Computing the Bracket Polynomial of Two and Three Strand Links

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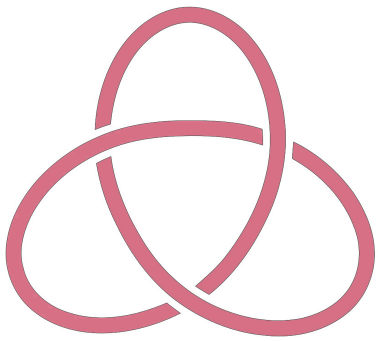

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Contents

1. Introduction

We will study invariants of knots and links such as the Kauffman bracket polynomial and the Jones polynomial.

In Sections 2 and 3, we recall the three Reidemeister moves, the definition and the properties of the bracket polynomial and the Jones polynomial following [1]. In particular, the curl relation in Proposition 3.3 describes the behavior of the bracket polynomial under the first Reidemeister move and is used frequently in this paper.

In Section 4, we compute the bracket polynomial for arbitrary $T(2,n)$ torus links which can be obtained as closures of two-strand braids σ_1^n. In particular, we get the following result (Theorem 4.2):

$$B_n = \langle T(2,n) \rangle = B_n = -A^{n+2} + \frac{-A^{n-6} - (-1)^n A^{-2-3n}}{1 + A^{-4}}.$$

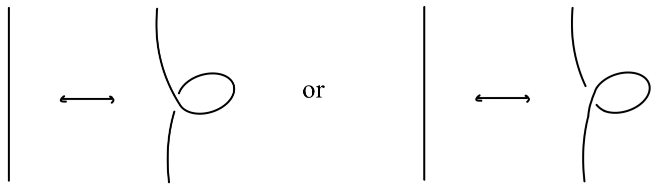
In Sections 5 and 6, we take a more abstract approach and define a vector space spanned by all crossingless matchings on 3 strands. This space is called the Temperley-Lieb algebra. Adding a crossing on top of a crossingless matching leads to a matrix which we compute in Section 6. There are two such matrices T_1, T_2 corresponding to the generators of the braid group. We verify that the braid relation $T_1T_2T_1 = T_2T_1T_2$ holds for such matrices.

Finally, in Theorem 6.6, we compute a matrix for the $(3, 3k)$ torus link by computing the powers $(T_1T_2)^{3k}$ inductively. This allows us to prove Theorem 6.7, which gives an explicit formula for the bracket polynomial of $(3, 3k)$ torus link for all $k \geq 0$.

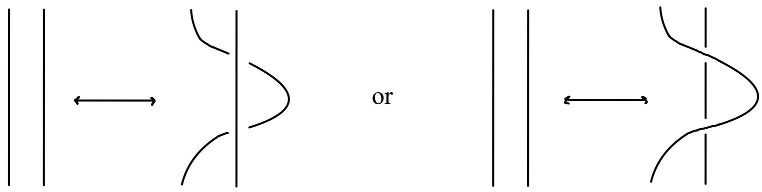
2. Background on Knots

Definition 2.1. *A* ***Reidemeister move*** *is one of three ways to change the projection of a knot that will change the relation between the crossings.*

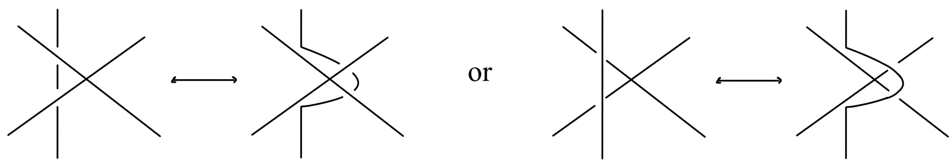
Definition 2.2. *The* ***first Reidemeister move*** *allows one to put in or take out a twist in the knot.*



Definition 2.3. *The* ***second Reidemeister move*** *allows one to either add two crossings or remove two crossings.*



Definition 2.4. *The* ***third Reidemeister move*** *allows one to slide a strand of the knot from one side of a crossing to the other side of the crossing.*



Theorem 2.5. *Two diagrams correspond to the same knot or link if and only if they are related by a sequence of Reidemeister moves.*

For more details see [1].

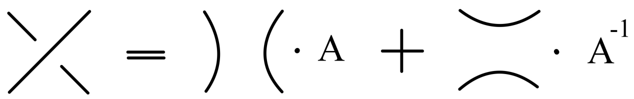
We will also need some facts about the braid group. The braid group on n strands has generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \sigma_i\sigma_j = \sigma_j\sigma_i\ (|i-j| \geq 2). \tag{1}$$

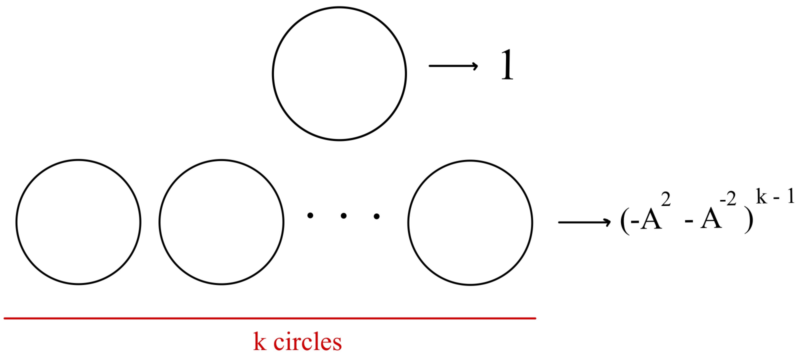
The second Reidemeister move corresponds to the relation $\sigma_i\sigma_i^{-1} = 1$ and the third Reidemeister move corresponds to the first equation in (1).

3. The Bracket Polynomial

Definition 3.1. *The* ***bracket polynomial*** *is defined by the following relations:*



and



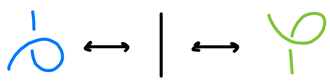
3.1. Properties.

Proposition 3.2. *The Bracket polynomial does not change under the second and third Reidemeister moves.*

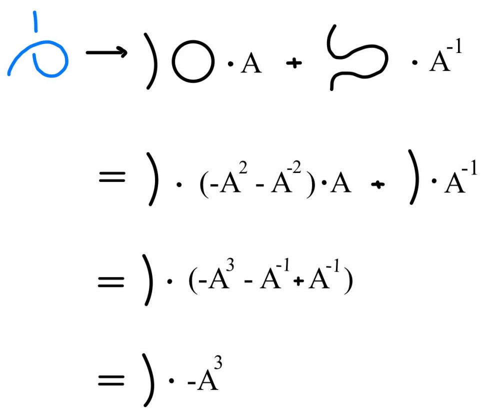
See Section 6.1 in [1].

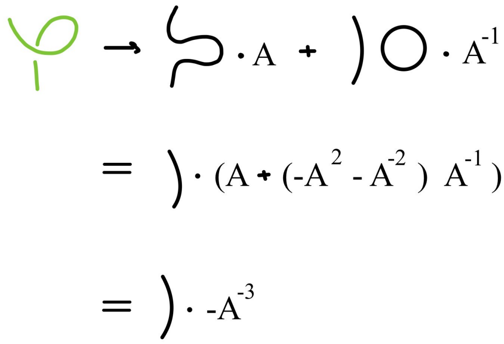
Proposition 3.3. *The Bracket polynomial does change under the first Reidemeister move, so a* ***Curl Relation*** *is used. The bracket polynomials before and after adding a curl differ by a factor* $-A^{\pm 3}$*.*

Proof. The definition of the first Reidemeister move is given by



Apply 3.1

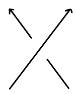


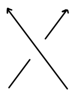


Thus, we obtain the Curl Relation $-A^{\pm 3}$. □

3.2. **The Jones Polynomial.**

How do we fix the issue with curls, and define an actual link invariant which does not change under *all* Reidemeister moves? We orient the link by choosing a direction for each component.

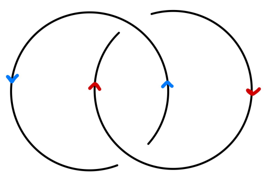
Definition 3.4. *A **positive crossing** (+1) is defined by* 

Definition 3.5. *A **negative crossing** (-1) is defined by* 

Definition 3.6. *The **writhe** of an oriented link $w(L)$ is the difference between the number of positive and negative crossings, or*

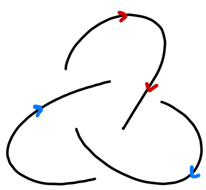
$$w\langle L\rangle = \text{(number of positive crossings) - (number of negative crossings)}$$

Example 3.7. *Orientation of the Hopf Link:*



Since there are two negative crossings, we have $w\langle L\rangle = -2$.

Example 3.8. *Orientation of the Trefoil Knot:*



Since there are three positive crossings, we have $w\langle L\rangle = 3$.

Definition 3.9. *Finally, we can define the **Jones polynomial** as*

$$J(L) = (-A^3)^{-w(L)}\langle L\rangle$$

Theorem 3.10. *J(L) is a link invariant and does not change under all three Reidemeister moves*

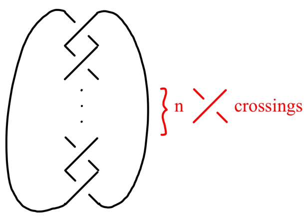
4. Computations of Two Strands

In this section we consider the $T(2,n)$ torus links which can be obtained as closures of two-strand braids σ_1^n. Let B_n be the bracket polynomial for $T(2,n)$.

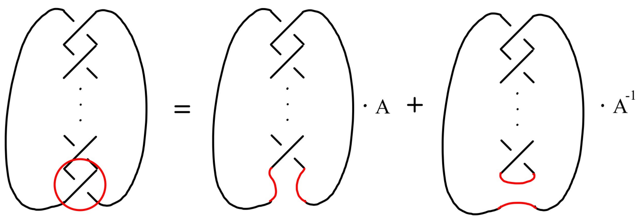
Lemma 4.1. *The polynomials B_n satisfy the recursion*

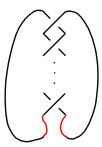
$$B_n = AB_{n-1} + A^{-1}(-A^{-3})^{n-1}$$

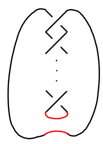
Proof. We present the following link B_n with n crossings:



We first apply the equation in Definition 3.1 to the bottom crossing, which gives us



Notice that  is the original knot, but with $n-1$ crossings. We call this B_{n-1}.

We can also note that we can apply the curl relation to . We will eventually obtain the unknot if we continuously apply the curl relation $(n-1)$ times to this knot.

Thus, we have obtained the recursive formula $B_n = AB_{n-1} + A^{-1}(-A^{-3})^{n-1}$ □

Theorem 4.2. *The bracket polynomial for the $T(2,n)$ torus link is*

$$B_n = -A^{n+2} + \frac{-A^{n-6} - (-1)^n A^{-2-3n}}{1 + A^{-4}}$$

Proof. We prove this by induction, using the previous lemma.

For the base case : $n = 0$, we have an unlink with two components, so $B_0 = -A^2 - A^{-2}$.
On the other hand, we have

$$-A^2 + \frac{-A^{-6} - (-1)^0 A^{-2}}{1 + A^{-4}} = -A^2 - \frac{A^{-2}(1 + A^{-4})}{1 + A^{-4}} = -A^2 - A^{-2}.$$

For the inductive step, we assume that the $n-1$ case is true:

$$B_{n-1} = -A^{n+1} + \frac{-A^{n-7} - (-1)^{n-1} \cdot A^{-2-3(n-1)}}{1 + A^{-4}}$$

So,

$$AB_{n-1} + A^{-1} \cdot (-A^{-3})^{n-1} = AB_{n-1} + A^{-1} \cdot (-1)^{n-1} \cdot A^{-3n+3}$$
$$= AB_{n-1} + (-1)^{n-1} \cdot A^{-3n+2}$$

Then we plug in B_{n-1}, which gives us

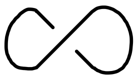
$$-A^{n+2} + \frac{-A^{n-6} - (-1)^{n-1} \cdot A^{-2-3(n-1)+1}}{1 + A^{-4}} + (-1)^{n-1} \cdot A^{-3n+2}$$

Simplifying this expression would give us

$$-A^{n+2} + \frac{-A^{n-6} - (-1)^{n-1} \cdot A^{-3n+2} + (-1)^{n-1} \cdot A^{-3n+2} + (-1)^{n-1} \cdot A^{-3n+2-4}}{1 + A^{-4}}$$
$$= -A^{n+2} + \frac{-A^{n-6} + (-1)^{n-1} \cdot A^{-2-3n}}{1 + A^{-4}}$$
$$= -A^{n+2} + \frac{-A^{n-6} - (-1)^{n} \cdot A^{-2-3n}}{1 + A^{-4}}.$$

Thus, the induction proof is completed. □

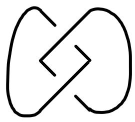
Example 4.3. *For $n = 1$ we get the unknot:*



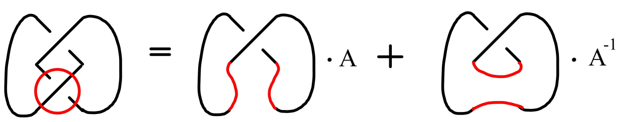
When we apply the curl relation, we get $B_1 = -A^3$. We can also plug in $n = 1$ into Theorem 4.2 and get

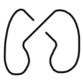
$$B_1 = -A^3 + \frac{-A^{-5} + A^{-5}}{1 + A^{-4}} = -A^3.$$

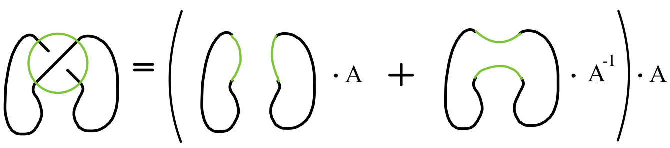
Example 4.4. *For $n = 2$ we get the Hopf link*



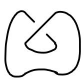
We apply Definition 3.1 to the bottom crossing to get

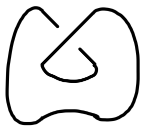


First, we look at  *and apply 3.1 again to get*



The equation we end up with is $((-A^2 - A^{-2}) \cdot A + 1 \cdot A^{-1}) \cdot A$, which then simplifies to $-A^4$

Then we apply the curl relation to *, and we get*



$$= -A^{-3} \cdot A^{-1}$$

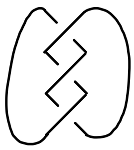
which simplifies to $-A^{-4}$. *Thus, the polynomial equals*

$$B_2 = -A^4 - A^{-4}.$$

We can also plug $n = 2$ *into Theorem 4.2 and get*

$$B_2 = -A^4 + \frac{-A^{-4} - A^{-8}}{1 + A^{-4}} = -A^4 - A^{-4}.$$

Example 4.5. *For* $n = 3$ *we get the trefoil knot*



We can follow the same process as the previous example where we apply 3.1 to the bottom crossing. Even better, we can use 4.1 to obtain the polynomial
Since there are three crossings, we have

$$B_3 = AB_2 + A^{-1}(-A^{-3})^2$$

We found that $B_2 = -A^4 - A^{-4}$ *in the previous example. Now,*

$$B_3 = A(-A^4 - A^{-4}) + A^{-1}(-A^{-3})^2$$

Thus, the polynomial equals $B_3 = -A^5 - A^3 + A^{-7}$.

We can also plug $n = 3$ *into Theorem 4.2 and get*

$$\begin{aligned} B_3 &= -A^5 + \frac{-A^{-3} + A^{-11}}{1 + A^{-4}} \\ &= -A^5 - A^{-3}\frac{1 - A^{-8}}{1 + A^{-4}} \\ &= -A^5 - A^{-3}(1 - A^{-4}) \\ &= -A^5 - A^3 + A^{-7} \end{aligned}$$

5. **Temperley-Lieb Matrices**

Let V be the 5 dimensional space spanned by the pictures in Figure 1 also known as crossingless matchings.

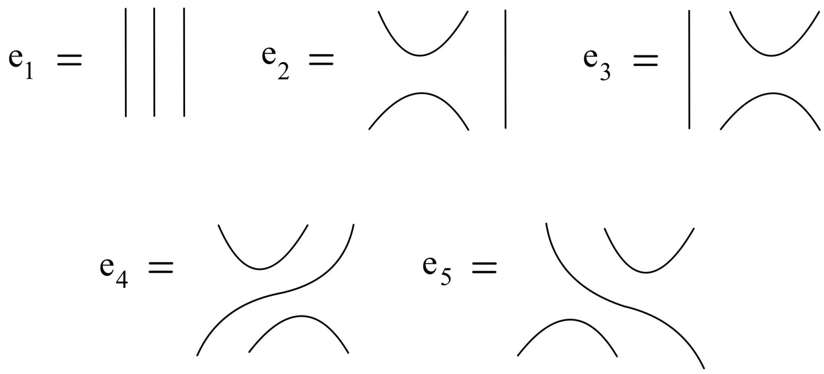
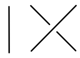


FIGURE 1.

Define T_1 to be the linear operator which adds the crossing σ_1 on top of a diagram in V: 

and define T_2 be the linear operator which adds the crossing σ_2 on top: 
We can use the defining relation for the bracket polynomial to write the matrix for T_1 in the basis of V, see Figure 2.

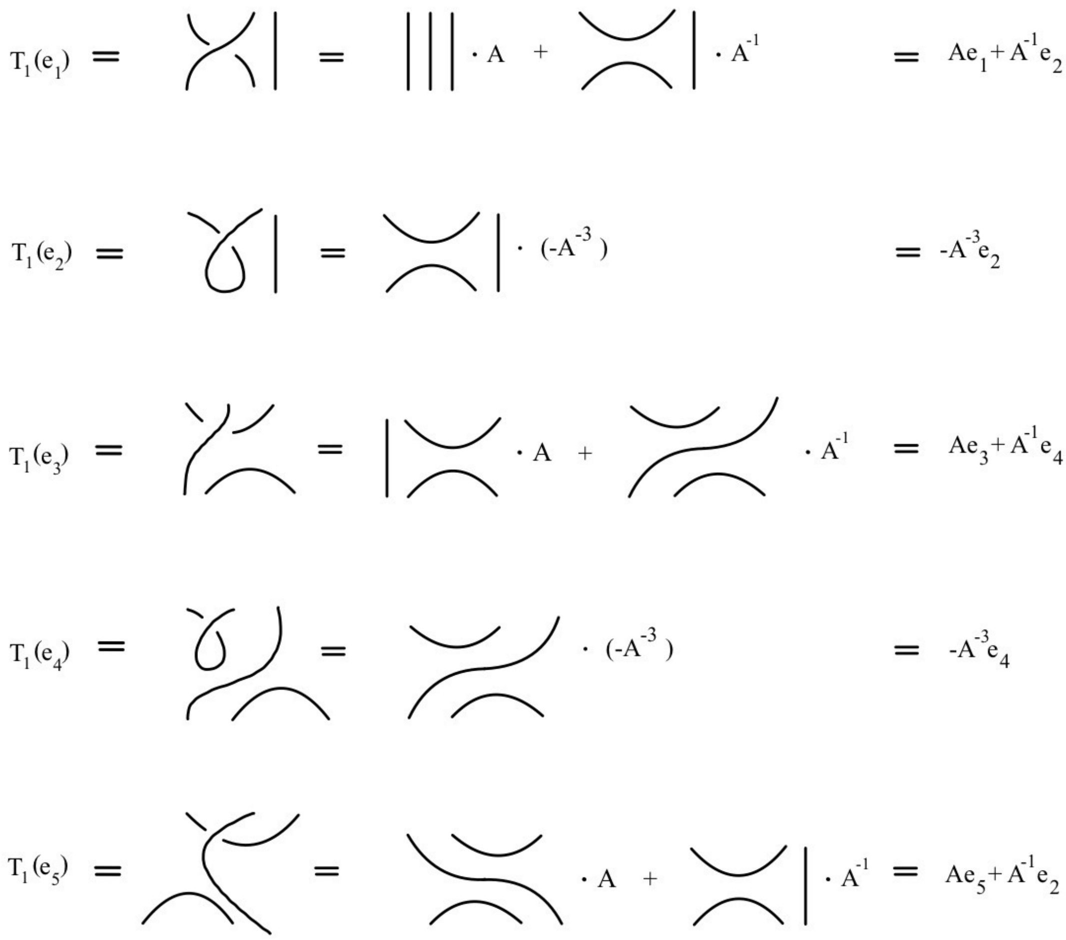


FIGURE 2.

6. Computations of Three Strands

Definition 6.1. *We define matrices T_1 and T_2 by the following equations*

$$T_1 = \begin{pmatrix} A & 0 & 0 & 0 & 0 \\ A^{-1} & -A^{-3} & 0 & 0 & A^{-1} \\ 0 & 0 & A & 0 & 0 \\ 0 & 0 & A^{-1} & -A^{-3} & 0 \\ 0 & 0 & 0 & 0 & A \end{pmatrix}$$

$$T_2 = \begin{pmatrix} A & 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 \\ A^{-1} & 0 & -A^{-3} & A^{-1} & 0 \\ 0 & 0 & 0 & A & 0 \\ 0 & A^{-1} & 0 & 0 & -A^{-3} \end{pmatrix}$$

It is easy to check the following formula for the product T_1T_2.

Proposition 6.2. *The product of T_1T_2 is the matrix*

$$T_1T_2 = \begin{pmatrix} A^2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -A^{-4} \\ 1 & 0 & -A^{-2} & 1 & 0 \\ A^{-2} & 0 & -A^{-4} & 0 & 0 \\ 0 & 1 & 0 & 0 & -A^{-2} \end{pmatrix}$$

To every braid on 3 strands, we associate a matrix by replacing σ_1 and σ_2 with T_1 and T_2.

The following proposition shows that T_1 and T_2 satisfy braid relations (1), and therefore this assignment of matrices to braids is well defined.

Proposition 6.3. $T_1T_2T_1 = T_2T_1T_2$

Proof. We compute this by matrix multiplication.

$$T_1T_2T_1 = \begin{pmatrix} A^3 & 0 & 0 & 0 & 0 \\ A & 0 & 0 & 0 & -A^{-3} \\ A & 0 & 0 & -A^{-3} & 0 \\ A^{-1} & 0 & -A^{-3} & 0 & 0 \\ A^{-1} & -A^{-3} & 0 & 0 & 0 \end{pmatrix}$$

$$T_2T_1T_2 = \begin{pmatrix} A^3 & 0 & 0 & 0 & 0 \\ A & 0 & 0 & 0 & -A^{-3} \\ A & 0 & 0 & -A^{-3} & 0 \\ A^{-1} & 0 & -A^{-3} & 0 & 0 \\ A^{-1} & -A^{-3} & 0 & 0 & 0 \end{pmatrix}$$

Thus, $T_1T_2T_1 = T_2T_1T_2$

□

Example 6.4. *The figure eight knot is obtained as the closure of the braid $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ and corresponds to the matrix*

$$T_1T_2^{-1}T_1T_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{2A^4-1}{A^6} & \frac{-A^{12}+(A^4-1)^2}{A^8} & 0 & 0 & \frac{A^8-A^4+1}{A^2} \\ 2A^2-A^6 & 0 & A^8-A^4 & \frac{-A^8+A^4-1}{A^2} & 0 \\ \frac{-A^8+2A^4-1}{A^4} & 0 & \frac{A^8-A^4+1}{A^2} & \frac{-A^{12}+(A^4-1)^2}{A^8} & 0 \\ 1 & \frac{-A^8+A^4-1}{A^2} & 0 & 0 & A^8-A^4 \end{pmatrix}$$

The $(3,m)$ torus links are obtained as the closures of the braids $(\sigma_1\sigma_2)^m$. The corresponding matrices are $(T_1T_2)^m$.

Example 6.5. *The braid $(\sigma_1\sigma_2)^3$ is a full twist, or a 360° rotation. The corresponding matrix is*

$$(T_1T_2)^3 = \begin{pmatrix} A^6 & 0 & 0 & 0 & 0 \\ \frac{A^8-1}{A^4} & A^{-6} & 0 & 0 & 0 \\ \frac{A^8-1}{A^4} & 0 & A^{-6} & 0 & 0 \\ \frac{A^4-1}{A^2} & 0 & 0 & A^{-6} & 0 \\ \frac{A^4-1}{A^2} & 0 & 0 & 0 & A^{-6} \end{pmatrix}$$

Theorem 6.6. *For $m = 3k$ we have*

$$(T_1T_2)^{3k} = \begin{pmatrix} A^{6k} & 0 & 0 & 0 & 0 \\ x_k & A^{-6k} & 0 & 0 & 0 \\ x_k & 0 & A^{-6k} & 0 & 0 \\ y_k & 0 & 0 & A^{-6k} & 0 \\ y_k & 0 & 0 & 0 & A^{-6k} \end{pmatrix} \tag{2}$$

where

$$x_k = \frac{\left(1 + A^{12} + A^{24} + A^{36} + \cdots + A^{12(k-1)}\right)(A^8 - 1)}{A^{6k-2}}$$

and

$$y_k = \frac{\left(1 + A^{12} + A^{24} + A^{36} + \cdots + A^{12(k-1)}\right)(A^4 - 1)}{A^{6k-4}}$$

Proof. We prove by induction in k. The base case $k = 1$ is covered by Example 6.5. To prove the inductive step, assume that $(T_1T_2)^{3k}$ satisfies (2). Then

$$(T_1T_2)^{3(k+1)} = (T_1T_2)^{3k}(T_1T_2)^3,$$

and by multiplying the matrices, we get the recursion relations

$$x_{k+1} = A^6x_k + A^{-6k}\frac{A^8-1}{A^4}, y_{k+1} = A^6y_k + A^{-6k}\frac{A^4-1}{A^2}.$$

It remains to prove that x_k and y_k are given by the above formulas.
Now, let's prove the formula for x_k.
Base case $k = 1$:

$$x_1 = \frac{(A^8-1)}{A^4}$$

For the inductive step, assume that the formula for x_k is true.

$$\begin{aligned} x_{k+1} &= \frac{\left(1 + A^{12} + \cdots + A^{12(k-1)}\right)(A^8-1)}{A^{6k-2}} \cdot A^6 + \frac{(A^8-1)}{A^4} \cdot A^{-6k} \\ &= (A^8-1)\left(\frac{\left(1 + A^{12} + \cdot + A^{12(k-1)}\right) \cdot A^6 \cdot A^6}{A^{6k-2} \cdot A^6} + \frac{1}{A^{6k-2} \cdot A^6}\right) \\ &= (A^8-1)\left(\frac{A^{12} + A^{24} + \cdot + A^{12k} + 1}{A^{6k-2} \cdot A^6}\right) \end{aligned}$$

The formula for y_k is proven similarly.
Thus, the proof is completed. □

Theorem 6.7. *The bracket polynomial for the torus link $T(3, 3k)$ is given by the equation*

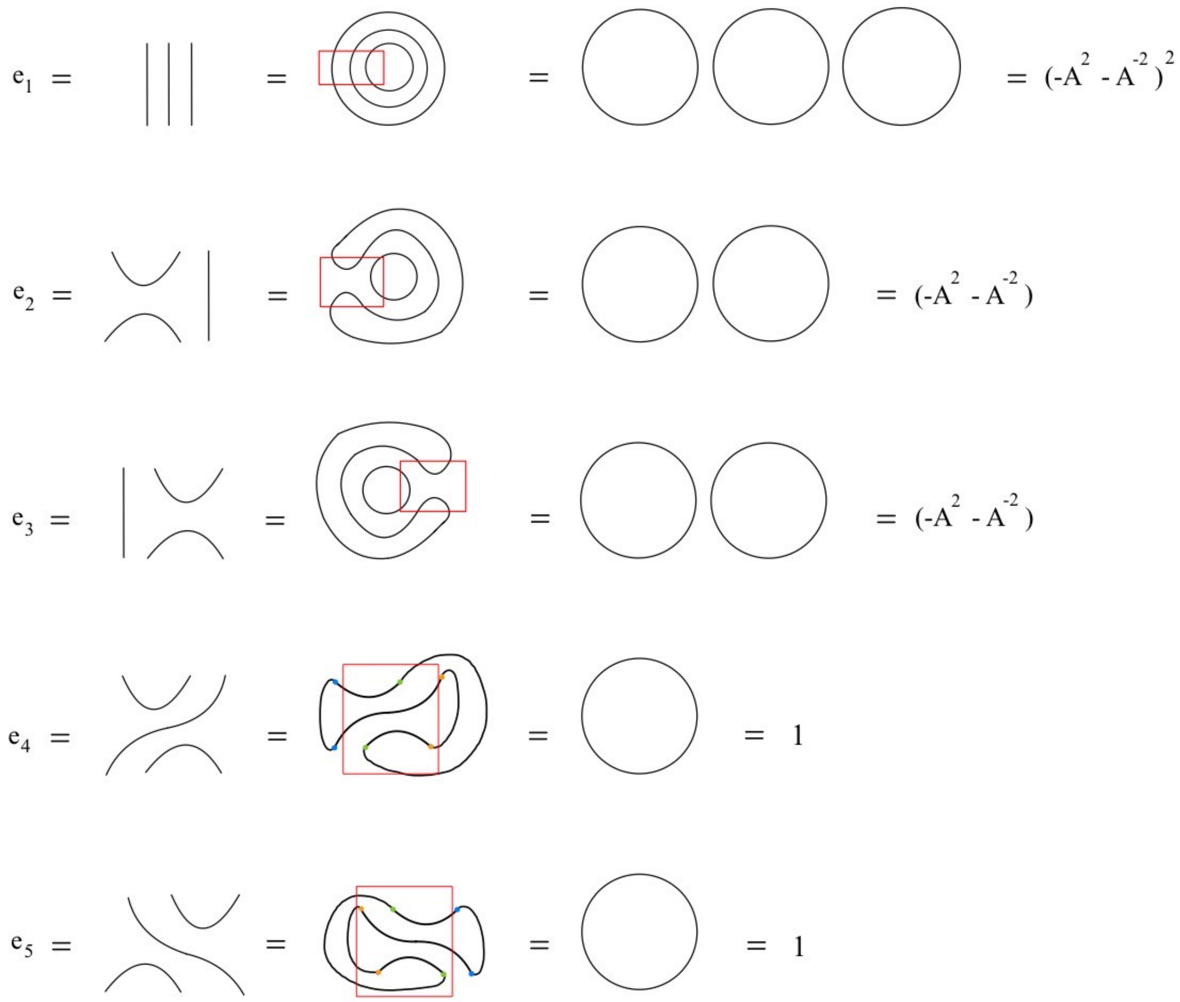
$$\langle T(3,3k)\rangle = A^{6k}(-A^2 - A^{-2})^2 + 2x_k(-A^2 - A^{-2}) + 2y_k$$

where x_k and y_k are as in Theorem 6.6.

Proof. First, we express $(T_1T_2)e_1$ in the basis of V using Theorem 6.6:

$$(T_1T_2)^{3k} \cdot e_1 = A^{6k}e_1 + x_ke_2 + x_ke_3 + y_ke_4 + y_ke_5 \tag{3}$$

Then, we compute the closure of each e_i for $e = 1, \cdots, 5$ by using Rules 1 and 2



Now the bracket polynomial of $T(3,3k)$ can be obtained as follows:

$$\langle T(3,3k)\rangle = \langle (T_1T_2)^{3k}\cdot e_1\rangle = A^{6k}\langle e_1\rangle + x_k\langle e_2\rangle + x_k\langle e_3\rangle + y_k\langle e_4\rangle + y_k\langle e_5\rangle$$

We then use the values for $\langle e_i\rangle$ to obtain the formula given in Theorem.

□

7. Acknowledgements

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References

[1] C. Adams. The Knot Book.