopic. Therefore, the braids themselves are not isotopic.

As a result, if we have two braids and we can show that any two of the strands have differing linking numbers in each braid, we know that they are not isotopic. We will apply this to the braids corresponding to Dehn twists. We will define the linking number between two interior boundary components to be the linking number between their associated strands.

We will now present a result that will allow us to easily calculate the linking numbers of any Dehn twist.

Theorem 2.10. *Given a product of Dehn twists over a collection of curves $\{\gamma_1, \ldots, \gamma_k\}$ on the surface $S_{0,0}^n$, the linking number between two strands b_x, b_y in the corresponding braid is equal to $-m$, where m is the number of curves γ_i that contain both b_x and b_y in their interior.*

Proof. Let $b_1, \ldots, b_n$ be the boundary components of $S_{0,0}^n$. First, we perform a homeomorphism so that $S_{0,0}^n$ lies flat in the plane with b_n as its outer boundary component. Take $B := \{b_{a_1}, \cdots, b_{a_k}\}$ to be a set of interior boundary components on $S_{0,0}^n$, and let γ be the convex curve containing exactly these boundary components. We claim that the Dehn twist over γ will give a linking number of -1 to each pair b_{a_i}, b_{a_j}, and a pairwise linking number of 0 to any pair of curves where one is not contained in γ in the corresponding braid.

We will consider the braid resulting from such a twist. To do this, first we will cap the $n-1$ interior boundary components and find the braid corresponding to the twist over γ on the resulting surface.

We will find such a braid by performing a homeomorphism that arranges the punctures in a vertical line such that the puncture corresponding to b_{a_1} is at the top and their indices are in order. We can find the braid corresponding to the twist over γ by skewing the punctures so that they form a semicircle and performing a swing on the set B.

As a convention, we will draw the crossing between any two strands to be level with the basepoints of the higher strand. Using this convention, the resulting braid is below:

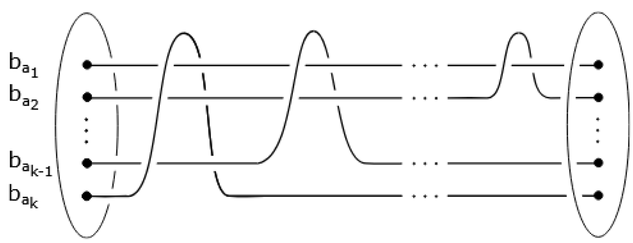


Figure 2.5

From the first section of this braid, we see that b_{a_k} has two crossings of the value -1 with each other strand, and it does not cross any other strands later in the braid. Thus, the linking number between b_{a_k} and any other hole in B is $\frac{-1-1}{2} = -1$.

Then $b_{a_{k-1}}$ crosses similarly over all holes b_{a_i} such that $i < k-1$, giving a linking number of -1 between $b_{a_{k-1}}$ and any other hole above it. Furthermore, we already know that b_{a_k} has a linking number of -1 with $b_{a_{k-1}}$, so $b_{a_{k-1}}$ has a linking number of -1 with every other element of B. Continuing inductively, we see that, for any choice of i and j, b_{a_i} and b_{a_j} will have a linking number of -1.

Now we will consider the linking numbers between holes in B and holes not in B. We will add three arbitrary strands to represent all possible cases: one above all b_{a_i}, one between them, and one below all of them.

Then, following the same procedure, we find the braid corresponding to the twist over γ.

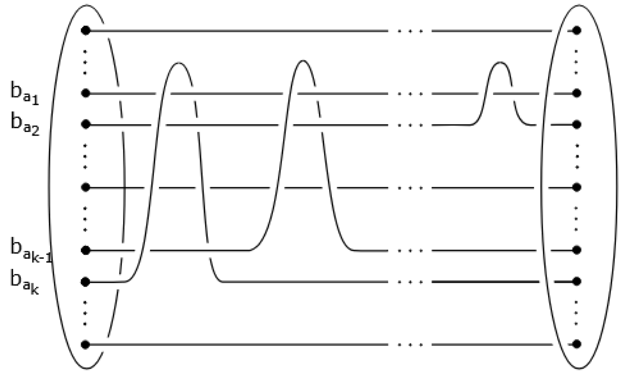


Figure 2.6

If we skew the punctures, we can see that the twist over γ would exclude the strands between b_{a_2} and $b_{a_{k-1}}$ because they will be farther back than the edge of γ that the points in γ will be following.

Let b_{a_i} be given. For each b_j not in B, either $b_j < a_i$, $a_i < b_j < a_1$, or $a_1 < b_j$.

In the first case, because b_{a_i} does not cross any strands below it, there are no crossings between b_{a_i} and b_j, so their linking number is 0. Similarly, if $a_1 < b_j$, then b_{a_i} will not cross b_j so their linking number will again be 0.

Finally, if $a_i < b_j < a_1$, then a_i crosses in front of b_j both on its way up and on its way down. The first crossing will have a value of -1 and the second will have a value of 1. Thus, the linking number between them will be $\frac{1-1}{2} = 0$.

Furthermore, given b_j, b_m that are both not in B, since both stay in place, they will also have a linking number of 0. Thus, a positive Dehn twist over B will yield a linking number of -1 between any two holes contained in it and a linking number of 0 between any other pair of holes.

Now we will consider the case where we are twisting over a non-convex curve containing $\{b_{a_1}, \ldots b_{a_k}\}$.

Let γ be the non-convex curve containing these boundary components. We will cut along γ and obtain two subsurfaces that are also genus 0 surfaces with boundary. Then we can take a homeomorphism on each one. For the subsurface containing $\{b_{a_1}, \ldots b_{a_k}\}$, we can reshape it so that its boundary has a convex shape. Similarly, we can take a homeomorphism on the other part to reshape the cavity left by cutting out the subdisk so that the boundary is the same shape as the subdisk under the chosen homeomorphism.

Now we can glue them back together on the boundary pieces that resulted from cutting. We claim that this is a homeomorphism on the whole surface. Indeed, it is a homeomorphism on each of its pieces, and the boundary is a circle, which means that all homeomorphisms on it are in one isotopy class. Therefore, if this function is not continuous on the boundary that is glued together, we can take an isotopy that will ensure that it is continuous on this curve. Thus, we have a homeomorphism of the surface that takes a non convex curve containing $\{b_{a_1}, \ldots b_{a_k}\}$ to a convex curve containing $\{b_{a_1}, \ldots b_{a_k}\}$.

This can be taken to be an element of the mapping class group of the surface, because it is a homeomorphism that fixes the boundary pointwise. Thus, up to isotopy, there exists a collection of curves $\{c_1, \ldots, c_m\}$ such that some product of positive and negative Dehn twists over curves in this collection will be equivalent to this homeomorphism in the mapping class group. Call this product of twists ϕ.

Then we know that $\phi(\gamma)$ gives the convex curve containing $\{b_{a_1}, \ldots b_{a_k}\}$. Call it c. Equivalently, we can say that $T_\gamma = T_{\phi(c)}$. From [2], we know that $T_{\phi(c)} = \phi \circ T_c \circ \phi^{-1}$.

Because ϕ is equivalent to a product of Dehn twists, we can represent this with a braid. By the properties of inverses in a group, the inverse of a product of Dehn twists $T_{c_1} \cdots T_{c_m}$ is equivalent to the product of the inverses $T_{c_m}^{-1} \cdots T_{c_1}^{-1}$. Let b_i and b_j be given. Assume they have a linking number of x under the twist over c and a linking number of y under ϕ. Then, under ϕ^{-1}, they will have a linking number of $-x$ because the inverse of a twist over a curve gives a linking number with opposite sign from the twist over the original curve. Thus, the total linking number between them under $T_{\phi(c)}$ is $y + (x) + (-y) = -1$. Therefore, twisting over a non-convex curve containing $\{b_{a_1}, \ldots b_{a_k}\}$ gives the same pairwise linking numbers to all boundary components as twisting over the convex curve containing the same b_{a_i}.

Therefore, in a product of convex Dehn twists, the linking number of any two holes b_i, b_j will be equal to -1 times the number of twists over curves that contain both b_i and b_j. □

Now we will present two results that will allow us to convert products of Dehn twists into words in the braid.

Theorem 2.11. *For $i < j$, the braid corresponding to $T_{b_{j-1},b_j} T_{b_{j-2},b_j} \cdots T_{b_{i+1},b_j} T_{b_i,b_j}$ is isotopic to the braid denoted by $\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\cdots\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}$*

Proof. First, we will find the word that represents T_{b_k,b_j} for arbitrary $k < j$.

By the notation, we know that we are twisting over the convex simple closed curve containing b_k and b_j.

This results in the following braid:

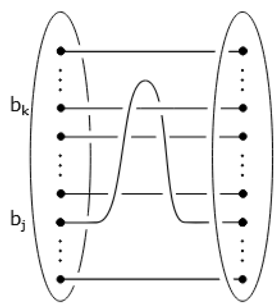


Figure 2.7

The word that denotes this braid is

$$\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\cdots\sigma_{k+1}^{-1}\sigma_k^{-1}\sigma_k^{-1}\sigma_{k+1}\cdots\sigma_{j-2}\sigma_{j-1}$$

Recall that Dehn twists are applied from right to left, while the word of a braid represents the braid from left to right. Taking this into account, for any k, $T_{b_{k+1},b_j}T_{b_k,b_j}$ corresponds to the braid

$$\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\cdots\sigma_{k+1}^{-1}\sigma_k^{-1}\sigma_k^{-1}\sigma_{k+1}\cdots\sigma_{j-2}\sigma_{j-1}\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\cdots\sigma_{k+2}^{-1}\sigma_{k+1}^{-1}\sigma_{k+1}^{-1}\sigma_{k+2}\cdots\sigma_{j-2}\sigma_{j-1}$$

Performing cancellation with inverses gives us the word

$$\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\cdots\sigma_{k+1}^{-1}\sigma_k^{-1}\sigma_k^{-1}\sigma_{k+1}^{-1}\sigma_{k+2}\cdots\sigma_{j-2}\sigma_{j-1}$$

We can see this visually as well.

$$\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\cdots\sigma_{k+1}^{-1}\sigma_k^{-1}\sigma_k^{-1}\sigma_{k+1}\cdots\sigma_{j-2}\sigma_{j-1}\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\cdots\sigma_{k+2}^{-1}\sigma_{k+1}^{-1}\sigma_{k+1}^{-1}\sigma_{k+2}\cdots\sigma_{j-2}\sigma_{j-1}$$

corresponds to the following braid:

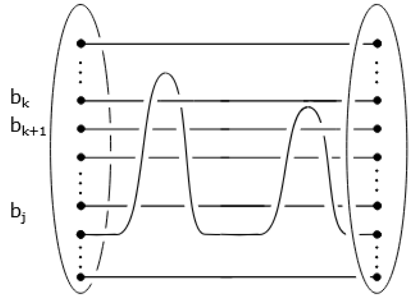


Figure 2.8

Cancelling out the inverses in an isotopy analogous to the Reidemeister 2 move for knots, we get

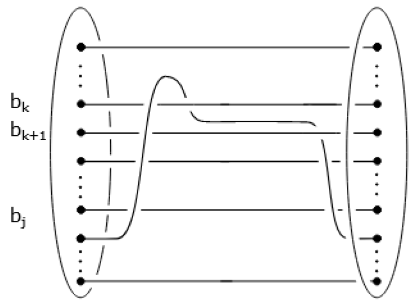


Figure 2.9

Then we can perform an isotopy that smooths the strands so that the braid follows our standard conventions. This is exactly the braid that is denoted by

$$\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\cdots\sigma_{k+1}^{-1}\sigma_{k}^{-1}\sigma_{k}^{-1}\sigma_{k+1}^{-1}\sigma_{k+2}\cdots\sigma_{j-2}\sigma_{j-1}$$

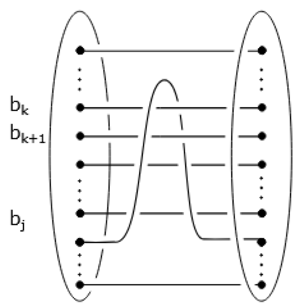


Figure 2.10

If we begin with $k = i$ and continue the process inductively, the resulting braid will be exactly that which is denoted by

$$\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\cdots\sigma_{i+1}^{-1}\sigma_{i}^{-1}\sigma_{i}^{-1}\sigma_{i+1}^{-1}\sigma_{i+2}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}$$

The braid will look like this:

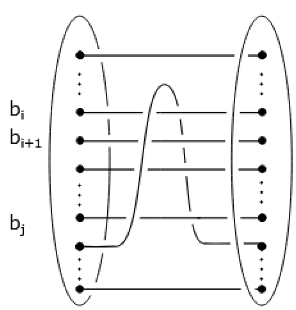


Figure 2.11

□

This result can also be applied to twists that do not contain all consecutive b_i's. In particular, for a product of Dehn twists of the form

$$T_{b_{j-1},b_j} T_{b_{j-2},b_j} \cdots T_{b_{k+1},b_j} T_{b_{k-1},j} \cdots T_{b_{i+1},b_j} T_{b_i,b_j}$$

we can first take the strand to be invisible, apply the twist as in the proof above, and add the kth strand behind the twisted strands. This will be denoted by the word

$$\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\cdots\sigma_{k+1}^{-1}\sigma_k^{-1}\sigma_{k-1}^{-1}\cdots\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1}\cdots\sigma_{k-1}^{-1}\sigma_k\sigma_{k+1}^{-1}\cdots\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}$$

We can also do this for multiple values of k. This gives us a way to simplify braids quickly, and will be used when proving various relations.

We will also present relations on the words representing braids that can be derived from the three known relations:

1. $\sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i$, and both are equivalent to no crossing at all.

2. $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

3. $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $i - j > 1$

In particular, we have the additional relations:

4. $\sigma_i^{-1}\sigma_{i+1}^{-1}\sigma_i^{-1} = \sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1}$

   This can be obtained by taking the inverse of both sides of relation (2).

5. $\sigma_i^{-1}\sigma_j^{-1} = \sigma_j^{-1}\sigma_i^{-1}$ for $i - j > 1$

   This can be obtained by taking the inverse of both sides of relation (3).

6. $\sigma_i\sigma_j^{-1} = \sigma_j^{-1}\sigma_i$

   In particular, we know that $\sigma_i\sigma_j = \sigma_j\sigma_i$. Right-multiplying both sides by σ_j gives us $\sigma_i = \sigma_j\sigma_i\sigma_j^{-1}$. Then left-multiplying both sides by σ_j^{-1} gives $\sigma_j^{-1}\sigma_i = \sigma_i\sigma_j^{-1}$

7. $\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1} = \sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}$

   We begin with $\sigma_{i+1}^{-1}\sigma_i^{-1} = \sigma_{i+1}^{-1}\sigma_i^{-1}$

   Then multiply the right hand side by $\sigma_{i+1}^{-1}\sigma_{i+1}$, which is equal to the identity.

   $\sigma_{i+1}^{-1}\sigma_i^{-1} = \sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1}\sigma_{i+1}$

   Applying relation (4) gives us

   $\sigma_{i+1}^{-1}\sigma_i^{-1} = \sigma_i^{-1}\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}$

Then left-multiplying both sides by σ_i gives

$\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1} = \sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}$

as desired.

8. $\sigma_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1} = \sigma_i^{-1}\sigma_{i+1}^{-1}\sigma_i$

We begin with $\sigma_i^{-1}\sigma_{i+1}^{-1} = \sigma_i^{-1}\sigma_{i+1}^{-1}$

Multiplying the left hand side with $\sigma_{i+1}\sigma_{i+1}^{-1}$ gives us

$\sigma_{i+1}\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1} = \sigma_i^{-1}\sigma_{i+1}^{-1}$

Then using relation (4), we get

$\sigma_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1}\sigma_i^{-1} = \sigma_i^{-1}\sigma_{i+1}^{-1}$

Then we right multiply both sides by σ_i to get

$\sigma_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1} = \sigma_i^{-1}\sigma_{i+1}^{-1}\sigma_i$

We have one more relation on braids to present:

Theorem 2.12. *The relation*

$$(\sigma_n^{-1}\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}\sigma_n^{-1})(\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{n-2}^{-1}\sigma_{n-1}^{-1})$$
$$= (\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{n-2}^{-1}\sigma_{n-1}^{-1})(\sigma_n^{-1}\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}\sigma_n^{-1})$$

holds.

Proof. We proceed by induction on n.

Base case: $n = 2$

We claim that $(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1}) = (\sigma_1^{-1}\sigma_1^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})$

We begin with the left hand side:

$\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1} = \sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1} = \sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1} = \sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1} = \sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}$

We see that this is equal to the right hand side.

Now we take the inductive step: Assume that the relation holds for $n = k - 1$. We will show that it holds for $n = k$.

The left hand side is equal to

$$(\sigma_k^{-1}\sigma_{k-1}^{-1}\sigma_{k-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{k-2}^{-1}\sigma_{k-1}^{-1}\sigma_k^{-1})(\sigma_{k-1}^{-1}\sigma_{k-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{k-2}^{-1}\sigma_{k-1}^{-1})$$

$$\sigma_k^{-1}\sigma_{k-1}^{-1}\sigma_{k-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{k-2}^{-1}\sigma_k^{-1}\sigma_{k-1}^{-1}\sigma_k^{-1}\sigma_{k-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{k-2}^{-1}\sigma_{k-1}^{-1}$$

We see that σ_k^{-1} commutes with each element to its right until it reaches σ_{k-1}^{-1}. Thus, we have

$$\sigma_k^{-1}\sigma_{k-1}^{-1}\sigma_{k-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{k-2}^{-1}\sigma_k^{-1}\sigma_{k-1}^{-1}\sigma_{k-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{k-2}^{-1}\sigma_k^{-1}\sigma_{k-1}^{-1}$$

Similarly, σ_k^{-1} commutes with every element to its left until it reaches σ_{k-1}^{-1}, resulting in

$$\sigma_k^{-1}\sigma_{k-1}^{-1}\sigma_k^{-1}\sigma_{k-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{k-2}^{-1}\sigma_{k-1}^{-1}\sigma_{k-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{k-2}^{-1}\sigma_k^{-1}\sigma_{k-1}^{-1}$$

$$\sigma_{k-1}^{-1}\sigma_k^{-1}(\sigma_{k-1}^{-1}\sigma_{k-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{k-2}^{-1}\sigma_{k-1}^{-1})(\sigma_{k-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{k-2}^{-1})\sigma_k^{-1}\sigma_{k-1}^{-1}$$

By the inductive hypothesis, the products in the parentheses commute with each other.

$$\sigma_{k-1}^{-1}\sigma_k^{-1}(\sigma_{k-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{k-2}^{-1})(\sigma_{k-1}^{-1}\sigma_{k-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{k-2}^{-1}\sigma_{k-1}^{-1})\sigma_k^{-1}\sigma_{k-1}^{-1}$$

Then σ_k^{-1} commutes with everything in the product in the leftmost set of parentheses, so we get

$$\sigma_{k-1}^{-1}(\sigma_{k-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{k-2}^{-1})(\sigma_k^{-1}\sigma_{k-1}^{-1}\sigma_{k-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{k-2}^{-1}\sigma_{k-1}^{-1})\sigma_k^{-1}\sigma_{k-1}^{-1}$$

$$\sigma_{k-1}^{-1}\sigma_{k-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{k-2}^{-1}\sigma_k^{-1}\sigma_{k-1}^{-1}\sigma_{k-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{k-2}^{-1}(\sigma_k^{-1}\sigma_{k-1}^{-1}\sigma_k^{-1})$$

Again, σ_k^{-1} commutes with everything on its left until it reaches σ_{k-1}^{-1}

$$\sigma_{k-1}^{-1}\sigma_{k-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{k-2}^{-1}\sigma_k^{-1}\sigma_{k-1}^{-1}\sigma_k^{-1}\sigma_{k-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{k-2}^{-1}\sigma_{k-1}^{-1}\sigma_k^{-1}$$

$$\sigma_{k-1}^{-1}\sigma_{k-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{k-2}^{-1}(\sigma_{k-1}^{-1}\sigma_k^{-1}\sigma_{k-1}^{-1})\sigma_{k-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{k-2}^{-1}\sigma_{k-1}^{-1}\sigma_k^{-1}$$

We see that this is exactly the right hand side of the relation for $n = k$.

Thus, the relation holds. □

We note that the argument in this proof does not rely on 1 being in the very center of each word. Rather, it relies on the indices being exactly adjacent. Therefore, we can obtain similar relations by adding a single positive integer to all indices in the relation.

Furthermore, we can also commute the terms in parentheses when given a product of the form

$$(\sigma_{n+j}^{-1}\cdots\sigma_{n+1}^{-1}\sigma_n^{-1}\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}\sigma_n^{-1}\sigma_{n+1}^{-1}\cdots\sigma_{n+j}^{-1})(\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{n-2}^{-1}\sigma_{n-1}^{-1})$$

by applying the relation $\sigma_i^{-1}\sigma_j^{-1} = \sigma_j^{-1}\sigma_i^{-1}$ for $i - j > 1$. In particular, we can commute the second expression in parentheses with the added terms of $\sigma_{n+1}^{-1}\cdots\sigma_{n+j}^{-1}$:

$$(\sigma_{n+j}^{-1}\cdots\sigma_{n+1}^{-1}\sigma_n^{-1}\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}\sigma_n^{-1})(\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{n-2}^{-1}\sigma_{n-1}^{-1})(\sigma_{n+1}^{-1}\cdots\sigma_{n+j}^{-1})$$

Then apply the relation proven above:

$$(\sigma_{n+j}^{-1}\cdots\sigma_{n+1}^{-1})(\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{n-2}^{-1}\sigma_{n-1}^{-1})(\sigma_n^{-1}\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}\sigma_n^{-1})(\sigma_{n+1}^{-1}\cdots\sigma_{n+j}^{-1})$$

Then applying $\sigma_i^{-1}\sigma_j^{-1} = \sigma_j^{-1}\sigma_i^{-1}$ for $i - j > 1$ again, we get

$$(\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{n-2}^{-1}\sigma_{n-1}^{-1})(\sigma_{n+j}^{-1}\cdots\sigma_{n+1}^{-1}\sigma_n^{-1}\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\cdots\sigma_1^{-1}\sigma_1^{-1}\cdots\sigma_{n-2}^{-1}\sigma_{n-1}^{-1}\sigma_n^{-1}\sigma_{n+1}^{-1}\cdots\sigma_{n+j}^{-1})$$

3 Relations on the Mapping Class Monoid of the Sphere with Boundary

This information can be used to find new relations between products of positive Dehn twists on genus 0 surfaces with boundary. One known relation is the lantern relation.

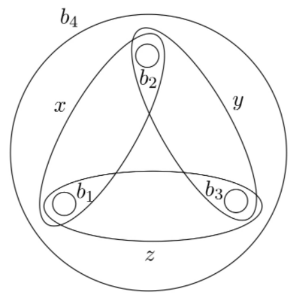


Figure 3.1

Using the labels in the above figure, the lantern relation says that

$$T_{b_1}T_{b_2}T_{b_3}T_{b_4} = T_xT_yT_z$$

where T_α denotes the Dehn twist over the curve α [2]. Since a Dehn twist is a function, this follows function notation, so such a product of twists is applied from right to left. Furthermore, as proven in [2],

$$T_xT_yT_z = T_yT_zT_x = T_zT_xT_y \neq T_xT_zT_y$$

For the rest of this thesis, we will be presenting relations that are analogous to the lantern relation in spheres with more boundary components. For clarity, we will use the notation $T_{b_{a_1},\dots,b_{a_n}}$ to denote the positive Dehn twist over the convex curve containing exactly $b_{a_1},\dots b_{a_n}$ in its interior. For example, $T_x = T_{b_1,b_2}$. Additionally, we will use 'twist" to mean 'positive Dehn twist" unless otherwise specified.

3.1 Relations that hold for general $S_{0,0}^n$

We have both general results that hold for all $n \geq 4$ and specific results for various values of n. We will present the general results first.

Let the boundary components of S_n be labeled as follows:

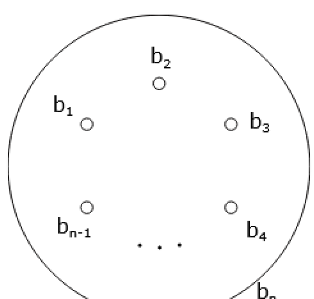


Figure 3.2

Then, for any $2 \leq i < n-1$ the following relation holds:

$$[T_{b_1}^{n-i-1}T_{b_2}^{n-i-1}\cdots T_{b_i}^{n-i-1}][T_{b_{i+1}}^{n-3}\cdots T_{b_{n-1}}^{n-3}]T_{b_n} = T_{(b_1,b_2,\dots,b_i)}[T_{(b_{i+1},b_i)}\cdots T_{(b_{i+1},b_1)}]\cdots[T_{(b_{n-1},b_{n-2})}\cdots T_{(b_{n-1},b_1)}]$$

Proof. We will first show that the multiplicities are equal.

Let b_j be an interior boundary component.

If $j \leq i$, then on the left hand side, the components of the product involving b_j are $T_{b_j}^{n-i-1}$ and T_{b_n}. T_{b_n} is a twist over a curve containing b_j because it is a twist that is parallel to the outer boundary component b_n, so all interior boundary components are contained in it. All components of the product involve twists over boundary-parallel curves that are not parallel to b_j.

Thus, the multiplicity of b_j is $n-i-1+1 = n-i$ on the left hand side.

On the right hand side, the components involving b_j are $T_{b_1,b_2,\ldots,b_i}$ and the T_{b_j,b_k} contained in $[T_{b_{k-1},b_k} \cdots T_{b_1,b_k}]$ for each $k > i$.

We know that there are $n-i-1$ values of k that are greater than i, so the multiplicity of b_j is $n-i-1+1 = n-i$.

Thus, the multiplicities are equal.

Now we will show that the corresponding braids are isotopic.

We will begin by finding the braid corresponding to the left hand side. We note that all twists except for T_{b_n} are boundary parallel. Because a twist along a boundary parallel curve corresponds to a braid that demonstrates a swing on the single puncture contained within that curve, it will not add any crossings to the braid. Therefore, the braid corresponding to T_{b_n} is exactly the braid corresponding to the left hand side.

T_{b_n} is represented by a swing on all interior boundary components, which looks like this:

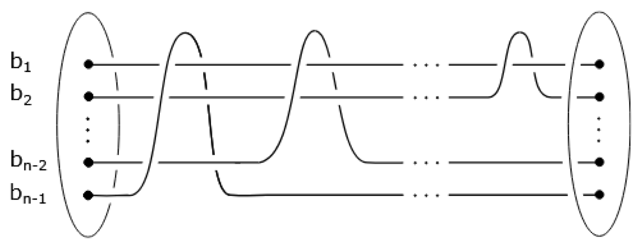


Figure 3.3

We note that each strand crosses in front of every strand above it, then crosses behind each again before returning to its position.

The word that denotes this braid is

$$(\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\cdots\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-3}^{-1}\sigma_{n-2}^{-1})(\sigma_{n-3}^{-1}\sigma_{n-4}^{-1}\cdots\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-4}^{-1}\sigma_{n-3}^{-1})\cdots(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

We will now find the braid corresponding to the right hand side.

Recall that Dehn twists are applied from right to left. Thus, the first portion of the braid will correspond to $[T_{b_{n-2},b_{n-1}} \cdots T_{b_1,b_{n-1}}]$.

By Theorem 2.11, the corresponding braid will be denoted by the word

$$\sigma_{n-2}^{-1}\sigma_{n-3}^{-1}\cdots\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\cdots\sigma_{n-3}^{-1}\sigma_{n-2}^{-1}$$

The following figure shows this braid.

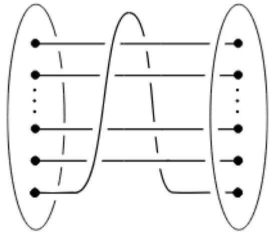


Figure 3.4

Then, applying Theorem 2.11 to each portion in brackets, we see that $[T_{b_i,b_{i+1}} \cdots T_{b_1,b_{i+1}}] \cdots [T_{b_{n-2},b_{n-1}} \cdots T_{b_1,b_{n-1}}]$ corresponds to

$$[\sigma_{n-2}^{-1}\sigma_{n-3}^{-1} \cdots \sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1} \cdots \sigma_{n-3}^{-1}\sigma_{n-2}^{-1}] \cdots [\sigma_i^{-1}\sigma_{i-1}^{-1} \cdots \sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1} \cdots \sigma_{i-1}^{-1}\sigma_i^{-1}]$$

The first two bracketed portions of this braid will look like this:

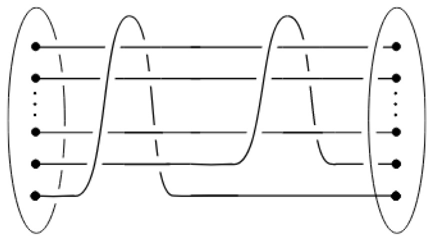


Figure 3.5

This will continue until the ith strand goes in front of and then behind every strand above it. Then we apply the final twist, $T_{b_1,b_2,\ldots,b_i}$. The corresponding braid will depict the swing of each of the first through the ith strand swinging around each other. This braid is denoted by

$$(\sigma_{i-1}^{-1} \cdots \sigma_1^{-1}\sigma_1^{-1} \cdots \sigma_{i-1}^{-1})(\sigma_{i-2}^{-1} \cdots \sigma_1^{-1}\sigma_1^{-1} \cdots \sigma_{i-2}^{-1}) \cdots (\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

Then, putting this together with the first part of the word, we get that the braid is

$$(\sigma_{n-2}^{-1}\sigma_{n-3}^{-1} \cdots \sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1} \cdots \sigma_{n-3}^{-1}\sigma_{n-2}^{-1}) \cdots (\sigma_i^{-1}\sigma_{i-1}^{-1} \cdots \sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1} \cdots \sigma_{i-1}^{-1}\sigma_i^{-1})$$

$$\cdot(\sigma_{i-1}^{-1} \cdots \sigma_1^{-1}\sigma_1^{-1} \cdots \sigma_{i-1}^{-1})(\sigma_{i-2}^{-1} \cdots \sigma_1^{-1}\sigma_1^{-1} \cdots \sigma_{i-2}^{-1}) \cdots (\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

This is exactly

$$(\sigma_{n-2}^{-1}\sigma_{n-3}^{-1} \cdots \sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1} \cdots \sigma_{n-3}^{-1}\sigma_{n-2}^{-1})(\sigma_{n-3}^{-1}\sigma_{n-4}^{-1} \cdots \sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1} \cdots \sigma_{n-4}^{-1}\sigma_{n-3}^{-1}) \cdots (\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

which is the word denoting the braid corresponding to the left hand side. Thus, the braids are isotopic. Therefore, the two products of Dehn twists are equivalent. □

To better understand this type of relation, we can consider the two most extreme cases for the choice of i. If $i = 2$, then the right hand side is a product of twists over all curves that contain exactly two interior boundary components. If $i = n-2$, then the right hand side is the product of all possible twists over the curves that contain exactly b_{n-1} and one other interior boundary component. In both cases, the left hand side is a product of boundary parallel curves, with powers that ensure that the multiplicity of each boundary component is equal on both sides.

We have an additional general result that rules out the existence of certain relations.

Theorem 3.1. *For $n \geq 5$, there are no relations between $T_{b_1}T_{b_2} \cdots T_{b_{n-1}}T_{b_n}$ and any product of twists over convex, non boundary parallel curves in the mapping class group of $S_{0,0}^n$.*

Proof. We proceed by contradiction. Assume that there is a product of Dehn twists over non boundary parallel curves that is equivalent to $T_{b_1}T_{b_2} \cdots T_{b_{n-1}}T_{b_n}$. Let $B : \{\alpha_1, \ldots, \alpha_k\}$ be the set of curves involved in this product of Dehn twists.

We know that the braid corresponding to $T_{b_1}T_{b_2} \cdots T_{b_{n-1}}T_{b_n}$ is equivalent to the braid corresponding to T_{b_n}. As determined above, this is the braid:

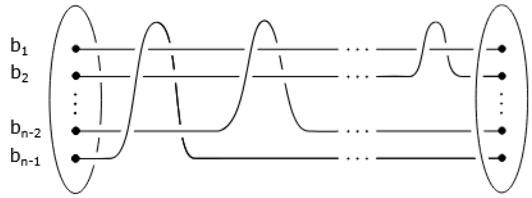


Figure 3.6

Additionally, by Theorem 2.10, the linking number between any two interior boundary components is -1. Since the linking number is a braid invariant, this means that each pair of interior boundary components must have a linking number of -1 under the product of twists over curves in B. Therefore, again by Theorem 2.10, for any two interior boundary components b_i and b_j, there must be exactly one curve $\alpha_k \in B$ such that both b_i and b_j are contained in its interior.

Furthermore, we see that, under $T_{b_1}T_{b_2}\cdots T_{b_{n-1}}T_{b_n}$, each boundary component has a multiplicity of 2. Therefore, for every boundary component b_i, there must exist exactly two curves $\alpha_j, \alpha_k \in B$ such that b_i is contained in their interiors.

We claim that, for $n > 4$, it is not possible to satisfy both conditions.

We will model the situation using a graph. Build a graph G such that each vertex v_i represents the interior boundary component b_i in $S_{0,0}^{n+1}$.

We will add edges, and colors of edges, to this graph in the following way: There exists an edge between vertices v_i and v_j if and only if b_i and b_j have a linking number of -1. All edges resulting from a particular curve α_i will be the same color, with a distinct color for each α_i.

Note that the subgraph of G containing all edges of a particular color, and their adjacent vertices, must be a complete subgraph. This is because, for any two boundary components b_i, b_j contained in a curve α_k, the edge between them will be the color corresponding to α_k. There is an edge between them because α_k is the only curve containing them both, so they have multiplicity -1 resulting from the twist over α_k. Additionally, we note that the number of colors of edges adjacent to a vertex is equal to the multiplicity of the corresponding hole.

Then, in order to have a relation with $T_{b_1}T_{b_2}\cdots T_{b_{n-1}}T_{b_n}$, we must have a complete graph (i.e. all holes have a linking number of -1 with every other hole) and every vertex must have adjacent edges of exactly two distinct colors (i.e. all holes have a multiplicity of 2). We will show this is not possible for $n \geq 4$.

Let $n \geq 4$ and consider the complete graph with n vertices, K_n. Let $v_1 \in K_n$ be given. We know that v_1 must have adjacent edges of exactly two distinct colors. Without loss of generality, assume they are blue and green. Then v_1 is part of two complete subgraphs, one with blue edges and one with green edges. Since $n \geq 4$, at least one of the subgraphs must contain two vertices other than v_1. Without loss of generality, assume it is the blue subgraph. Let v_2, v_3 be distinct vertices in the blue subgraph and let v_4 be a vertex in the green subgraph such that none are equal to v_1.

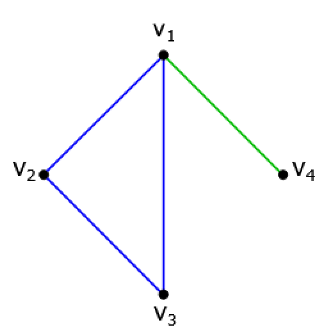


Figure 3.7

Because the graph is complete, we know that there must be an edge between v_4 and v_2, and it must be a third color, call it red.

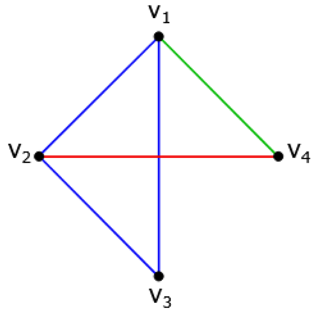


Figure 3.8

There must also be an edge between v_4 and v_3, and it cannot be blue or green because the two vertices are not both in either subgraph. Furthermore, it cannot be red because the edge between v_2 and v_3 is blue, so the red subgraph would not be complete. Thus, it must be a fourth color, call it black.

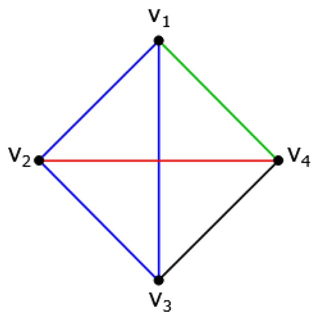


Figure 3.9

Now we see that v_4 is adjacent to edges of three distinct colors, which contradicts the original conditions. Therefore, such a graph is not possible. □

Now we can discuss the validity of particular permutations on a relation.

In particular, the following result will show that we can reorder the right hand side of a relation up to permutation. Furthermore, all boundary parallel curves are disjoint, so the left hand side can be permuted in any way.

Theorem 3.2. *On the surface $S^n_{0,0}$, given its boundary components $\{b_1, \ldots, b_n\}$ such that b_n is taken to be the outer boundary component, if there is a collection of curves $\{c_1, \ldots, c_k\}$ such that*

$$T_{b_1}^{a_1} \cdots T_{b_{n-1}}^{a_{n-1}} T_{b_n} = T_{c_1} \cdots T_{c_{k-1}} T_{c_k}$$

then

$$T_{b_1}^{a_1} \cdots T_{b_{n-1}}^{a_{n-1}} T_{b_n} = T_{c_1} \cdots T_{c_k} = T_{c_k} T_{c_1} \cdots T_{c_{k-1}}$$

Proof. We can obtain the second relation by conjuating the first. If we take the top equation and conjugate both sides by T_{c_k}, then we have the following:

$$T_{c_k} T_{b_1}^{a_1} \cdots T_{b_{n-1}}^{a_{n-1}} T_{b_n} T_{c_k}^{-1} = T_{c_k} T_{c_1} \cdots T_{c_k} T_{c_k}^{-1}$$

We know that Dehn twists over disjoint curves commute [2], so we can commute T_{c_k} with each of the boundary parallel twists:

$$T_{b_1}^{a_1}\cdots T_{b_{n-1}}^{a_{n-1}}T_{b_n}T_{c_k}T_{c_k}^{-1} = T_{c_k}T_{c_1}\cdots T_{c_{k-1}}T_{c_k}T_{c_k}^{-1}$$

Then T_{c_k} and $T_{c_k}^{-1}$ cancel each other out on both sides.

$$T_{b_1}^{a_1}\cdots T_{b_{n-1}}^{a_{n-1}}T_{b_n}T_{c_k}T_{c_k}^{-1} = T_{c_k}T_{c_1}\cdots T_{c_k}T_{c_k}^{-1}$$

$$T_{b_1}^{a_1}\cdots T_{b_{n-1}}^{a_{n-1}}T_{b_n} = T_{c_k}T_{c_1}\cdots T_{c_{k-1}}$$

as desired. □

We have another strategy to determine which orders of twists will preserve a relation. Recall that *capping* refers to the gluing of disks on the interior boundary components along their boundary. If we are given a relation on $S_{0,0}^n$ of the form

$$T_{b_1}^{a_1}\cdots T_{b_{n-1}}^{a_{n-1}}T_{b_n} = T_{c_1}\cdots T_{c_k}$$

then we can take the collection of curves that are used on the right hand side and cap particular boundary components so that we have three left, and the only curves c_i that do not go to boundary parallel curves each contain two of the remaining boundary components, as in the lantern relation. Then we can label each curve as x, y, and z according to the diagram of the lantern relation. In order to preserve the existing relation, we cannot have the order $T_xT_zT_y$ or any rotations of this, according to the lantern relation. If we can reorder the elements of the product of twists on the right hand side so that these curves are in this order, then we can use capping to make the rest of the strands in the braids of both sides of this relation in the braid invisible. Then this would imply that $T_xT_zT_y$ is equivalent to the lantern relation, because the braid for the left hand side will correspond to $T_xT_yT_z$ by the lantern relation. This contradicts the fact that $T_xT_yT_z \neq T_xT_zT_y$.

Therefore, we can use capping to rule out potential permutations of the twists in an existing relation.

Finally, we will show that any relation holds up to isometry. In particular, given a relation between products of twists over a set of curves, we can perform a reflection or rotation on both sets of curves curves while keeping the labels on the boundary components fixed by position, and the products of twists in the same order will preserve the relation.

For example, by the general relation proven above, we have the following relation on $S_{0,0}^5$:

$$T_{b_1}T_{b_2}T_{b_3}T_{b_4}^2T_{b_5} = T_{b_3,b_4}T_{b_2,b_4}T_{b_1,b_4}T_{b_1,b_2,b_3}$$

involving the set of curves shown below:

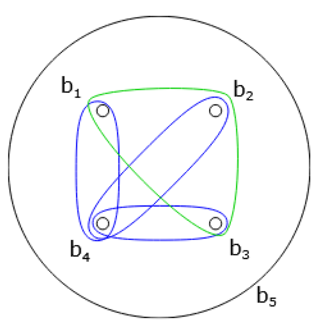


Figure 3.10

Then we can perform an isometry on the curves on both sides. The boundary-parallel curves will not be affected visually, but the curves on the right hand side will look like this:

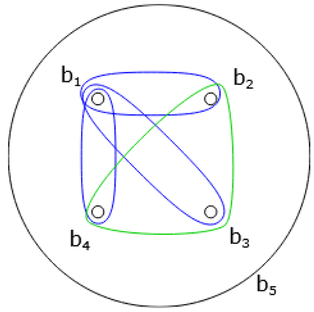


Figure 3.11

Then, to get an equivalent relation, we can relabel the boundary components in the first relation so that we are twisting over the images of the curves under this rotation in the same order as in the original relation.

This gives

$$T_{b_1}^2 T_{b_2} T_{b_3} T_{b_4} T_{b_5} = T_{b_1,b_2} T_{b_1,b_3} T_{b_1,b_4} T_{b_2,b_3,b_4}$$

The following argument explains why relations work up to isometry:

Assume we have a relation, and apply an isometry to the disk. This gives a bijection between boundary components, giving a relabelling of the boundary components that preserves adjacency between them.

In particular, the multiplicities under both sides of the relation will be the same for each boundary component, and the strands in the braid will have the same adjacency, although the order might be flipped and the strand that is at the top may change. However, the isometry affects both sides of the relation in the same way, and it will preserve convex curves, so the braids will still be isotopic, so the relation holds.

In the next sections, we will be classifying relations that exist on spheres with various numbers of boundary components, up to isometry, with the knowledge that other relations can be derived from applying an isometry to a surface on which a relation is already defined.

3.2 Relations on the Mapping Class Monoid of $S^5_{0,0}$

Our aim in this section is to completely describe the relations that exist in the mapping class group of $S^5_{0,0}$ between products of twists of the form $T^{a_1}_{b_1}T^{a_2}_{b_2}T^{a_3}_{b_3}T^{a_4}_{b_4}T_{b_5}$, where each a_i is nonzero, and products of positive Dehn twists over non-boundary parallel curves.

First, we can apply the general relation proven in section 3.1 to $S^5_{0,0}$ for each possible value of i. For $i = 2$, we have

$$T^2_{b_1}T^2_{b_2}T^2_{b_3}T^2_{b_4}T_{b_5} = [T_{b_1,b_2}][T_{b_2,b_3}T_{b_1,b_3}][T_{b_3,b_4}T_{b_2,b_4}T_{b_1,b_4}] \tag{1}$$

The collection of curves used in the right hand side are shown below:

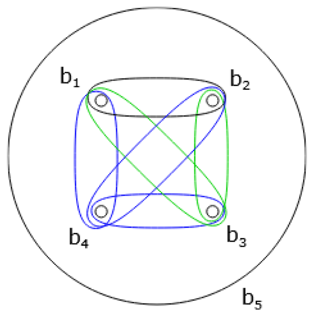


Figure 3.12

We use color to group the curves corresponding to twists in each set of brackets.

We can also consider permutations of the right hand side and determine which permutations also satisfy the relation. In particular, Theorem 3.2 gives us a class of additional relations that come from this one based on permutation of the Dehn twists on the right hand side. Additionally, any two twists along disjoint curves commute with each other [2].

Additionally, applying Theorems 2.11 and 2.12, we see that each product of twists in brackets commute with each other.

Conjecture: *The only valid permutations of the twists on the right hand side of this relation can be derived from the relation given in Theorem 3.2, the commuting of twists along disjoint curves, and the permutation of the products of twists in brackets.*

For $i = 3$, we have

$$T_{b_1}T_{b_2}T_{b_3}T^2_{b_4}T_{b_5} = T_{b_3,b_4}T_{b_2,b_4}T_{b_1,b_4}T_{b_1,b_2,b_3} \tag{2}$$

The collection of curves used on the right hand side is shown below:

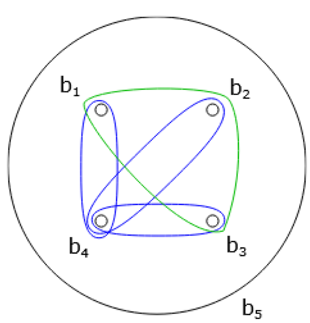


Figure 3.13

Now we can consider other possible relations of this type. In particular, we want to find relations between products of twists of the form

$$T_{b_1}^{a_1}T_{b_2}^{a_2}T_{b_3}^{a_3}T_{b_4}^{a_4}T_{b_5},$$

where each a_i is a nonzero integer, and products of twists over convex, non boundary parallel curves.

For any braid of the form $T_{b_1}^{a_1}T_{b_2}^{a_2}T_{b_3}^{a_3}T_{b_4}^{a_4}T_{b_5}$, we know that the corresponding braid will be the one shown below.

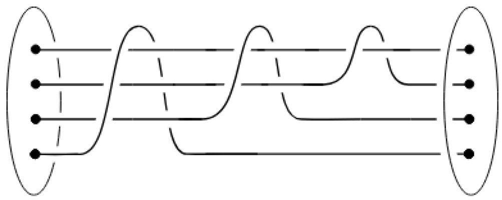


Figure 3.14

In this image, we follow the standard convention of ordering the strands in descending order, with b_1 at the top and b_4 at the bottom.

Now we will first consider all possible combinations of twists over non-boundary parallel curves that will yield this braid.

We will do this by first considering the linking number of each pair of strands. By Theorem 2.10, each pair of strands has linking number -1. Therefore, for any choice of a_i's, if there exists a relation between $T_{b_1}^{a_1}T_{b_2}^{a_2}T_{b_3}^{a_3}T_{b_4}^{a_4}T_{b_5}$ and a product of non-boundary parallel twists, then the latter must correspond to a braid with a linking number of -1 between every pair of boundary components.

In particular, by Theorem 2.10, this means that the product of twists must twist over a set of curves $\{c_1, \ldots, c_k\}$ such that, for any pair of boundary components b_i, b_j, there is exactly one c_k such that both b_i and b_j are contained in its interior. There are only finitely many ways to choose a set of curves that follows this restriction, so we consider all possibilities.

Since we are not considering products of twists that include boundary-parallel twists, each curve c_i must contain at least 2 interior boundary components. Furthermore, any curve containing b_1, b_2, b_3, and b_4 is isotopic to the outer boundary, so, because Dehn twists are well-defined on isotopy classes of curves, each curve must contain at most 3 boundary components.

One possible choice for the set of curves would be all curves that contain exactly two boundary components. Relation (1) shown in this section covers this case, so we will consider other possible cases.

Because all pairs of boundary components must have a linking number of -1, the only other possible case is one where at least one of the curves in the collection contains exactly three boundary components.

Up to isometry, this means that we would twist over a curve like this:

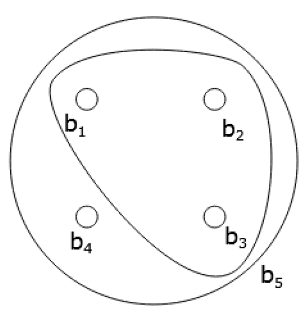


Figure 3.15

We note that b_1, b_2, and b_3 are all contained in the same curve, so we may not include any more curves containing two of them. Because we are looking for a product of twists that does not include boundary parallel curves, the rest of the twists must include twists over curves containing exactly one of the aforementioned boundary components as well as b_4. Furthermore, each of these curves must be included in order to have a nonzero linking number for every pair of boundary components. This gives the collection of curves involved in relation (2) above.

Therefore, up to isometry, (1) and (2) are the only relations of the form specified.

3.3 Considering additional relations in $S_{0,0}^5$

We can now consider relations between a product of twists over each boundary component at least once, as well as some number of disjoint non-boundary parallel curves, and products of twists over only non-boundary parallel curves.

First, we will consider the possibilities for the additional non-boundary parallel curves that will be used in the former. Any non-boundary parallel curve must contain two or three boundary components. There is one option for the curve with three and two options for the curve with two, up to isometry. Furthermore, if we add twists along multiple non-boundary parallel disjoint curves, there are two choices for the set of two curves, up to isometry. We show one case below.

Let c_1 and c_2 be as labeled below:

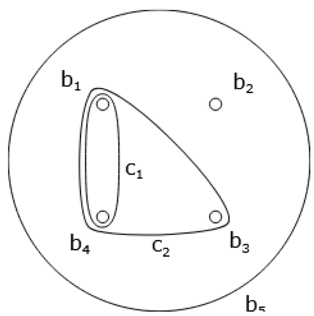


Figure 3.16

We will consider relations with a product of curves of the form $T_{b_1}^{a_1}T_{b_2}^{a_2}T_{b_3}^{a_3}T_{b_4}^{a_4}T_{b_5}T_{c_1}$

We know that we can get a relation of the desired form by adding a twist over c_1 at the beginning of the product of twists on both sides of the previously mentioned relations, and we claim that there are no relations that do not include a twist over c_1 on both sides.

In particular, we claim that there is no collection of curves $\{d_1, \ldots, d_k\}$ such that a product of twists over each d_i will give a relation that does not include T_{c_1} in the product of only non-boundary parallel curves.

By Theorem 2.10, under this product of twists, b_1 and b_4 have a linking number of -2 and all other pairs have a linking number of -1.

In order to choose d_i so that the product of twists over each d_i give the same pairwise winding numbers, we know that exactly two elements of $\{d_1, \ldots, d_k\}$ must contain both b_1 and b_4. Because we cannot include c_1 in this set, we know that both of them must contain at least three interior boundary components. Furthermore, since we are looking for non-boundary parallel curves, they can contain at most three interior boundary components. There are only two such curves, so both must be included.

In particular, the two curves are the convex curve containing b_1, b_2, and b_4, and the convex curve containing b_1, b_3, and b_4. After twisting along both of these curves, b_1 will have a linking number of -2 with b_4 and -1 with every other boundary component. Additionally, it has a multiplicity of 2. For any twist of the form $T_{b_1}^{a_1}T_{b_2}^{a_2}T_{b_3}^{a_3}T_{b_4}^{a_4}T_{b_5}T_{c_1}$, b_1 has a multiplicity of at least 3 because it is contained in b_1, b_5, and c_1. Thus, we must add at least one other curve containing b_1 to $\{d_1, \ldots, d_k\}$ in order to give it a multiplicity of 3 as well. However, since each d_i is not boundary parallel, it must contain another boundary component b_i, and this would change the linking number of b_1 and b_i, which would mean there is no relation.

Therefore, there are no relations of this form.

Now we will consider analogous relations involving $T_{b_1}^{a_1}T_{b_2}^{a_2}T_{b_3}^{a_3}T_{b_4}^{a_4}T_{b_5}T_{c_2}$.

We will again consider a set of convex, non boundary parallel curves $\{d_1, \ldots, d_k\}$ such that twisting over each will be equivalent to $T_{b_1}^{a_1}T_{b_2}^{a_2}T_{b_3}^{a_3}T_{b_4}^{a_4}T_{b_5}T_{c_2}$.

First, we can apply T_{c_2} first on both sides of each of the previous relations.

This gives the relations

$$T_{b_1}^2T_{b_2}^2T_{b_3}^2T_{b_4}^2T_{b_5}T_{c_2} = T_{b_1,b_2}[T_{b_2,b_3}T_{b_1,b_3}][T_{b_3,b_4}T_{b_2,b_4}T_{b_1,b_4}]T_{c_2}$$

$$T_{b_1}T_{b_2}T_{b_3}T_{b_4}^2T_{b_5}T_{c_2} = T_{b_3,b_4}T_{b_2,b_4}T_{b_1,b_4}T_{b_1,b_2,b_3}T_{c_2}$$

Additionally, we can apply the lantern relation just to the region enclosed by c_2.

In particular, the lantern relation tells us that the braids for T_{c_2} and $T_{b_1,b_3}T_{b_1,b_4}T_{b_3,b_4}$ are isotopic. Therefore, on the left hand side, we can replace T_{c_2} with $T_{b_1,b_3}T_{b_1,b_4}T_{b_3,b_4}$ and adjust the multiplicities on the left hand side of each of these relations to get another relation.

$$T_{b_1}^3T_{b_2}^2T_{b_3}^3T_{b_4}^3T_{b_5}T_{c_2} = T_{b_1,b_2}[T_{b_2,b_3}T_{b_1,b_3}][T_{b_3,b_4}T_{b_2,b_4}T_{b_1,b_4}]T_{b_1,b_3}T_{b_1,b_4}T_{b_3,b_4}$$

$$T_{b_1}^2T_{b_2}T_{b_3}^2T_{b_4}^3T_{b_5}T_{c_2} = T_{b_3,b_4}T_{b_2,b_4}T_{b_1,b_4}T_{b_1,b_2,b_3}T_{b_1,b_3}T_{b_1,b_4}T_{b_3,b_4}$$

We note that applying an isometry to one of these relations may yield another relation; however, we choose to focus on relations that cannot be derived from other relations. In particular, the second relation here shows a choice of $\{d_1, \ldots, d_k\}$ such that each d_i contains exactly two boundary components. Therefore, we now shift our focus to sets where at least one contains three boundary components. Up to isometry, we can consider the curve containing b_2, b_3, and b_4 to represent this case. Adding another curve containing three boundary components will either mean that we are using c_2 or we are adding -1 to the linking number of one of the following pairs: b_2 and b_3 or b_2 and b_4. This would give a total linking number of -2, which is not equal to their linking number on the left hand side. Thus, all other curves must contain two boundary components.

In particular, in order to get a linking number of -1 for b_2 and b_3, we cannot add any further curves containing them both. To get a linking number of -2 for b_3 with every other boundary component, we must include a twist over the curve containing just b_3 and b_4, as well as two twists over the curve containing b_1 and b_3. Finally, we need to include curves containing b_1 and each of b_2 and b_4.

This yields the following relation:

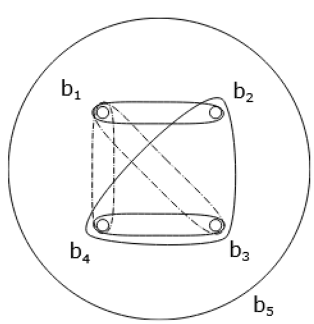


Figure 3.17

$$T_{b_1}^3 T_{b_2} T_{b_3}^2 T_{b_4}^2 T_{b_5} T_{c_2} = T_{b_1,b_2} T_{b_2,b_3,b_4} T_{b_1,b_4} T_{b_1,b_3} T_{b_1,b_3} T_{b_3,b_4} T_{b_1,b_4}$$

In the figure above, the dotted line denotes a curve which is used in two Dehn twists.

Now we will consider similar relations, this time with $T_{b_1}^{a_1} T_{b_2}^{a_2} T_{b_3}^{a_3} T_{b_4}^{a_4} T_{b_5} T_{c_1} T_{c_2}$.

As with the other cases, we can derive new relations by applying twists over one or both of c_1 and c_2 to both sides of the products of twists in the relations shown in section 3.2, as well as the relations shown above. We will focus on the case where we do not include twists over c_1 or c_2 on the other side.

In particular, b_1 and b_4 must have a linking number of -3. Since we are not including twists over c_1 or c_2, or over curves containing all four boundary components, the only curve that contains both b_1 and b_4 must be the one that also contains b_2 and not b_3. However, twisting over this curve once will give b_1 and b_4 a linking number of -1. In order to get to -3, we would have to twist over this curve three times in total. However, this would cause the linking number of b_2 and b_3 to be -3. Since they should have a linking number of -1, we know that no such relation exists.

Now we can consider the case where c_1 does not contain adjacent boundary components.

Excluding relations derived from those already proven, there are no relations between

$T_{b_1}^{a_1} T_{b_2}^{a_2} T_{b_3}^{a_3} T_{b_4}^{a_4} T_{b_5} T_{c_1}$ of $T_{b_1}^{a_1} T_{b_2}^{a_2} T_{b_3}^{a_3} T_{b_4}^{a_4} T_{b_5} T_{c_1} T_{c_2}$ and any product of non boundary parallel twists. This follows from the arguments of the same claims for the case where c_1 contains b_1 and b_4. This is because the arguments are based on the linking numbers and does not rely on the relative position of the boundary components.

Because c_2 is the same, we have found all possible relations of the specified type.

3.4 Relations on $S_{0,0}^6$

Now we will classify all relations on the mapping class group of $S_{0,0}^6$ between products of boundary parallel twists of the form $T_{b_1}^{a_1} T_{b_2}^{a_2} T_{b_3}^{a_3} T_{b_4}^{a_4} T_{b_5}^{a_5} T_{b_6}$ and products of positive twists along non boundary parallel curves.

We begin by applying the general relation to $S_{0,0}^6$. We will state the relations themselves and present images of the set of curves involved on the right hand side of each relation.

For $i = 2$, we have

$$T_{b_1}^3 T_{b_2}^3 T_{b_3}^3 T_{b_4}^3 T_{b_5}^3 T_{b_6} = T_{(b_5,b_4)} T_{(b_5,b_3)} T_{(b_5,b_2)} T_{(b_5,b_1)} T_{(b_4,b_3)} T_{(b_4,b_2)} T_{(b_4,b_1)} T_{(b_3,b_2)} T_{(b_3,b_1)} T_{(b_2,b_1)} \quad (3)$$

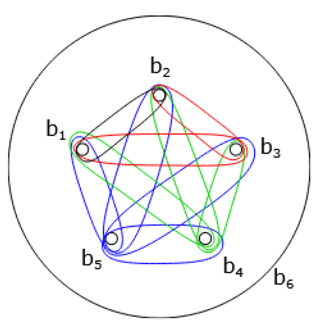


Figure 3.18

Then for $i = 3$, we have

$$T_{b_1}^2 T_{b_2}^2 T_{b_3}^2 T_{b_4}^3 T_{b_5}^3 T_{b_6} = T_{(b_1,b_2,b_3)} T_{(b_3,b_2)} T_{(b_3,b_1)} T_{(b_4,b_3)} T_{(b_4,b_2)} T_{(b_4,b_1)} \quad (4)$$

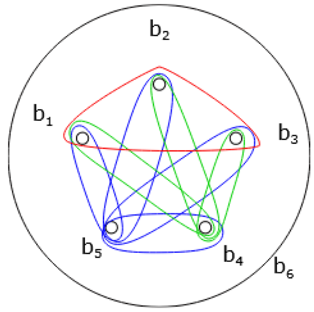


Figure 3.19

Finally, taking $i = 4$ gives us the relation

$$T_{b_1}T_{b_2}T_{b_3}T_{b_4}T_{b_5}^3T_{b_6} = T_{(b_1,b_2,b_3,b_4)}T_{(b_4,b_5)}T_{(b_3,b_5)}T_{(b_2,b_5)}T_{(b_1,b_5)} \tag{5}$$

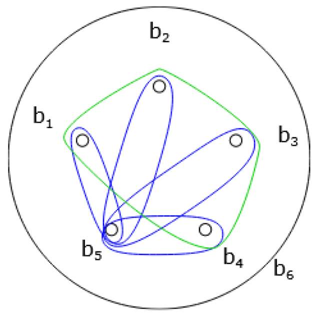


Figure 3.20

Now we will consider additional relations. Using the same argument as in the case of $S_{0,0}^5$, we want to find a collection of non boundary parallel, convex curves $\{c_1, \ldots, c_k\}$ such that, for any two interior boundary components b_i, b_j, there is exactly one c_m that contains them both.

In the case that there is at least one curve containing four boundary components, we know that all other curves must contain two boundary components. Up to isotopy, this is exactly the collection of curves used in relation (5). Furthermore, relation (3) uses a collection of curves that each contain exactly two boundary components each. Therefore, we consider the cases where there is at least one curve with three boundary components, where all others contain two or three boundary components each.

First, we will consider products of twists involving at least one curve that contains three non-adjacent holes. Up to isometry, such a curve will look like this:

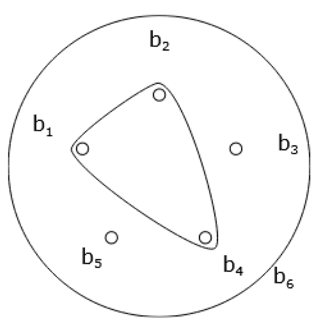


Figure 3.21

We will now consider the case where there is a second curve containing three boundary components. If this curve contains three non adjacent boundary components, then, up to isometry, together the two curves must be arranged as in the figure below.

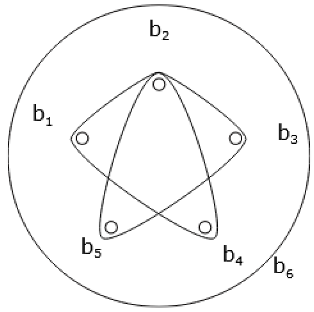


Figure 3.22

We see that all boundary components are contained in at least one of the two curves. Therefore, by the Pigeonhole principle, given any three boundary components, two of them must be contained in one of the two curves. Therefore, we cannot add any more curves that contain three boundary components.

In order to ensure that any two boundary components have exactly one curve in common, we must add the blue curves to this collection:

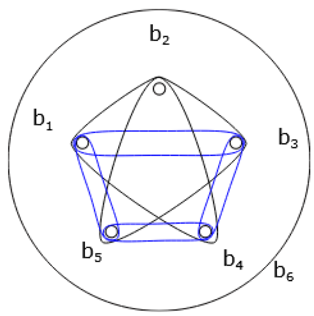


Figure 3.23

We claim that the following relation involving these curves holds:

$$T_{b_1}^2 T_{b_2} T_{b_3}^2 T_{b_4}^2 T_{b_5}^2 T_{b_6} = T_{(b_3,b_4)} T_{(b_1,b_2,b_4)} T_{(b_4,b_5)} T_{(b_2,b_3,b_5)} T_{(b_1,b_5)} T_{(b_1,b_3)} \tag{6}$$

Proof. First, we note that the multiplicity of each boundary component is equal on both sides.

Then we need only to show that the braids are isotopic.

By the argument in section 3.1, the braid for the left hand side is as follows:

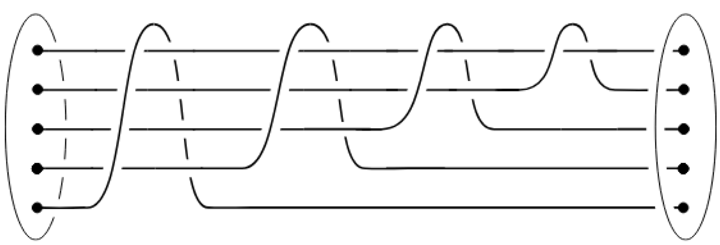


Figure 3.24

The word representing this braid is

$$\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}$$

Now we will find the braid for the right hand side.

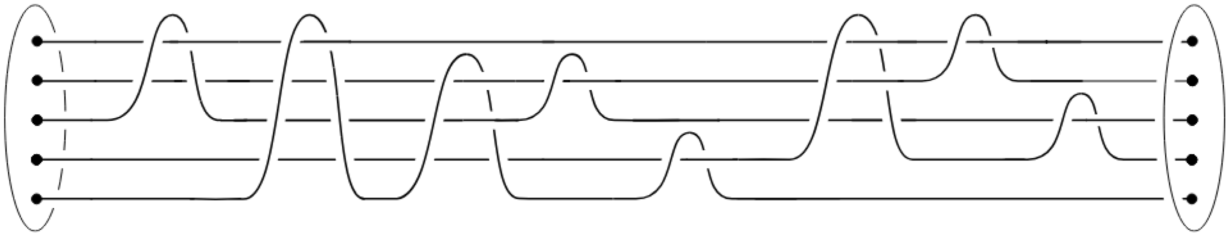


Figure 3.25

The word denoting this braid is

$$\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_3\sigma_4\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_2^{-1}\sigma_2^{-1}\sigma_4^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_3^{-1}$$

We see that we can perform cancellation using inverses on the portion $\sigma_2\sigma_3\sigma_4\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}$, which yields the braid

$$\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_2^{-1}\sigma_2^{-1}\sigma_4^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_3^{-1}$$

Visually, this cancellation will give the braid below:

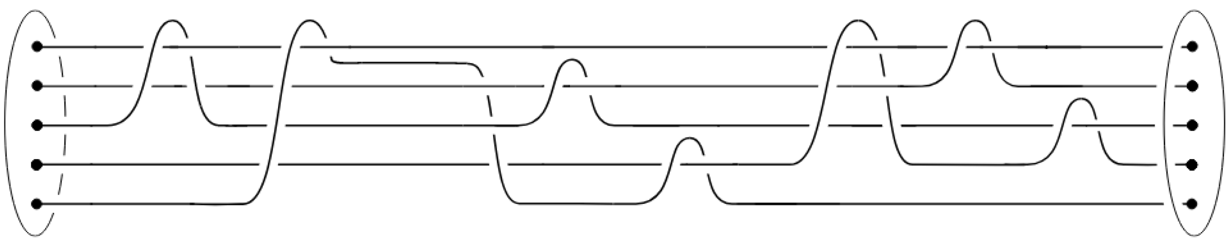


Figure 3.26

Using the relation $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $i - j > 1$, we get

$$\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_2^{-1}\sigma_2^{-1}\sigma_4^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_3^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_1^{-1}$$

which denotes the braid

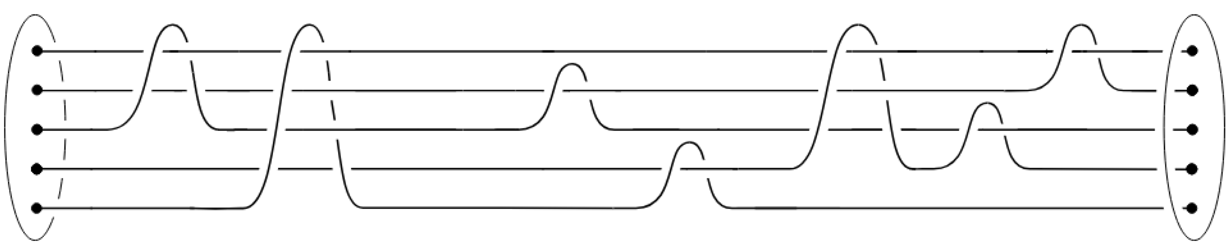


Figure 3.27

Again, cancellation by inverses gives

$$\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_2^{-1}\sigma_2^{-1}\sigma_4^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_1^{-1}$$

Using $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $i - j > 1$, we can commute σ_4 and σ_2^{-1}

$$\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_4\sigma_2^{-1}\sigma_4^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_1^{-1}$$

Then we use the fact that $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}$

$$\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_2^{-1}\sigma_4^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_1^{-1}$$

Again, we use the fact that $1-3 \ > 1$ to commute σ_3^{-1} with σ_1^{-1} twice:

$$\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_2^{-1}\sigma_4^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_1^{-1}$$

Then we apply $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $i = 2$:

$$\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_4^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_2^{-1}\sigma_4^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_1^{-1}$$

Finally, we can commute σ_2^{-1} with σ_4^{-1} and with σ_4

$$\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_2^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_4\sigma_4^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_1^{-1}$$

The visual effect of this calculation on the braid is shown below:

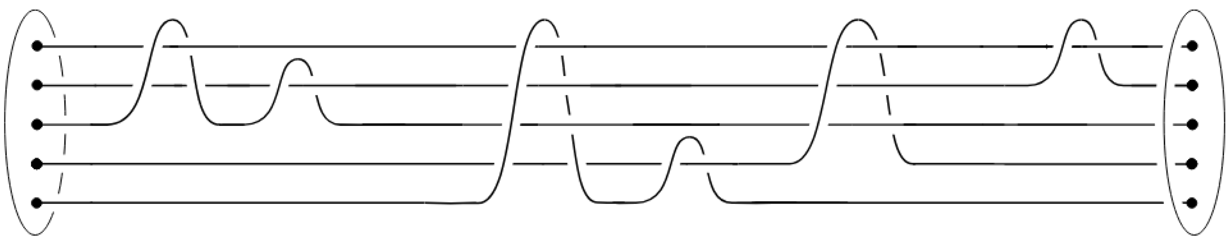


Figure 3.28

We can now perform cancellation again:

$$\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_1^{-1}$$

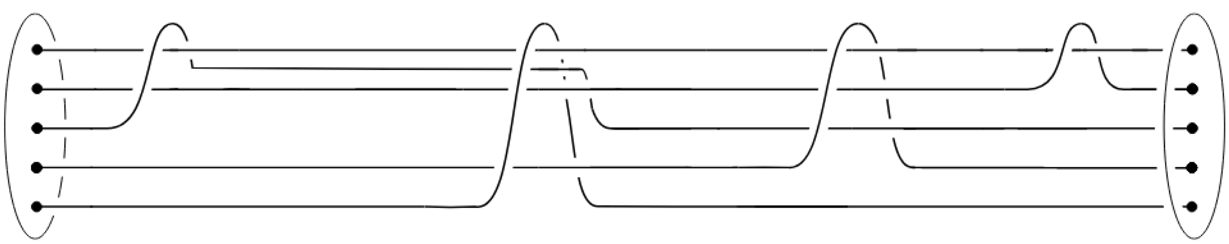


Figure 3.29

Repeatedly applying $\sigma_{i+1}\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $i - j \ > 1$, we can first get

$$\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_1^{-1}$$

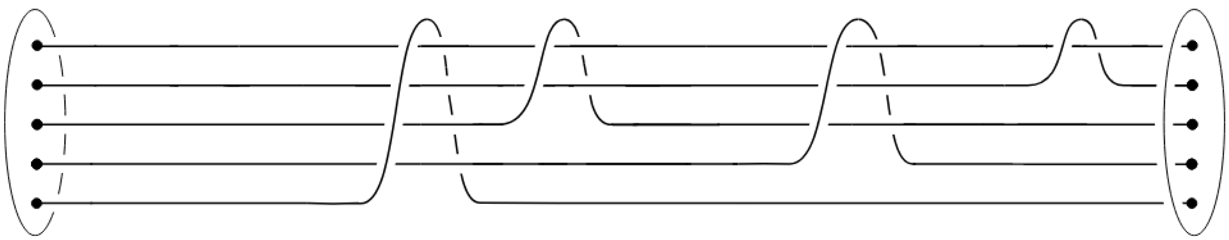


Figure 3.30

Applying those relations again, we can get the crossings in the desired order:

$$\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}$$

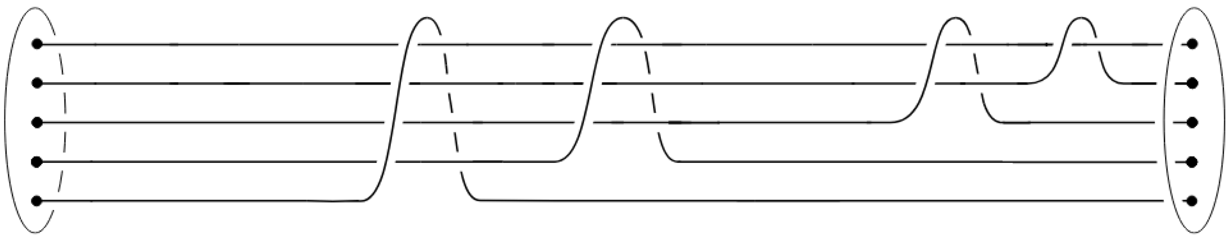


Figure 3.31

This is exactly the braid representing the left hand side, so we are done. □

The other possible case involving a second curve with three boundary components is one where such a curve contains adjacent boundary components. In particular, it must contain b_3 and b_5, since they are the only two boundary components that are outside of the first curve, so the third must be b_4.

Visually, two such curves are arranged as shown:

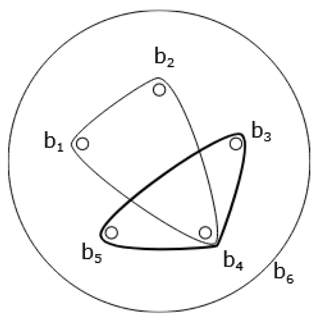


Figure 3.32

Then, in order to satisfy the condition that, for any two boundary components there is exactly one curve that contains both, we must add the following curves:

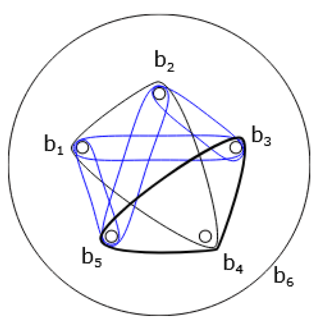


Figure 3.33

We have the following relation involving twists over this set of curves:

$$T_{b_1}^2 T_{b_2}^2 T_{b_3}^2 T_{b_4} T_{b_5}^2 T_{b_6} = T_{(b_2,b_3)} T_{(b_1,b_3)} T_{(b_3,b_4,b_5)} T_{(b_2,b_5)} T_{(b_1,b_5)} T_{(b_1,b_2,b_4)} \tag{7}$$

Proof. We see that each boundary component has the same multiplicity on both sides of this proposed relation. Thus, we need only to show that the corresponding braids are isotopic.

As with all products of twists over boundary parallel curves including exactly one twist over the outer boundary, the braid corresponding to the left hand side is denoted by the word

$$\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}$$

The braid for the right hand side is

$$\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_1^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}(\sigma_3\sigma_4\sigma_4^{-1}\sigma_3^{-1})\sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}(\sigma_2\sigma_2^{-1})\sigma_2^{-1}$$

Cancelling out the parts in parentheses gives us

$$\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_1^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}$$

Using relation (3), we can commute σ_1^{-1} with other elements to get

$$\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_4^{-1}\sigma_3^{-1}(\sigma_1^{-1}\sigma_1^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})\sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}$$

Using Theorem 2.12, we can again commute products in this word:

$$\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_4^{-1}\sigma_3^{-1}(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})\sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}$$

Then, commuting $(\sigma_1^{-1}\sigma_1^{-1})$ with more elements using these two relations again, we get

$$\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_3^{-1}(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

Using relation (7), we can perform a replacement:

$$\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}(\sigma_4^{-1}\sigma_3^{-1}\sigma_4)\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_3^{-1}(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

By commutativity, we can move σ_4:

$$\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_4\sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_3^{-1}(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

Applying braid relation 7, we get

$$\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}(\sigma_3^{-1}\sigma_4^{-1}\sigma_3)\sigma_3^{-1}\sigma_3^{-1}(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

Continuing to use the standard braid relations gives us the following lines:

$$\sigma_3^{-1}(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-1}(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

$$\sigma_3^{-1}\sigma_4^{-1}(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})\sigma_4^{-1}\sigma_3^{-1}(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

$$\sigma_3^{-1}\sigma_4^{-1}(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})\sigma_4^{-1}\sigma_3^{-1}(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

$$\sigma_4^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})\sigma_4^{-1}\sigma_3^{-1}(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

$$\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_4^{-1}\sigma_3^{-1}(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})\sigma_4^{-1}\sigma_3^{-1}(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

$$\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_4^{-1}(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})\sigma_3^{-1}(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

And finally, applying $\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1} = \sigma_i^{-1}\sigma_{i+1}^{-1}\sigma_i^{-1}$, we get the following word:

$$(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1})(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

We see that this is equal to the word for the braid corresponding to the left hand side. □

Now we will consider the case where we include two curves containing three adjacent boundary components. We note that this is the final possible case involving two curves with three boundary components.

Up to isometry, we can say that one contains b_1, b_2, and b_3 and the other contains b_3, b_4, and b_5. As described above, the Pigeonhole Principle dictates that all other curves must only contain two boundary components each. Therefore, the collection of curves corresponding to this case is as shown below.

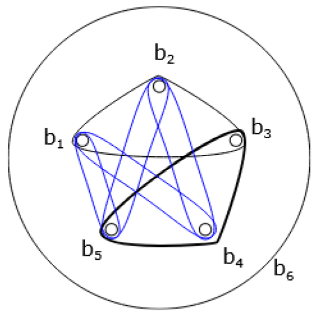


Figure 3.34

$$T_{b_1}^2 T_{b_2}^2 T_{b_3} T_{b_4}^2 T_{b_5}^2 T_{b_6} = T_{(b_1,b_2,b_3)} T_{b_3,b_4,b_5} T_{b_2,b_5} T_{b_1,b_5} T_{b_2,b_4} T_{b_1,b_4}$$

Proof. We see that each boundary component has the same multiplicity on both sides of this relation. Thus, we need only to show that the corresponding braids are isotopic.

As with all products of twists over boundary parallel curves including exactly one twist over the outer boundary, the braid corresponding to the left hand side is denoted by the word

$$\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}$$

Additionally, the word denoting the braid for the right hand side is

$$\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}(\sigma_3\sigma_4\sigma_4^{-1}\sigma_3^{-1})\sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}$$

Cancelling the inverses in parentheses gives

$$\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}$$

In the proof of the previous relation, we saw that this word is equivalent to the word that denotes the braid given by a twist along the outer boundary.

Therefore, we are done. □

Now we shift our focus to collections of curves where exactly one contains three boundary components. Up to isometry, there are two distinct possibilities for such a curve: one that contains adjacent boundary components and another that contains non-adjacent boundary components. The former case is addressed by the application of the general relation, so we only consider the latter.

In this case, the collection of curves must be as follows:

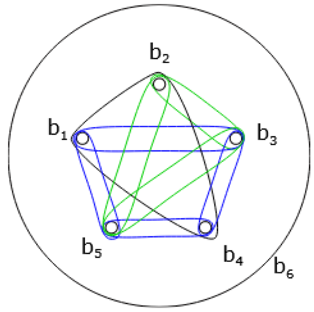


Figure 3.35

For this collection of curves, we have the following relation:

$$T_{b_1}^2 T_{b_2}^2 T_{b_3}^3 T_{b_4}^2 T_{b_5}^3 T_{b_6} = T_{b_2,b_3} T_{b_1,b_3} T_{b_3,b_4} T_{b_4,b_5} T_{b_3,b_5} T_{b_2,b_5} T_{b_1,b_5} T_{b_1,b_2,b_4}$$

Proof. We see that the multiplicity of each boundary component is equal on both sides, so we only need to prove that the braids are isotopic.

Recall that we have the relation

$$T_{b_1}^2 T_{b_2}^2 T_{b_3}^2 T_{b_4} T_{b_5}^2 T_{b_6} = T_{(b_2,b_3)} T_{(b_1,b_3)} T_{(b_3,b_4,b_5)} T_{(b_2,b_5)} T_{(b_1,b_5)} T_{(b_1,b_2,b_4)}$$

By the lantern relation, if we restrict our consideration of the braid to just the strands 3, 4, and 5, we know that the braid representing T_{b_3,b_4,b_5} is isotopic to the braid representing $T_{b_3,b_4} T_{b_4,b_5} T_{b_3,b_5}$

Thus, because the braid representing $T_{(b_2,b_3)} T_{(b_1,b_3)} T_{(b_3,b_4,b_5)} T_{(b_2,b_5)} T_{(b_1,b_5)} T_{(b_1,b_2,b_4)}$ is isotopic to the braid resulting from a Dehn twist along the outer boundary, the braid corresponding to $T_{(b_2,b_3)} T_{(b_1,b_3)} T_{b_3,b_4} T_{b_4,b_5} T_{b_3,b_5} T_{(b_2,b_5)} T_{(b_1,b_5)} T_{(b_1,b_2,b_4)}$ is isotopic to the same braid. □

This covers all cases, so we have found all relations between products of boundary parallel twists of the form $T_{b_1}^{a_1} T_{b_2}^{a_2} T_{b_3}^{a_3} T_{b_4}^{a_4} T_{b_5}^{a_5} T_{b_6}$ and products of twists over non-boundary parallel curves, up to isometry and permutations of the products.

3.5 Relations on $S_{0,0}^7$

From the relations that were proven for all $n > 0$, we can say the following relations hold on the sphere with 7 boundary components:

$$T_{b_1}^4 T_{b_2}^4 T_{b_3}^4 T_{b_4}^4 T_{b_5}^4 T_{b_5}^4 T_{b_7} = T_{b_1,b_2} T_{b_2,b_3} T_{b_1,b_3} T_{b_3,b_4} T_{b_2,b_4} T_{b_1,b_4} T_{b_4,b_5} T_{b_3,b_5} T_{b_2,b_5} T_{b_1,b_5} T_{b_5,b_6} T_{b_4,b_6} T_{b_3,b_6} T_{b_2,b_6} T_{b_1,b_6}$$

$$T_{b_1} T_{b_2} T_{b_3} T_{b_4} T_{b_5} T_{b_6}^4 T_{b_7} = T_{b_1,b_2,b_3,b_4,b_5} T_{b_5,b_6} T_{b_4,b_6} T_{b_3,b_6} T_{b_2,b_6} T_{b_1,b_6}$$

$$T_{b_1}^3 T_{b_2}^3 T_{b_3}^3 T_{b_4}^4 T_{b_5}^4 T_{b_6}^4 T_{b_7} = T_{b_1,b_2,b_3} T_{b_3,b_4} T_{b_2,b_4} T_{b_1,b_4} T_{b_4,b_5} T_{b_3,b_5} T_{b_2,b_5} T_{b_1,b_5} T_{b_5,b_6} T_{b_4,b_6} T_{b_3,b_6} T_{b_2,b_6} T_{b_1,b_6}$$

$$T_{b_1}^2 T_{b_2}^2 T_{b_3}^2 T_{b_4}^2 T_{b_5}^4 T_{b_6}^4 T_{b_7} = T_{b_1,b_2,b_3,b_4} T_{b_4,b_5} T_{b_3,b_5} T_{b_2,b_5} T_{b_1,b_5} T_{b_5,b_6} T_{b_4,b_6} T_{b_3,b_6} T_{b_2,b_6} T_{b_1,b_6}$$

We will now consider all other collections of curves that, when twisting over each, may give a relation with a product of twists of the form $T_{b_1}^{a_1} T_{b_2}^{a_2} T_{b_3}^{a_3} T_{b_4}^{a_4} T_{b_5}^{a_5} T_{b_6}^{a_6} T_{b_7}$. We know that, for a product of twists of this form, the pairwise linking number between any two interior boundary components is -1. Any product of non-boundary parallel twists that is equivalent to such a product must have the same pairwise linking numbers, so, in particular, we must consider only collections of curves such that, for any two holes, there is exactly one curve that contains both of them.

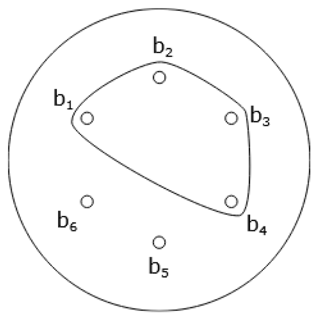
We note that, in each case, we are looking for a product of twists that corresponds to a braid that is isotopic to the braid representing the twist T_{b_6}. The word for this braid is

$$(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1})(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1})$$
$$(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

We will consider each case by the maximum number of holes contained by any curve. In the case that one curve contains five boundary components, the only additional curves that may be twisted over will be the ones that contain the one hole not in the curve and one other boundary component. This case is covered, up to isometry, by a relation described above.

Now we will consider collections of convex curves where at least one contains four holes. First, we will discuss the case where the holes contained in this curve are adjacent.

Up to isometry, we are considering the curve shown below:



Because there are six interior holes, all other curves can contain at most three holes. If we twist over this curve and a curve containing three holes, they may have at most one hole in common. Therefore, all holes will be contained in one or the other. Thus, by the Pigeonhole Principle, in order to achieve a linking number of -1 for all pairs of holes, we must only add twists containing at most two holes.

We will now consider all possible cases involving a curve that contains exactly three boundary components in its interior. Such a curve may contain adjacent or non adjacent holes.

We begin by considering the case where it contains three adjacent holes. Up to isometry, we can say that it contains b_4, b_5, and b_6. Because any pair of boundary components must have exactly one curve in common, we know that the whole collection of curves must be as follows:

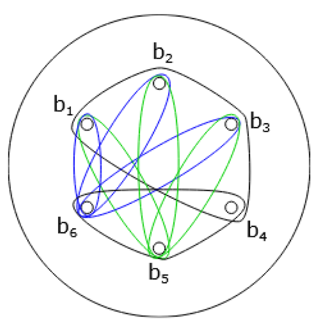


Figure 3.36

We claim that the following relation involving such a collection of curves holds:

$$T_{b_1}^2 T_{b_2}^2 T_{b_3}^2 T_{b_4} T_{b_5}^3 T_{b_6}^3 T_{b_7} = T_{(b_1,b_2,b_3,b_4)} T_{(b_4,b_5,b_6)} T_{(b_3,b_6)} T_{(b_2,b_6)} T_{(b_1,b_6)} T_{(b_3,b_5)} T_{(b_2,b_5)} T_{(b_1,b_5)}$$

Proof. We see that the multiplicity of each boundary component is equal on both sides, so we need only compare the braids.

We will find the braid for the right hand side.

By Theorem 2.11, we can simplify the braid to be

$$\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}(\sigma_4\sigma_5\sigma_5^{-1}\sigma_4^{-1})$$
$$\sigma_4^{-1}\sigma_5^{-1}\sigma_4^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}$$

Note that this is similar to the braid shown in relation (7) on the sphere with 6 boundary components. Thus, applying the same relations in the same order, we can find this to be equal to

$$\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}$$
$$\sigma_4\sigma_4^{-1}\sigma_5^{-1}\sigma_4^{-1}\sigma_4^{-1}(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

$$\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}$$
$$\sigma_4^{-1}\sigma_5^{-1}\sigma_4\sigma_4^{-1}\sigma_4^{-1}(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

$$\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}$$
$$\sigma_4^{-1}\sigma_5^{-1}\sigma_4^{-1}(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

Again, using the relations used in the proof of relation (7) in an analogous way, we find that this is equal to

$$(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1})(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1})$$
$$(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

Thus, we are done. □

Now we consider the case where the curve containing three holes contains holes that are not adjacent. Up to isometry, this is the only possible choice for such a curve:

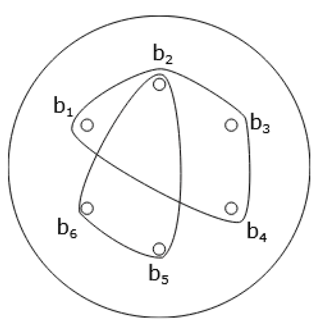


Figure 3.37

For ease of calculation, we will perform a rotation before presenting the relation:

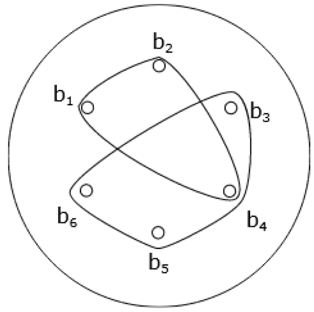


Figure 3.38

This is the full set of curves that will be included in the product of twists:

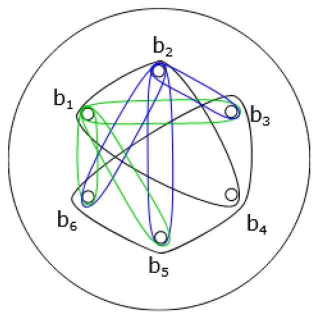


Figure 3.39

Using a product of twists over these curves, we have the following relation:

$$T_{b_1}^3 T_{b_2}^3 T_{b_3}^2 T_{b_4} T_{b_5}^2 T_{b_6}^2 T_{b_7} = T_{(b_2,b_3)} T_{(b_1,b_3)} T_{(b_3,b_4,b_5,b_6)} T_{(b_2,b_6)} T_{(b_1,b_6)} T_{(b_2,b_5)} T_{(b_1,b_5)} T_{(b_1,b_2,b_4)}$$

Proof. We see that each boundary component has the same multiplicity, so we show that the associated braids are isotopic.

We see that the braid for the right hand side is

$$\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_1^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_4\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1}$$
$$\sigma_4^{-1}\sigma_3^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}$$

Repeatedly applying braid relations 7 and 8, we find that this braid is isotopic to

$$\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_1^{-1}\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}$$
$$\sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}$$

Following a similar process again, we get

$$\sigma_1^{-1}\sigma_1^{-1}\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}$$
$$\sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}$$

Finally, applying Theorem 2.12, we can rearrange this word into the braid desired:

$$(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1})(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1})$$
$$(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

□

This covers all possible combinations of the chosen curve with four boundary components and a curve with three boundary components, so the only other case involving this curve is the one given by additional twists over curves containing exactly two holes, which is covered by the general case.

Now we may consider relations involving one curve that contains four non adjacent holes. First, we consider the case where such a curve has three holes that are adjacent and one that is not, as in the figure below:

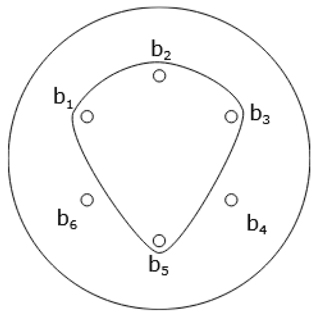


Figure 3.40

We will now consider the possible curves containing three holes that will not give a linking number of less than -1 between any two holes. As described above, we know that there may not be more than one curve containing three holes, so we need only consider the possibilities for one choice of such a curve.

Furthermore, we know that any additional curve over which we twist must contain b_4 and b_6, because otherwise it would contain two boundary components within the curve pictured above, which would lead to a linking number of -2 for that pair.

Then the third hole can be b_1, b_2, b_3, or b_5. An isometry will relate the cases involving b_1 and b_3, so we will only consider the case where b_3 is the third hole. The other two cases are distinct and will also be considered.

In all cases, we know that all other curves over which twists occur must contain exactly two holes. Thus, for the first case, we have the following collection of curves:

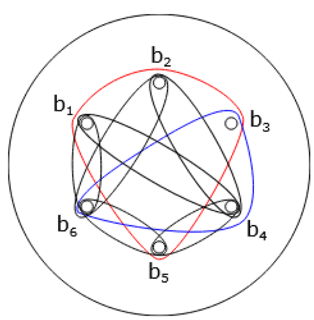


Figure 3.41

We have the following relation involving this set of curves:

$$T_{b_1}^2 T_{b_2}^2 T_{b_3} T_{b_4}^3 T_{b_5}^2 T_{b_6}^3 T_{b_7} = T_{(b_3,b_4)} T_{(b_2,b_4)} T_{(b_1,b_4)} T_{(b_4,b_5)} T_{(b_1,b_2,b_3,b_5)} T_{(b_5,b_6)} T_{(b_3,b_4,b_6)} T_{(b_2,b_6)} T_{(b_1,b_6)} T_{(b_2,b_4)} T_{b_1,b_4}$$

For the case where the curve with three holes contains b_5, we have the following complete set of curves:

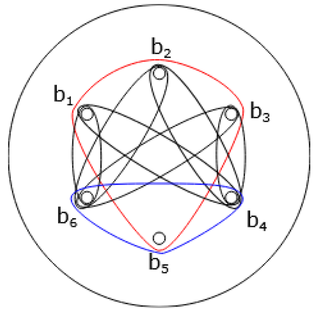


Figure 3.42

We have the following relation involving such curves:

$$T_{b_1}^2 T_{b_2}^2 T_{b_3}^2 T_{b_4}^3 T_{b_5} T_{b_6}^3 T_{b_7} = T_{(b_4,b_3)} T_{(b_4,b_2)} T_{(b_4,b_1)} T_{(b_4,b_5,b_6)} T_{(b_6,b_3)} T_{(b_6,b_2)} T_{(b_6,b_1)} T_{(b_1,b_2,b_3,b_5)}$$

Proof. We see that the braid for the right hand side is

$$\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_5\sigma_5^{-1}\sigma_4^{-1}$$
$$\sigma_4^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}$$

Using cancellation by inverses, we get

$$\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1}$$
$$\sigma_4^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}$$

Using the commutativity proven in Theorem 2.12 and the identity $\sigma_i^{-1}\sigma_j^{-1} = \sigma_j^{-1}\sigma_i^{-1}$ if $|i-j| > 1$, we get

$$\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1}$$
$$\sigma_4^{-1}\sigma_4^{-1}(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

Now, applying braid relations 7 and 8, and rearranging elements that commute, we get

$$(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1})(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1})$$
$$(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

□

Finally, we consider the case where the curve with three holes contains b_2 in its interior. Then we will have the following collection of curves:

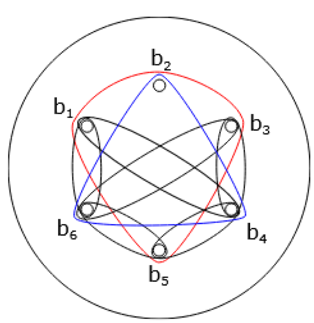


Figure 3.43

We suspect that there exist no relations between products of boundary parallel twists and a product of non boundary parallel curves that includes the blue curve pictured above. However, we can consider a case where each curve contains the same boundary components, but one or more curves involved are not convex. In particular, let c be the curve depicted below.

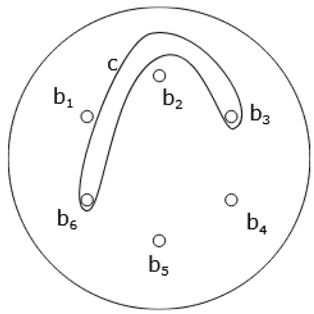


Figure 3.44

Then we will show that the following relation holds:

$$T_{b_1}^2 T_{b_2} T_{b_3}^2 T_{b_4}^3 T_{b_5}^2 T_{b_6}^3 T_{b_7} = T_{b_4,b_5} T_{b_1,b_2,b_3,b_5} T_{b_5,b_6} T_{b_3,b_4} T_{b_2,b_4,b_6} T_c T_{b_1,b_6} T_{b_1,b_4}$$

Proof. We see that the multiplicities are the same.

Furthermore, as mentioned in the proof of Theorem 2.10, every non-convex curve on a convexly punctured surface is the image of a convex curve under some homeomorphism of the surface.

In this case, we want to find the homeomorphism that takes c to the convex curve containing b_3 and b_6.

We claim that $c = T_{b_2,b_6}((b_3, b_6))$, where (b_3, b_6) denotes the convex curve containing b_3 and b_6.

We can show this by taking the Dehn twist. We want to show that c is the image of the blue curve under a Dehn twist along the green curve.

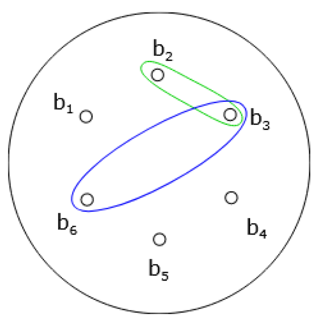


Figure 3.45

Indeed, we see that a Dehn twist yields

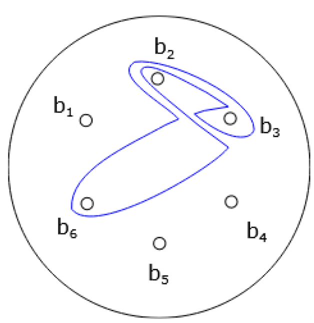


Figure 3.46

This is isotopic to c. Because Dehn twists are well-defined on isotopy classes of curves, the claim is true.

As mentioned earlier, Fact 3.7 from [2] tells us that $T_{f(a)} = f \circ T_a \circ f^{-1}$ for a homeomorphism on a surface.

Thus, we can rewrite $T_c = T_{T_{b_2,b_6}(b_1,b_6)} = T_{b_2,b_6} \circ T_{b_1,b_6} \circ T^{-1}_{b_2,b_6}$.

Now we can find the word for the braid associated with the product of twists:

$$(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_3)(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_3\sigma_4\sigma_5)(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_3\sigma_4\sigma_5)(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_3^{-1}\sigma_4\sigma_5)$$
$$(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2\sigma_2\sigma_3\sigma_4\sigma_5)(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_3\sigma_4^{-1}\sigma_5)(\sigma_3^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_3)(\sigma_3^{-1}\sigma_3^{-1})(\sigma_5^{-1}\sigma_5^{-1})$$
$$(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4)(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})(\sigma_4^{-1}\sigma_4^{-1})$$

Cancelling by inverses, we get

$$(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_3)(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5)(\sigma_3^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_3^{-1})(\sigma_5^{-1}\sigma_5^{-1})$$

$$(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4)(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})(\sigma_4^{-1}\sigma_4^{-1})$$

Now, using braid relation (5), we can commute elements to perform more cancellations:

$$(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_3)(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5)(\sigma_5^{-1}\sigma_5^{-1})(\sigma_3^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_3^{-1})$$
$$(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4)(\sigma_4^{-1}\sigma_4^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

Then the cancellation gives:

$$(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_3)(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1})(\sigma_3^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_3^{-1})$$
$$(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

Finally, using braid relations 7 and 8, as well as 5 to commute terms, we get

$$(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1})(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})$$

$$(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

Finally, using Theorem 2.12 and braid relation 5, we can commute the terms in parentheses to get

$$(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1})(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1})$$
$$(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

as desired. □

We have now considered every possible choice of a curve with three boundary components to be included in the collection along with the curve containing b_1, b_2, b_3, and b_5.

The only remaining case for this particular curve with four boundary components is the case where all other curves contain two boundary components. In such a case, the following relation holds:

$$T^2_{b_1}T^2_{b_2}T^2_{b_3}T^4_{b_4}T^2_{b_5}T^4_{b_6}T_{b_7} = T_{(b_3,b_4)}T_{(b_2,b_4)}T_{(b_1,b_4)}T_{(b_5,b_6)}T_{(b_4,b_6)}T_{(b_3,b_6)}T_{(b_2,b_6)}T_{(b_1,b_6)}T_{(b_4,b_5)}T_{(b_1,b_2,b_3,b_5)}$$

Now we will finish our consideration of collections of curves where one contains four boundary components. Up to isometry, the only other possible curve is the one that contains exactly b_1, b_2, b_4, and b_5.

We suspect that, like the curve above, no product of twists along convex curves including a twist over this curve will be equivalent to a product of boundary parallel twists. However, we do have a relation involving twists over non convex curves.

In particular, we claim that the twist over these curves in the following order, together with a twist over each of the convex curves containing b_6 and one other interior boundary component, is equivalent to a twist over boundary parallel curves.

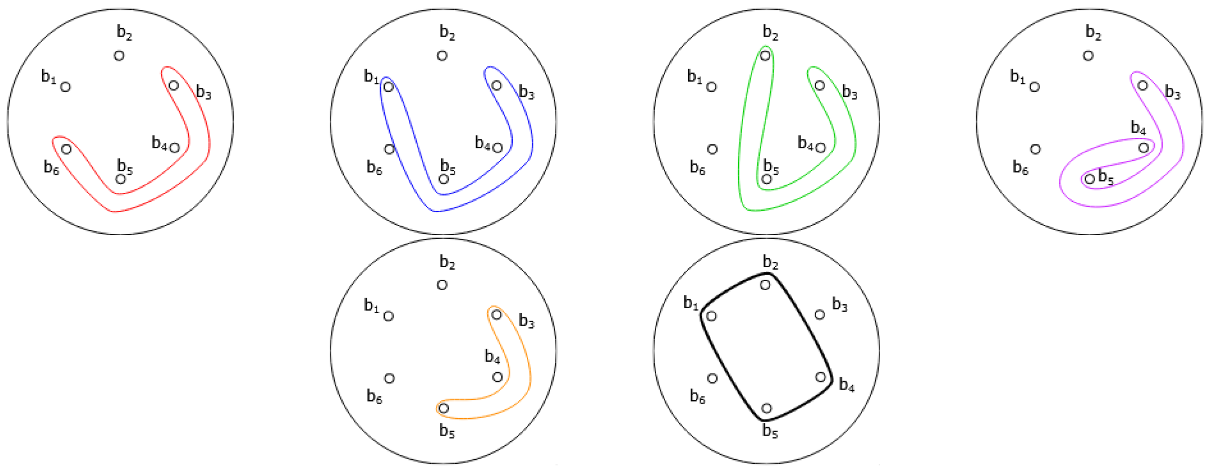


Figure 3.47

More precisely, we label these curves a through f, from left to right, and we claim that the following relation holds:

$$T_{b_1}^2 T_{b_2}^2 T_{b_3}^5 T_{b_4}^2 T_{b_5}^2 T_{b_6}^4 T_{b_7} = T_f T_e T_d T_c T_b T_a T_{b_5,b_6} T_{b_4,b_6} T_{b_3,b_6} T_{b_2,b_6} T_{b_1,b_6}$$

.

We will now find the word denoting the braid on the left hand side. In order to do this, we must first write each non-convex curve as the image of a convex curve under a homeomorphism. As we did in the case above, it can be verified that the following equalities hold:

$$\begin{aligned} a &= T_{b_4,b_5,b_6}((b_3, b_6)) \\ b &= T_{b_1,b_4,b_5}((b_1, b_3)) \\ c &= T_{b_2,b_4,b_5}((b_2, b_3)) \\ d &= T_{b_4,b_5}((b_3, b_4)) \\ e &= T_{b_4,b_5}((b_3, b_5)) \end{aligned}$$

Then, using the fact that $T_{f(a)} = f \circ T_a \circ f^{-1}$, we can rewrite the product of twists on the right hand side as

$$T_{b_1,b_2,b_4,b_5} T_{b_4,b_5} T_{b_3,b_5} T_{b_4,b_5}^{-1} T_{b_4,b_5} T_{b_3,b_4} T_{b_4,b_5}^{-1} T_{b_2,b_4,b_5} T_{b_2,b_3} T_{b_2,b_4,b_5}^{-1} T_{b_1,b_4,b_5} T_{b_2,b_3} T_{b_1,b_4,b_5}^{-1} T_{b_4,b_5,b_6} T_{b_3,b_6} T_{b_4,b_5,b_6} T_{b_3,b_6}^{-1}$$

$$T_{b_5,b_6} T_{b_4,b_6} T_{b_3,b_6} T_{b_2,b_6} T_{b_1,b_6}$$

$$= T_{b_1,b_2,b_4,b_5} T_{b_4,b_5} T_{b_3,b_5} T_{b_3,b_4} T_{b_4,b_5}^{-1} T_{b_2,b_4,b_5} T_{b_2,b_3} T_{b_2,b_4,b_5}^{-1} T_{b_1,b_4,b_5} T_{b_2,b_3} T_{b_1,b_4,b_5}^{-1} T_{b_4,b_5,b_6} T_{b_3,b_6} T_{b_4,b_5,b_6} T_{b_3,b_6}^{-1}$$

$$T_{b_5,b_6}T_{b_4,b_6}T_{b_3,b_6}T_{b_2,b_6}T_{b_1,b_6}$$

The word corresponding to this braid is

$$(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1})(\sigma_4\sigma_4)(\sigma_5\sigma_4\sigma_4\sigma_5)(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_3^{-1}\sigma_4\sigma_5)(\sigma_5^{-1}\sigma_4^{-1}\sigma_4^{-1}\sigma_5^{-1})(\sigma_4^{-1}\sigma_4^{-1})$$
$$(\sigma_3^{-1}\sigma_2^{-1}\sigma_1\sigma_1\sigma_2\sigma_3)(\sigma_4\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2)(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_3\sigma_4^{-1})$$
$$(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_3)(\sigma_3^{-1}\sigma_2\sigma_2\sigma_3)(\sigma_2^{-1}\sigma_2^{-1})(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_3\sigma_4^{-1})(\sigma_3^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_3)$$
$$(\sigma_4\sigma_4)(\sigma_3^{-1}\sigma_3^{-1})(\sigma_4^{-1}\sigma_4^{-1})(\sigma_4\sigma_4)(\sigma_4^{-1}\sigma_3^{-1}\sigma_3^{-1}\sigma_4)(\sigma_4^{-1}\sigma_4^{-1})$$

$$(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1})(\sigma_4\sigma_4)(\sigma_5\sigma_4\sigma_4\sigma_5)(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1})(\sigma_4^{-1}\sigma_4^{-1})$$
$$(\sigma_3^{-1}\sigma_2^{-1}\sigma_1\sigma_1\sigma_2\sigma_3)(\sigma_4\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2)(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_3\sigma_4^{-1})$$
$$(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_3)(\sigma_3^{-1}\sigma_2\sigma_2\sigma_3)(\sigma_2^{-1}\sigma_2^{-1})(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_3\sigma_4^{-1})(\sigma_3^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_3)$$
$$(\sigma_4\sigma_4)(\sigma_3^{-1}\sigma_3^{-1})(\sigma_4^{-1}\sigma_3^{-1}\sigma_3^{-1}\sigma_4)(\sigma_4^{-1}\sigma_4^{-1})$$

Using the known braid relations, we know that this is isotopic to the braid on the right hand side.

We will now present all relations of the form we are considering which include two curves containing three boundary components. We will present each by first showing the set of curves involved in the product of twists along non-boundary parallel curves. We can see that each relation gives the same multiplicity to each boundary component. Furthermore, the braid that corresponds to theleft hand side of each of these relations is equivalent to that which represents the twist over the outer boundary, so we need only to show that the right hand sides of each relation correspond to braids that are isotopic to this braid.

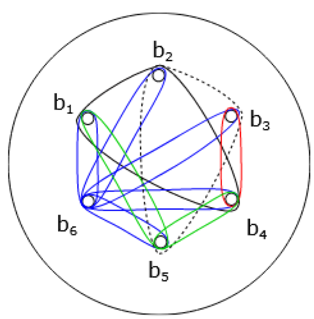


Figure 3.48

$$T_{b_1}^3T_{b_2}^2T_{b_3}^3T_{b_4}^3T_{b_5}^3T_{b_6}^4T_{b_7} = T_{b_3,b_4}T_{b_1,b_2,b_4}T_{b_4,b_5}T_{b_2,b_3,b_5}T_{b_1,b_5}T_{b_5,b_6}T_{b_4,b_6}T_{b_3,b_6}T_{b_2,b_6}T_{b_1,b_6}T_{b_1,b_3}$$

Proof. Applying Theorem 2.11 directly, we see that the word for the right hand side is

$$(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2)(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1})(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_3\sigma_4)(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4)$$
$$(\sigma_2^{-1}\sigma_2^{-1})(\sigma_4^{-1}\sigma_4^{-1})(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3)(\sigma_1^{-1}\sigma_1^{-1})(\sigma_3^{-1}\sigma_3^{-1})$$

Now we can commute elements using the relation $\sigma_i^{-1}\sigma_j^{-1} = \sigma_i^{-1}\sigma_j^{-1}$ for $i - j > 1$.

$$(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2)(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1})(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_3\sigma_4)(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4)$$

$$(\sigma_4^{-1}\sigma_4^{-1})(\sigma_2^{-1}\sigma_2^{-1})(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3)(\sigma_3^{-1}\sigma_3^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

and perform cancellation using inverses:

$$(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2)(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1})(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1})$$
$$(\sigma_2^{-1}\sigma_2^{-1})(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

Commuting σ_2 with elements whose indices are greater than 1 away from 2 and applying braid relations 7 and 8 repeatedly, we get

$$(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1})(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1})$$
$$(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_2^{-1})(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

Then, applying Theorem 2.12, we can get the terms in the parentheses in the correct order:

$$(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1})(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1})$$
$$(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

□

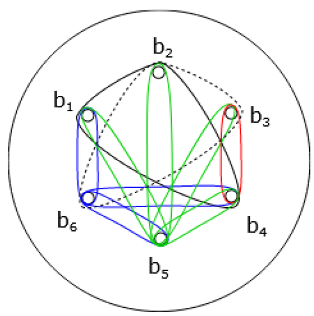


Figure 3.49

$$T_{b_1}^3T_{b_2}^2T_{b_3}^3T_{b_4}^3T_{b_5}^3T_{b_6}^4T_{b_7} = T_{(b_4,b_3)}T_{(b_1,b_2,b_4)}T_{(b_5,b_4)}T_{(b_5,b_3)}T_{(b_5,b_2)}T_{(b_5,b_1)}T_{(b_5,b_6)}T_{(b_4,b_6)}T_{(b_2,b_3,b_6)}T_{(b_1,b_6)}T_{(b_1,b_3)}$$

Proof. The word for the braid corresponding to the right hand side is

$$(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2)(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_3\sigma_4\sigma_5)(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_5)$$
$$(\sigma_2^{-1}\sigma_2^{-1})(\sigma_5^{-1}\sigma_4^{-1}\sigma_4^{-1}\sigma_5^{-1})(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1})(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3)(\sigma_1^{-1}\sigma_1^{-1})(\sigma_3^{-1}\sigma_3^{-1})$$

Using braid relation 5 to commute elements, and cancelling inverses gives us

$$(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2)(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1})$$
$$(\sigma_2^{-1}\sigma_2^{-1})(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1})(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

Similarly to the procedure in the previous proof, use braid relations 7 and 8 and theorem 2.12 to get

$$(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1})$$
$$(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1})(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

Then we can again apply Theorem 2.12 to get

$$(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1})(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1})$$
$$(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

as desired. □

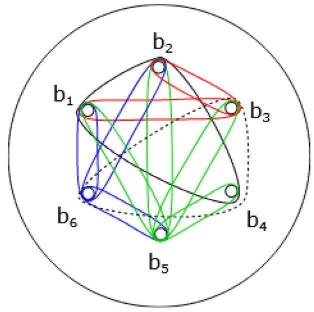


Figure 3.50

$$T_{b_1}^3T_{b_2}^3T_{b_3}^3T_{b_4}^2T_{b_5}^4T_{b_6}^3T_{b_7} = T_{(b_3,b_2)}T_{(b_3,b_1)}T_{(b_5,b_4)}\cdots T_{(b_5,b_1)}T_{(b_5,b_6)}T_{(b_3,b_4,b_6)}T_{(b_6,b_2)}T_{(b_6,b_1)}T_{(b_1,b_2,b_4)}$$

Proof. Using Theorem 2.11, we see that the word for the braid corresponding to the right hand side is

$$(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3)(\sigma_1^{-1}\sigma_1^{-1})(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_4\sigma_5)(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5)(\sigma_3^{-1}\sigma_3^{-1})$$
$$(\sigma_5^{-1}\sigma_5^{-1})(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})$$

Commuting using braid relation 5 and cancelling inverses gives us

$$(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3)(\sigma_1^{-1}\sigma_1^{-1})(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1})(\sigma_3^{-1}\sigma_3^{-1})$$
$$(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})$$

Then we can apply braid relations 7 and 8, using 5 to commute, and Theorem 2.12 to get

$$(\sigma_1^{-1}\sigma_1^{-1})(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1})(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})$$
$$(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})$$

Then we can apply Theorem 2.12 to commute the portions in parentheses and get

$$(\sigma_5^{-1}\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{-1})(\sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1})$$
$$(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})(\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1})(\sigma_1^{-1}\sigma_1^{-1})$$

□

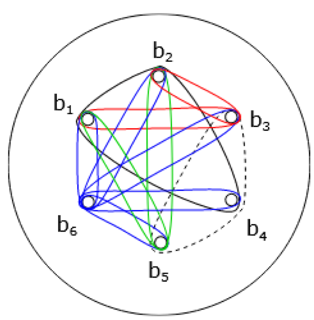


Figure 3.51

$$T_{b_1}^3T_{b_2}^3T_{b_3}^3T_{b_4}^2T_{b_5}^3T_{b_6}^4T_{b_7} = T_{b_2,b_3}T_{b_1,b_3}T_{b_3,b_6}T_{b_3,b_4,b_5}T_{b_2,b_5}T_{b_1,b_5}T_{b_5,b_6}T_{b_1,b_2,b_4}T_{b_4,b_6}T_{b_2,b_6}T_{b_1,b_6}$$

Proof. Recall that we proved the relation

$$T_{b_1}^2 T_{b_2}^2 T_{b_3}^2 T_{b_4}^3 T_{b_5} T_{b_6}^3 T_{b_7} = T_{(b_4,b_3)} T_{(b_4,b_2)} T_{(b_4,b_1)} T_{(b_4,b_5,b_6)} T_{(b_6,b_3)} T_{(b_6,b_2)} T_{(b_6,b_1)} T_{(b_1,b_2,b_3,b_5)}$$

Performing an isometry that takes each b_i to b_{i-1} for $i \leq 5$, and b_1 to b_6, we get the following relation:

$$T_{b_6}^2 T_{b_1}^2 T_{b_2}^2 T_{b_3}^3 T_{b_4} T_{b_5}^3 T_{b_7} = T_{(b_2,b_3)} T_{(b_1,b_3)} T_{(b_3,b_6)} T_{(b_3,b_4,b_5)} T_{(b_2,b_5)} T_{(b_1,b_5)} T_{(b_5,b_6)} T_{(b_1,b_2,b_4,b_6)}$$

Therefore, the right hand side of this relation corresponds to a braid that is isotopic to the braid corresponding to any product of twists over boundary parallel curves with exactly one twist over the outer boundary component.
Furthermore, applying relation (2) from $S_{0,0}^5$ to the subdisk containing b_1, b_2, b_4, and b_6, we see that the braid corresponding to

$$T_{(b_1,b_2,b_4,b_6)}$$

is isotopic to the braid corresponding to

$$T_{b_1,b_2,b_4} T_{b_4,b_6} T_{b_2,b_6} T_{b_1,b_6}$$

Therefore, using the relation derived by an isometry, we know that the braid corresponding to

$$T_{(b_2,b_3)} T_{(b_1,b_3)} T_{(b_3,b_6)} T_{(b_3,b_4,b_5)} T_{(b_2,b_5)} T_{(b_1,b_5)} T_{(b_5,b_6)} T_{(b_2,b_3,b_4,b_6)}$$

is isotopic to the braid corresponding to

$$T_{b_2,b_3} T_{b_1,b_3} T_{b_3,b_6} T_{b_3,b_4,b_5} T_{b_2,b_5} T_{b_1,b_5} T_{b_5,b_6} T_{b_1,b_2,b_4} T_{b_4,b_6} T_{b_2,b_6} T_{b_1,b_6}$$

which is isotopic to the braid corresponding to the left hand side of the proposed relation.
Therefore, we need only check multiplicity. We see that the multiplicity of each boundary component is equal on both sides, so the relation indeed holds. ☐

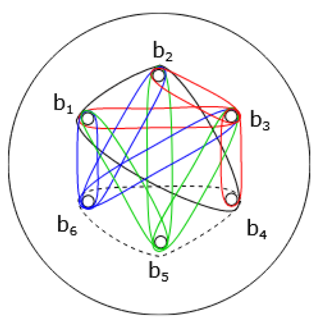


Figure 3.52

$$T_{b_1}^3 T_{b_2}^3 T_{b_3}^4 T_{b_4}^2 T_{b_5}^3 T_{b_6}^3 T_{b_7} = T_{(b_1,b_2,b_4)} T_{(b_2,b_3)} T_{(b_1,b_3)} T_{(b_3,b_4)} T_{(b_4,b_5,b_6)} T_{(b_3,b_6)} T_{(b_2,b_6)} T_{(b_1,b_6)} T_{(b_3,b_5)} T_{(b_2,b_5)} T_{(b_1,b_5)}$$

Proof. Recall the earlier relation

$$T_{b_1}^2 T_{b_2}^2 T_{b_3}^2 T_{b_4} T_{b_5}^3 T_{b_6}^3 T_{b_7} = T_{(b_1,b_2,b_3,b_4)} T_{(b_4,b_5,b_6)} T_{(b_3,b_6)} T_{(b_2,b_6)} T_{(b_1,b_6)} T_{(b_3,b_5)} T_{(b_2,b_5)} T_{(b_1,b_5)}$$

This, in particular, implies that the braids corresponding to both sides are isotopic.
Additionally, applying the relation (2) on the mapping class group of $S_{0,0}^5$ to the subdisk bounded by the curve (b_1, b_2, b_4), we can see that the braids corresponding to $T_{(b_1,b_2,b_3,b_4)}$ and $T_{(b_1,b_2,b_4)} T_{(b_2,b_3)} T_{(b_1,b_3)} T_{(b_3,b_4)}$ are isotopic.

Therefore, the braid corresponding to the left hand side is isotopic to the braid corresponding to the right hand side after substituting $T_{(b_1,b_2,b_3,b_4)}$ with $T_{(b_1,b_2,b_4)}T_{(b_2,b_3)}T_{(b_1,b_3)}T_{(b_3,b_4)}$. This gives the right hand side of the relation we are claiming to be true.
Since all products of twists boundary parallel curves involving exactly one twist along the outer boundary have the same braid, we see that the braids for the left hand and right hand side in the proposed relation are isotopic. Therefore, we need only to check that the multiplicities are the same for each boundary component, which can be seen easily.
Thus, this relation holds. □

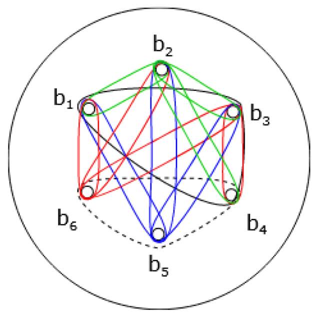


Figure 3.53

$$T_{b_1}^4 T_{b_2}^3 T_{b_3}^2 T_{b_4}^3 T_{b_5}^3 T_{b_6}^3 T_{b_7} = T_{b_1,b_3,b_4} T_{b_1,b_2} T_{b_2,b_4} T_{b_2,b_3} T_{(b_4,b_5,b_6)} T_{(b_3,b_6)} T_{(b_2,b_6)} T_{(b_1,b_6)} T_{(b_3,b_5)} T_{(b_2,b_5)} T_{(b_1,b_5)}$$

Proof. As in the previous proof, we will prove this using a previously proven relation.
In this case, we have the relation

$$T_{b_1}^2 T_{b_2}^2 T_{b_3}^2 T_{b_4} T_{b_5}^3 T_{b_6}^3 T_{b_7} = T_{(b_1,b_2,b_3,b_4)} T_{(b_4,b_5,b_6)} T_{(b_3,b_6)} T_{(b_2,b_6)} T_{(b_1,b_6)} T_{(b_3,b_5)} T_{(b_2,b_5)} T_{(b_1,b_5)}$$

Thus, the braid representing the right hand side of this relation has a braid which is isotopic to the left hand side of the proposed relation.
Furthermore, by relation (2) on the sphere with 5 boundary components, we know that T_{b_1,b_2,b_3,b_4} corresponds to a braid that is isotopic to $T_{b_1,b_3,b_4} T_{b_1,b_2} T_{b_2,b_4} T_{b_2,b_3}$.
Thus, we can substitute this into the above relation to find that

$$T_{b_1,b_3,b_4} T_{b_1,b_2} T_{b_2,b_4} T_{b_2,b_3} T_{(b_4,b_5,b_6)} T_{(b_3,b_6)} T_{(b_2,b_6)} T_{(b_1,b_6)} T_{(b_3,b_5)} T_{(b_2,b_5)} T_{(b_1,b_5)}$$

corresponds to a braid that is isotopic to the desired braid.
Thus, both sides of the proposed relation correspond to isotopic braids. Furthermore, we see that the multiplicities of the boundary components are the same on both sides, so the relation holds. □

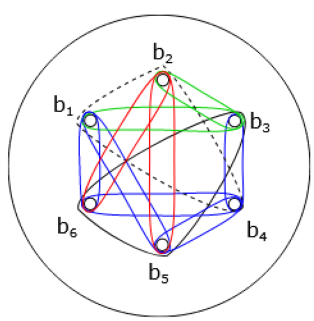


Figure 3.54

$$T_{b_1}^3 T_{b_2}^3 T_{b_3}^3 T_{b_4}^3 T_{b_5}^3 T_{b_6}^3 T_{b_7} = T_{(b_3,b_2)} T_{(b_3,b_1)} T_{(b_5,b_4)} T_{(b_5,b_3)} T_{(b_1,b_2,b_5)} T_{(b_5,b_6)} T_{(b_3,b_4,b_6)} T_{(b_6,b_2)} T_{(b_6,b_1)} T_{(b_4,b_2)} T_{(b_4,b_1)}$$

Proof. We will prove this using another relation. In particular, recall the relation

$$T_{b_1}^3 T_{b_2}^3 T_{b_3}^2 T_{b_4} T_{b_5}^2 T_{b_6}^2 T_{b_7} = T_{(b_2,b_3)} T_{(b_1,b_3)} T_{(b_3,b_4,b_5,b_6)} T_{(b_2,b_6)} T_{(b_1,b_6)} T_{(b_2,b_5)} T_{(b_1,b_5)} T_{(b_1,b_2,b_4)}$$

Using an isometry of relation (2) on the sphere with 5 boundary components, we know that the braid corresponding to T_{b_3,b_4,b_5,b_6} is isotopic to the braid corresponding to $T_{b_3,b_5,b_6} T_{b_3,b_4} T_{b_4,b_6} T_{b_4,b_5}$
Thus,

$$T_{(b_2,b_3)} T_{(b_1,b_3)} T_{b_3,b_5,b_6} T_{b_3,b_4} T_{b_4,b_6} T_{b_4,b_5} T_{(b_2,b_6)} T_{(b_1,b_6)} T_{(b_2,b_5)} T_{(b_1,b_5)} T_{(b_1,b_2,b_4)}$$

corresponds to a braid that is isotopic to the desired braid.
Additionally, the multiplicity of each boundary component is equal on both sides, so the relation holds. □

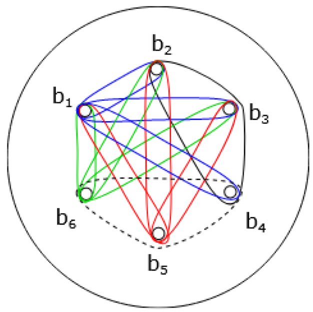


Figure 3.55

$$T_{b_1}^3 T_{b_2}^3 T_{b_3}^2 T_{b_4}^3 T_{b_5}^2 T_{b_6}^4 T_{b_7} = T_{b_2,b_3,b_4} T_{b_1,b_4} T_{b_1,b_3} T_{b_1,b_2} T_{(b_4,b_5,b_6)} T_{(b_3,b_6)} T_{(b_2,b_6)} T_{(b_1,b_6)} T_{(b_3,b_5)} T_{(b_2,b_5)} T_{(b_1,b_5)}$$

Proof. Recall the relation

$$T_{b_1}^2 T_{b_2}^2 T_{b_3}^2 T_{b_4} T_{b_5}^3 T_{b_6}^3 T_{b_7} = T_{(b_1,b_2,b_3,b_4)} T_{(b_4,b_5,b_6)} T_{(b_3,b_6)} T_{(b_2,b_6)} T_{(b_1,b_6)} T_{(b_3,b_5)} T_{(b_2,b_5)} T_{(b_1,b_5)}$$

Furthermore, using relation (2) on the mapping class group of the sphere with 5 boundary components, we can say that the braid corresponding to $T_{(b_1,b_2,b_3,b_4)}$ is isotopic to the braid corresponding to $T_{b_2,b_3,b_4} T_{b_1,b_4} T_{b_1,b_3} T_{b_1,b_2}$
Thus, the braid corresponding to the left hand side of the proposed relation is isotopic ot the braid corresponding to

$$T_{b_2,b_3,b_4} T_{b_1,b_4} T_{b_1,b_3} T_{b_1,b_2} T_{(b_4,b_5,b_6)} T_{(b_3,b_6)} T_{(b_2,b_6)} T_{(b_1,b_6)} T_{(b_3,b_5)} T_{(b_2,b_5)} T_{(b_1,b_5)}$$

Furthermore, we see that the multiplicities are equal, so the relation holds. □

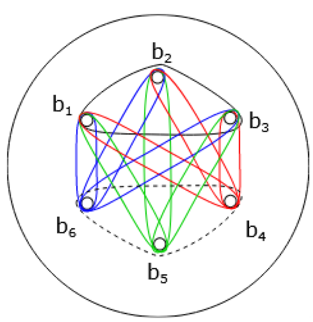


Figure 3.56

$$T_{b_1}^3 T_{b_2}^3 T_{b_3}^2 T_{b_4}^3 T_{b_5}^2 T_{b_6}^4 T_{b_7} = T_{b_1,b_2,b_3} T_{b_3,b_4} T_{b_2,b_4} T_{b_1,b_4} T_{b_4,b_5,b_6} T_{b_3,b_6} T_{b_2,b_6} T_{b_1,b_6} T_{b_3,b_5} T_{b_2,b_5} T_{b_1,b_5}$$

Proof. Recall the relation

$$T_{b_1}^2 T_{b_2}^2 T_{b_3}^2 T_{b_4} T_{b_5}^3 T_{b_6}^3 T_{b_7} = T_{(b_1,b_2,b_3,b_4)} T_{(b_4,b_5,b_6)} T_{(b_3,b_6)} T_{(b_2,b_6)} T_{(b_1,b_6)} T_{(b_3,b_5)} T_{(b_2,b_5)} T_{(b_1,b_5)}$$

This tells us that the right hand side of the equation corresponds to a braid that is isotopic to the braid corresponding to a twist over the outer boundary.

Furthermore, using relation (2) on the sphere with 5 boundary components, we can say that the braid corresponding to T_{b_1,b_2,b_3,b_4} is isotopic to the braid corresponding to $T_{b_1,b_2,b_3} T_{b_3,b_4} T_{b_2,b_4} T_{b_1,b_4}$

Therefore, the braid corresponding to

$$T_{b_1,b_2,b_3} T_{b_3,b_4} T_{b_2,b_4} T_{b_1,b_4} T_{(b_4,b_5,b_6)} T_{(b_3,b_6)} T_{(b_2,b_6)} T_{(b_1,b_6)} T_{(b_3,b_5)} T_{(b_2,b_5)} T_{(b_1,b_5)}$$

is isotopic to the braid corresponding to the left hand side of the proposed relation. □

The only other possible collections of curves where two contain three holes will involve a curve of this shape:

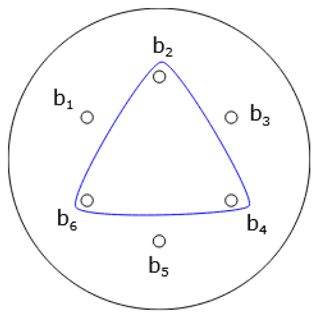


Figure 3.57

As mentioned earlier, we suspect that there exist no relations between products of boundary parallel curves and products of boundary parallel curves that contain a twist over this curve. However, we can use the relation

$$T_{b_1}^2 T_{b_2} T_{b_3}^2 T_{b_4}^3 T_{b_5}^2 T_{b_6}^3 T_{b_7} = T_{b_4,b_5} T_{b_1,b_2,b_3,b_5} T_{b_5,b_6} T_{b_3,b_4} T_{b_2,b_4,b_6} T_c T_{b_1,b_6} T_{b_1,b_4}$$

where c is as defined earlier in this section, together with relation (2) from the sphere with 5 boundary components, to derive further relations involving this curve, as we have done with the other relations in this section.

Future work can be done in finding additional relations involving non-convex curves with twists that include this curve and the one containing b_1, b_2, b_4, and b_5.

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