Tricolorablity of Pretzel Knots

Samantha Shepherd

Supervised by Dr. Jennifer Schultens

Department of Mathematics
University of California, Davis
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1 Overview/Introduction

In 1771, Alexandre-Théophile Vandermonde created one of the first mathematical theories of knots. In the following centuries, mathematicians expanded the study of knots, knot transformations, and applications of knot theory. Since then, knot theory in combination with topology, have been widely used to describe phenomena such as modeling DNA and circuit topology.

A *knot* can be thought of as a free-moving knotted loop of one-dimensional string that does not intersect itself and exists in 3-dimensional space. A *knot invariant* is a function used to measure and determine knots' topological equivalence. An example would be the *unknotting number*, the minimum number of times two points of the knot need to pass through each other to make it the unknot. While knots are 1-dimensional objects living in 3-dimensional space, we often think of them in terms of diagrams in the plane. A *knot diagram* is a projection of a knot onto a plane consisting of a circle with points called *crossings*, where one subarc of the circle crosses over another. A *strand* in a projection of a link is a piece of the link that goes from one under crossing to another over crossing in between. Planar isotopy describes the stretching deformation a knot may experience as if it were made of rubber.

This thesis grew out of my interest in more theoretical fields of mathematics, especially those that also provide for interesting computations. I have never explored knot theory before so this has been a fun exploration.

2 Definitions, concepts, key examples

A *knot* is the image of a smooth map of a circle into 3-space, considered up to certain allowable deformations. The *unknot* is the equivalence class of the unit circle in the xy-plane considered in 3-dimensional space. A *knot invariant* is a characteristic that is the same for all equivalent knots. Since knots are an equivalence classes, and since the deformations allowed are extensive, it is often difficult to decide whether two given knots are equivalent. However, some deformations are easier to detect. Given two knot projections, a *planar isotopy* is when the projection plane of a knot projection experiences a deformation, as if it were made of rubber. [Fig. 1] A knot is *tricolorable* if each strand in the projection can be colored one of three different colors such that at each crossing either all three colors are present or only one color. It is also required that the projection is colored with at least two colors.

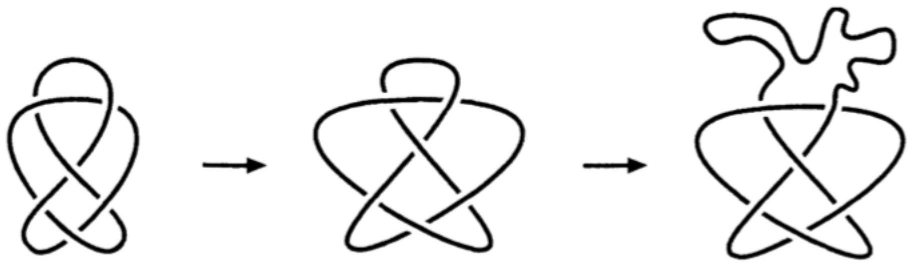

Figure 1: An example of planar isotopies [1]

Projections

The unknot is the simplest knot. By definition, it can be deformed into the unit circle in the plane where it has no crossings. [Fig. 2] However, certain projections of the unknot can have crossings. [Fig. 3]

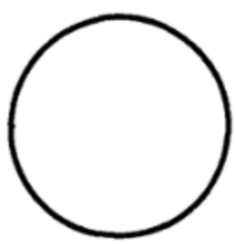

Figure 2: The most common projection of the Unknot [1]

Figure 3: Another projection of the Unknot, but with crossings [1]

Any knot, such as the Figure-eight knot, [Fig. 4] can have multiple projections. The most common projection is the one on the far left.

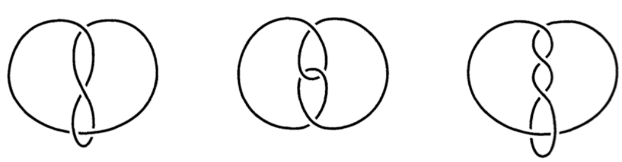

Figure 4: Various Projections of the Figure-eight knot [1]

Pretzel Knots

There are types of knots that have been studied in a variety of contexts. The knots considered in this thesis are *pretzel knots*. We will focus on those with three connected twists, where each twist has a finite number of crossings.

These particular pretzel knots are characterized by the number of crossings and the orientation of the crossings in each of the three twists. As shown in [Fig. 5], a left-hand crossing arises when the strand that starts in the bottom right is an over crossing, and a right-hand crossing is when the strand that starts in the bottom right is an under crossing.

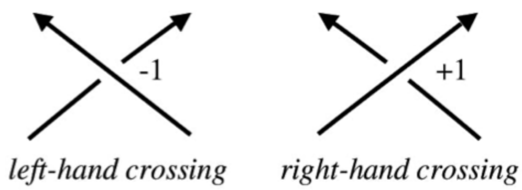

Figure 5: Left hand and right hand crossings [3]

Thus, each crossing is assigned a number which officially denotes the orientation of each crossing. To get the number of the twist, just add up all the crossings in the twist. For example, [Fig. 6] is considered a (-2,3,7)-pretzel knot, since there are two left handed crossings in the left twist, 3 right hand crossings in the middle twist, and 7 right hand crossings in the right twist. Thus, it is denoted as a (-2,3,7)-pretzel knot.

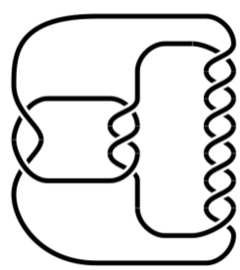

Figure 6: A (-2,3,7)-Pretzel Knot [4]

3 Important Theorems and Results

Reidemeister Moves

There are three types of Reidemeister Moves. Type I twists or untwists a strand in either direction; this adds or removes a crossing [Fig 7]. Type II moves a loop over or under a strand. Essentially, this move takes two parallel strands and moves one over or under the other. Type II adds two crossings or removes two crossings [Fig 8]. Type III describes when a stand is moved over a crossing. The crossing stays in the same place, but the strand changes location without making any additional loops or crossings [Fig 9].

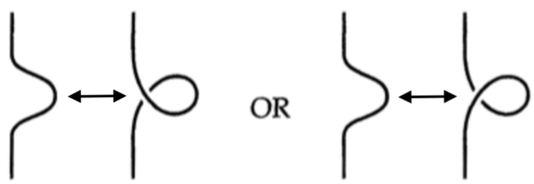

Figure 7: Reidemeister Type I move [1]

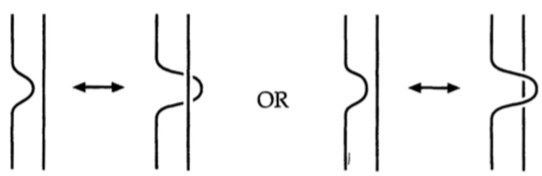

Figure 8: Reidemeister Type II move [1]

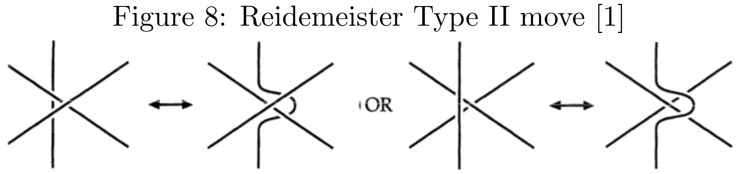

Figure 9: Reidemeister Type III move [1]

As seen within [Fig 7, Fig 8, Fig 9], the Reidemeister move type number also denotes the number of strands involved within each move respectively. [2]

Reidemeister's Theorem

Kurt Reidemeister proved that if two projections represent the same knot, one can get from one projection to the other through a series of Reidemeister moves and planar isotopies. [1] For example, in [Fig. 4], since all the projections are of the Figure-eight knot, one can use Reidemeister moves to get from one projection to another.

The other important part of Reidemeister's theorem is that planar isotopies can also be used to manipulate projections, as seen in [Fig. 10]'s last two steps. But, neither Reidemeister moves nor planar isotopies change the knot.

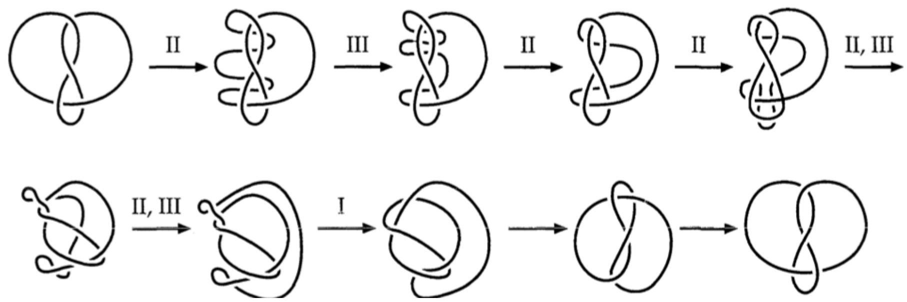

Figure 10: Using Reidemeister moves to get from the Figure-eight knot to its mirror image [1]

The most relevant result is that tricolorability is unaffected by Reidemeister moves. A Type I move simply takes a strand and adds or removes a twist, thus if a twist was added, the new crossing will be all the same color. If a crossing was removed, only one color occurred in the crossing, so instead of a twisted strand there is now only a strand of that one color. [Fig. 11] A Type II move involves two strands that are either both the same color, or different colors. If the strands are the same color, the Type II move does not affect tricolorability. This is because the pair of crossings that are introduced or removed involve only one color. If the strands are different colors, then pulling one strand over another and introducing a pair of crossings introduces a strand with the third color. Conversely, removing the two crossings by pulling a strand away from the other removes the third color. However, since there are still two strands of opposite colors, and since the knot is connected, it must be the case that somewhere in the diagram, there is a crossing involving all three colors. [Fig. 12] A Type III move is slightly more complicated. There are several cases to consider. The simplest case is when the strands are all the same color, because tricolorability is not affected. I depicted the two most interesting cases in [Fig. 13]. Pulling the strand under the crossing simply moves the under-crossing to the other side of the crossing, and vice versa. For moving a strand over a crossing, the strand that is moving is the same color before and after the move. In all cases, tricolorability is not affected.

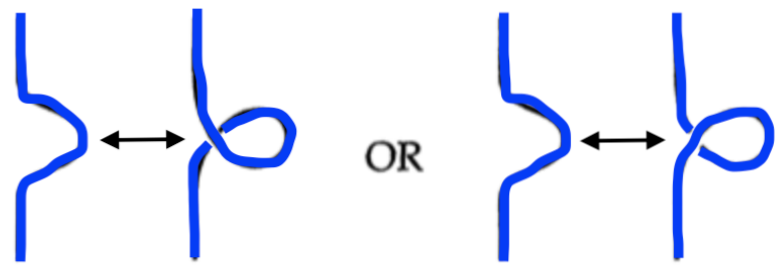

Figure 11: Type I Reidemeister move tricolored

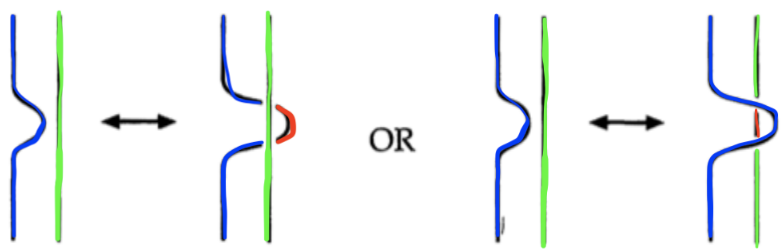

Figure 12: Type II Reidemeister move tricolored

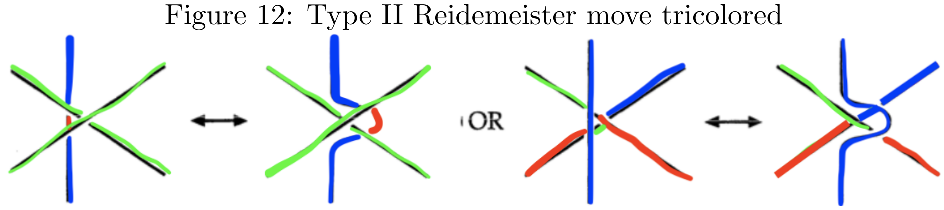

Figure 13: Type III Reidemeister move tricolored

[Fig. 14], we can see the trefoil knot is tricolorable in both projections, and is thus tricolorable.

Figure 14: Two tricolored projections of the trefoil knot

4 Tricolorability

Various Cases to Test Tricolorability of the (-2,3,7)-Pretzel Knot

In [Fig. 15] we have the top three strands as color one. Then, the top right, top middle, and top left crossings all have to be color one. It then follows that the other crossings have to be color one, since each crossing will have the same color for two strands of a crossing, which leads the third strand to be the same color. This does not meet the definition of tricolorability.

Next, [Fig. 16] has the top three strands as different colors. Starting in the top right crossing, this leads the twist to alternate colors according to tricolorabilty rules. This leads to the bottom strand being color three. Then, the colors on the left strand are colored according to the tricolorability rules. This leads the strand at the very bottom to be color three and the other to be color one. Coloring the twist on the left also leads to the strand at the very bottom to be color three and the other to be color one. In the middle twist, two distinct colors enter the crossings at the top and the bottom, but two equal colors enter the middle crossing. There is no way to color the remaining strand in accordance with tricoloring rules.

Finally, [Fig. 17, 18, and 19] have various permutations of 2 strands with color one, and the other strand with color two. Tricolorabilty issues arise when the two top strands of color one meet, because that strand is forced to be color one, or a fourth color must be used. In each case, I started coloring the right twist. This led to only the right twist in [Fig. 18] being color one all the way through, since the top two strands are color one, and thus the last strand must be color one, and thus the twist is color one. However for [Fig. 17 and 19], while the right twist follows the tricolorabilty rules, the other two twists do not. In the right twist, the top two strands are color one and color two, which leads color three to be part of the twist, and thus

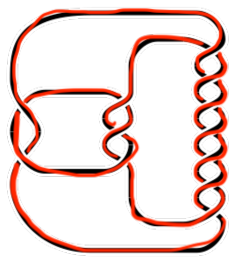

Figure 15: All top three strands the same color

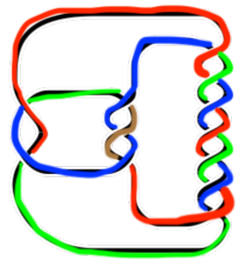

Figure 16: Three different colors for top strands

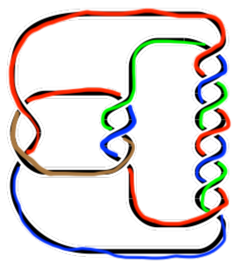

Figure 17: Two of the top strands are color one, and the other is color two

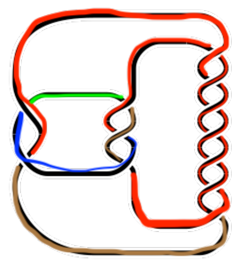

Figure 18: Two of the top strands are color one, and the other is color two

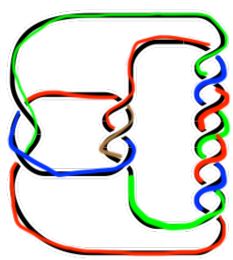

Figure 19: Two of the top strands are color one, and the other is color two

the right twist follows tricolorabilty rules. Then, the middle and left twists violate tricolorabilty because it is necessary to add a fourth color. In the middle twist of [Fig. 19], two distinct colors enter the crossings at the top and the bottom, but two equal colors enter the middle crossing. There is no way to color the remaining strand in accordance with tricoloring rules, hence the fourth color being introduced. The same issue arises for [Fig. 17]. In the left twist, two equal colors enter the top crossing, and thus color one enters the bottom crossing. There is no way to color the remaining strand in accordance with tricoloring rules, hence the fourth color being introduced.

As we can see, this knot is not tricolorable, but it can be considered 4-colorable. This aligns with the theorem we will prove, because the knot is not tricolorable, and the number of crossings do not have the correct relationship.

Various Cases to Test Tricolorability of the (-2,3,8)-Pretzel Knot

In [Fig. 20] we have the top three strands as color one. Then, the top right, top middle, and top left crossings all have to be color one. It then follows that the other crossings have to be color one, since each crossing will have the same color for two strands of a crossing, which leads the third strand to be the same color. This does not meet the definition of tricolorability.

Next, [Fig. 21] has the top three strands as different colors. Starting in the top right crossing, this leads the twist to alternate colors according to tricolorabilty rules. This leads to the bottom strand being the middle strand's color. The top crossing of the middle twist follows the rules, but the rest of the crossings do not. This is because the left twist introduces a fourth color in the bottom crossing, which then leads to a fifth color being introduced in the middle twist. This does not meet the definition of tricolorability.

Finally, [Fig. 22, 23, and 24] have various permutations of 2 strands with color one, and the other strand with color two. Tricolorabilty issues arise when the two top strands of color one meet, because that strand is forced to be color one, or a fourth color must be used. In each case, begin by coloring the right twist. Therefore color one is applied on the entire right twist of [Fig. 23]. However for [Fig. 22 and 24], while the right twist follows the tricolorabilty rules, the other two twists do not. In the right twist, the top two strands are color one and color two, which leads color three to be part of the twist, and thus the right twist follows tricolorabilty rules. Then, the middle and left twists violate tricolorabilty because it is necessary to add a fourth color. In the middle twist of [Fig. 24], two equal colors enter the crossings at the top and continue for the rest of the twist. There is no way to color the remaining strand in accordance with tricoloring rules,

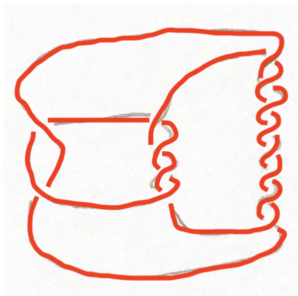

Figure 20: All top three strands the same color

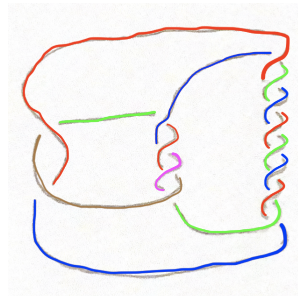

Figure 21: Three different colors for top strands

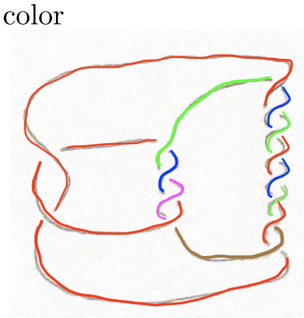

Figure 22: Two of the top strands are color one, and the other is color two

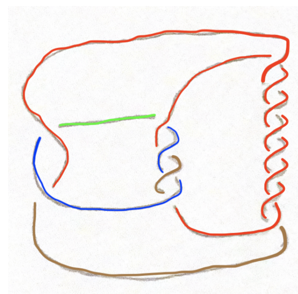

Figure 23: Two of the top strands are color one, and the other is color two

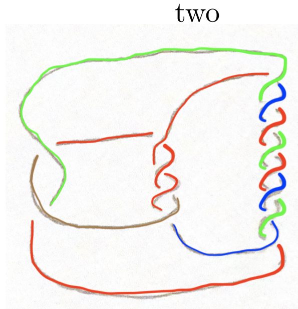

Figure 24: Two of the top strands are color one, and the other is color two

hence the fourth color being introduced. The same issue arises for [Fig. 22]. In the middle twist, the top crossing follows tricolorabilty rules. However, for the remaining strands, there is no way to color the remaining strand in accordance with tricoloring rules, hence the fourth and fifth colors being introduced.

As we can see, this knot it not tricolorable, but it can be considered 5-colorable. This aligns with the theorem we will prove, because the knot is not tricolorable, and the number of crossings do not have the correct relationship.

Examples of Knots That Are Tricolorable

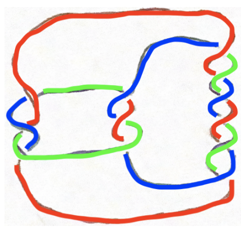

Figure 25: (-3,3,6)-Pretzel Knot

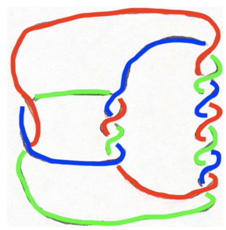

Figure 26: (-2,4,7)-Pretzel Knot

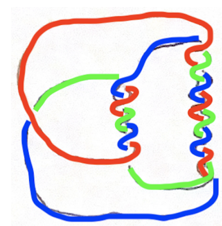

Figure 27: (-1,5,8)-Pretzel Knot

Theorem 1. *The* (p, q, r)*-pretzel link is tricolorable if and only if*

1. $p = q \mod 3$ *and* $q = r \mod 3$*; or*
2. $p = -q \mod 3$.

Proof. We consider each of the three twists separately in order to understand how colorings can match up. If only one color goes in at the top of a single twist, then the entire twist is that one color. If two distinct colors go in at the top, we label them with variables. The variables represent the three distinct colors in each distinct twist. Either each twist is a single color, or it contains all three colors. This means that if, for instance, x, y are the same, then x, y, z are the same. There are then three cases for each twist.

Case 1: In the right twist, the number of crossings is 0 mod 3. The colors entering are y from the top left and x from the top right. The colors exiting on the bottom are y at the bottom left and x at the bottom right.

Case 2: In the right twist, the number of crossings is 1 mod 3. The colors entering are y from the top left and x from the top right. The colors exiting on the bottom are x at the bottom left and z at the bottom right.

Case 3: In the right twist, the number of crossings is 2 mod 3. The colors entering the top are y on the top left and x on the top right. The colors exiting on the bottom are z at the bottom left and y at the bottom right.

Case 4: In the middle twist, the number of crossings is 0 mod 3. The colors entering are v from the top left and u from the top right. The colors exiting on the bottom are v at the bottom left and u at the bottom right.

Case 5: In the middle twist, the number of crossings is 1 mod 3. The colors entering the top are v from the top left and u from the top right. The colors exiting on the bottom are u at the bottom left and t at the bottom right.

Case 6: In the middle twist, the number of crossings is 2 mod 3. The colors entering are v on the top left and u on the top right. The colors exiting on the bottom are t at the bottom left and v at the bottom right.

Case 7: In the left twist, the number of crossings is 0 mod 3. The colors entering are q from the top left and s from the top right. The colors exiting on the bottom are q at the bottom left and s at the bottom right.

Case 8: In the left twist, the number of crossings is 1 mod 3. The colors entering are q from the top left and s from the top right. The colors exiting on the bottom are s at the bottom left and r at the bottom right.

Case 9: In the left twist, the number of crossings is 2 mod 3. The colors entering are q from the top left and s from the top right. The colors exiting on the bottom are r at the bottom left and q at the bottom right.

Now, we want to create a system of equations to illustrate the relationship between the colors in each twist. We can model the 6 horizontal strands with a system of equations and hope it works out. See Figure 28.

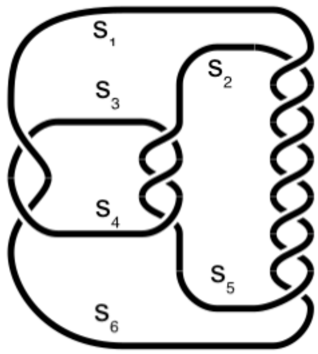

Figure 28: A (-2,3,7)-pretzel knot with each horizontal strand labeled

For $p, q,$ and r being 0 mod 3 we have: $\begin{cases} s_1 : q = x \\ s_2 : u = y \\ s_3 : s = v \\ s_4 : s = v \\ s_5 : u = y \\ s_6 : q = x \end{cases}$

This system shows us that colors q and x are the same; u and y are the same; and s and v are the same. We also have some free variables, z, r and t. They don't interact with any of the other variables, which means they represent the third color that shows up in the twist, but doesn't go in the top or come out the bottom.

For $p, q,$ and r being 1 mod 3 we have: $\begin{cases} s_1 : q = x \\ s_2 : u = y \\ s_3 : s = v \\ s_4 : r = u \\ s_5 : t = x \\ s_6 : s = z \end{cases}$

This system shows us that colors q, x and t are the same; u, y and r are the same; and s, v and z are the same. This is exactly what we wanted to show.

For $p, q,$ and r being 2 mod 3 we have: $\begin{cases} s_1 : q = x \\ s_2 : u = y \\ s_3 : s = v \\ s_4 : q = t \\ s_5 : v = z \\ s_6 : r = y \end{cases}$

This system shows us that colors q, x and t are the same; u, y and r are the same; and s, v and z are the same. This is exactly what we wanted to show. We see that in all three cases where p, q, r are the same mod 3, the (p, q, r)-pretzel knot is tricolorable.

According to our theorem, if a (p, q, r)-pretzel knot is not tricolorable, then $p, q,$ and r are not the same mod 3. We see this when we examine a (-2,3,7)-pretzel knot.

We have: $\begin{cases} s_1 : q = x \\ s_2 : u = y \\ s_3 : s = v \\ s_4 : q = v \\ s_5 : u = x \\ s_6 : r = z \end{cases}$

This system shows us that colors x, u and y are the same; q, s, and v are the same; and u, x, v and q are the same. This is a contradiction because we got $x = y$, which forces $x = y = z$. We also got $q = s$, which forces $q = r = s$. Since $q = x$, $x = y = z = q = r = s$. Finally, $u = v$, which forces $t = u = v$. Earlier, we had $x = u$, which means $x = y = z = q = r = s = t = u = v$. Thus all colors are the same and this is not a tricoloring.

This same phenomenon is present in a (-2,3,8)-pretzel knot.

We have: $\begin{cases} s_1 : q = x \\ s_2 : u = y \\ s_3 : s = v \\ s_4 : q = v \\ s_5 : u = x \\ s_6 : r = z \end{cases}$

This system shows us that colors x, u and y are the same; q, s, and v are the same; and u, x, v and q are the same. This is a contradiction because we got $x = y$, which forces $x = y = z$. We also got $q = s$, which forces $q = r = s$. Since $q = x$, $x = y = z = q = r = s$. Finally, $u = v$, which forces $t = u = v$. Earlier, we had $x = u$, which means $x = y = z = q = r = s = t = u = v$. Thus all colors are the same and this is not a tricoloring.

In these two cases where p, q, r are not the same mod 3, we see that there can be no tricoloring. A similar contradiction to the existence of a tricoloring arises in the other cases where p, q, r are not the same mod 3. Since we have the top strands being color one, color two, and color three, respectively, a (p, q, r)-pretzel knot is tricolorable and the strands at the bottom match up if and only if (p, q, r) are the same mod 3.

□

5 Conclusions

As demonstrated in this thesis, tricolorability is a knot invariant that serves to distinguish knots. As it turns out, tricolorability is one of a family of knot invariants including 4-colorability, 5-colorability, etc., and, more generally, *quandles*. A *quandle* is a set X with a binary operation $\circ$ that satisfies certain axioms. In one of the simplest cases, we have the following: (a) $x \circ x = x$; (b) $x \circ y = z$; (c) $y \circ z = x$; (d) $z \circ x = y$. Translating x, y, z into colors establishes a correspondence between the quandle rules and the rules for coloring strands of a knot at a crossing. Quandles are associated with knots in 3-dimensional space and also with so-called surface knots in 4-dimensional space. They constitute a rapidly evolving area of research.

6 Acknowledgements

I would like to take this opportunity to thank my advisor, Jennifer Schultens for her patience and incredible support throughout this process. This was my first introduction to knot theory, and I'm so grateful to have a leader in this field to guide me through this endeavor. I would also like to thank my friends for putting up with my random rants about pretzel knots. I know you guys didn't understand a word I was saying, but your support was appreciated nonetheless.

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