HEEGAARD SPLITTINGS OF SURFACE BUNDLES OVER THE CIRCLE

Undergraduate Thesis
University of California, Davis

Patrick Valle

Advisors: Gabriel Islambouli and Jennifer Schultens

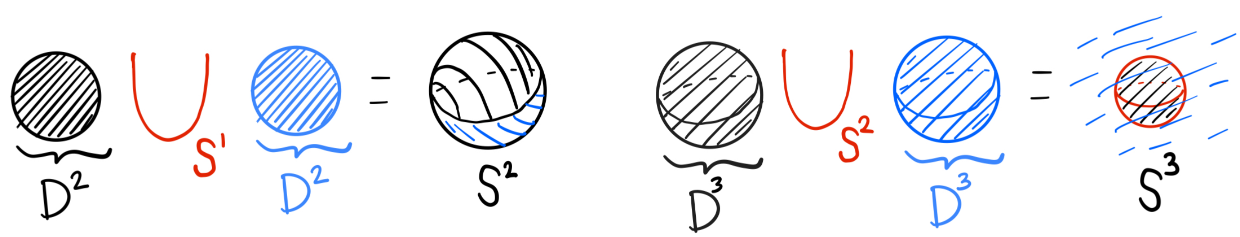
June 2025

Introduction

This thesis serves as an exposition of a few topics on 3-manifolds that are needed to build up to an understanding of Heegaard Splittings of Surface Bundles over S^1. We mainly reference Schultens' book on 3-manifolds and Jesse Johnson's notes on Heegaard Splittings (Schultens [12], Johnson [5]). We will explore Heegaard splittings through the lens of surface bundles over the circle, that is, Heegaard splittings of 3-manifolds that admit a fibration whose fibers are surfaces and whose base space is S^1. These manifolds represent a natural next step in complexity beyond simpler product manifolds.

As for Heegaard Splittings, over the past century they have become a fundamental tool in understanding the topology of 3-manifolds. They are a way of decomposing 3-manifolds, and as Hempel puts it, they have a "simple description" and "visual geometric appeal," (Hempel [2]). This simple construction is central to many modern approaches in 3-manifold topology, such as Heegaard Floer homology, Dehn Surgery, and thin position techniques. What follows is a motivating, albeit informal, example of Heegaard Splittings:

Example 0.1. Take two 2-disks D^2's, and union them along their circle boundary S^1 to make S^2, the 2-dimensional sphere. When this decomposition is done analogously for the 3-dimensional sphere S^3, specifically for 3-manifolds, it is known as a Heegaard Splitting.



In the following chapter, we seek to formalize our language for these disks, spheres, and more. Thus, keeping this example in mind, let's take many steps back to cover some preliminaries. We start with asking ourselves...

1. What is a Manifold?

Manifolds will serve to set the stage, as many theorems in low-dimensional topology are formulated in terms of manifolds. In the first part of our example, our D^2's and S^2's were

2-manifolds, otherwise known as *surfaces*. Intuitively, we thought of the D^2's as bent, as if made of rubber to form the hemispheres of S^2. This bendability is due to their topological nature, specifically to their homeomorphisms. Formally defined:

Definition 1.1. A *homeomorphism* is a continuous map between topological spaces with a continuous inverse.

Example 1.2. Stereographic projection is the homeomorphism given by:

$$f : S^n \setminus \{pt\} \to \mathbb{R}^n, \quad \text{where} \quad x \mapsto \frac{1}{1 - x_{n+1}} \cdot (x_1, \ldots, x_n).$$

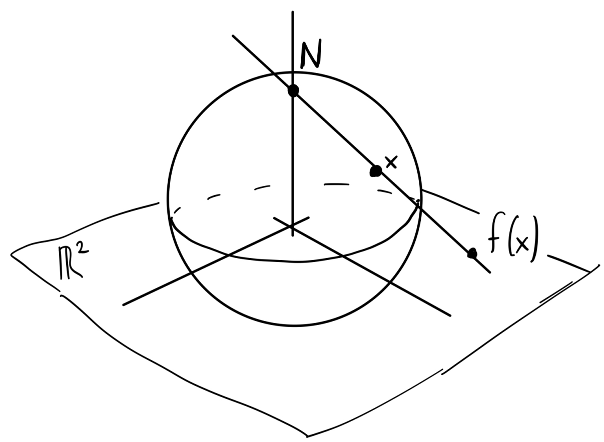


FIGURE 1. Stereographic Projection of S^2

Definition 1.3. A *topological n-manifold* is defined as a topological space that is second countable, Hausdorff, and *locally Euclidean of dimension n* (that is, it is locally homeomorphic to $\mathbb{R}^n$).

Our manifolds of interest will satisfy Hausdorff and second countability properties by virtue of being compact. Although second countable and Hausdorff are essential properties (see Jänich [4], pg. 17), we focus mainly on the locally Euclidean aspect (eg., Extrinsically, $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ gives a nice visible visualization, but it relies on $\mathbb{R}^3$ to inherit its topology via the subspace topology.)

Intuitively, "locally euclidean of dimension n" means that it looks like $\mathbb{R}^n$ near any point in our manifold (eg., S^2 has global spherical geometry, but near every point it resembles $\mathbb{R}^2$ with Euclidean geometry). Formally:

Definition 1.4. A manifold M is *locally Euclidean* if and only if $\forall x \in M$, there is an open subset U of M such that $x \in U$ and there are n-dimensional charts, i.e., homeomorphisms $f : U \to V$, where V is an open subset of $\mathbb{R}^n$.

Additionally, U is known as the *chart domain*, and a set of charts whose chart domains cover the whole of M is called an *Atlas* (i.e., an open cover $\{U_i\}_{i\in I}$ such that $\bigcup U_i = M$, with a collection of homeomorphisms $f_i : U_i \to V_i$, where V_i are open subsets of $\mathbb{R}^n$. Hence, a topological n-manifold admits an n-dimensional atlas. We redefine it the following way:

Definition 1.5. A *topological n-manifold* is a defined as a topological space that is second countable, Hausdorff, and admits an n-dimensional atlas.

Example 1.6. $S^n = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \;\middle|\; x_1^2 + \cdots + x_{n+1}^2 = 1\right\}$ is an n-manifold. We equip $S^n \subset \mathbb{R}^{n+1}$ with the subspace topology inherited from $\mathbb{R}^{n+1}$. Since $\mathbb{R}^{n+1}$ is Hausdorff and second-countable, S^n is thus Hausdorff and second-countable. Now let

$$U_+ = S^n \setminus \{N\}, \quad \text{where} \quad N = (0, \ldots, 0, 1),$$

$$U_- = S^n \setminus \{S\}, \quad \text{where} \quad S = (0, \ldots, 0, -1).$$

Then stereographic projection gives us our homeomorphisms:

$$\varphi_+ : U_+ \longrightarrow \mathbb{R}^n, \quad \varphi_- : U_- \longrightarrow \mathbb{R}^n.$$

A complete proof would show that the stereographic projections are indeed homeomorphisms and thus valid charts. Hence S^n admits the atlas:

$$\mathcal{A} = \{ (U_+, \varphi_+), (U_-, \varphi_-)\}, \quad \text{where} \quad U_+ \cup U_- = S^n.$$

Example 1.7. The n-dimensional torus $T^n = S^1 \times \cdots \times S^1$ is a manifold. We can view it as a quotient space $T^n = \mathbb{R}^n/\mathbb{Z}^n$, with quotient map $q : \mathbb{R}^n \to T^n$: which identifies all points in the same orbit $\{x + k : k \in \mathbb{Z}^n\}$. The orbit corresponds to the group action $\mathbb{Z}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, where $((k_1, \ldots, k_n), (x_1, \ldots, x_n)) \longmapsto (x_1 + k_1, \ldots, x_n + k_n)$.

That is, $\mathbb{Z}^n$ acts on $\mathbb{R}^n$ by discrete integer translations. We mention group actions because we can consider the fundamental domain for this action as the unit n-cube:

$$C^n = [0, 1]^n \subset \mathbb{R}^n.$$

We can see T^n is a manifold by picking an open ball $U \subset \mathbb{R}^n$ around $x \in \mathbb{R}^n$, $\forall [x] \in T^n$, of radius less than $\frac{1}{2}$. Then no nonzero integer translate of the ball overlaps U. This restricts

q to a homomorphism from U onto its image $q(U)$, and $q|_{q(U)}^{-} : q(U) \to U$ serves as a chart. The collection of these charts forms the Atlas for T^n.

Remark. Every compact surface has a polygonal representation. This can be proven using triangulations of manifolds, which we will view in chapter 3.

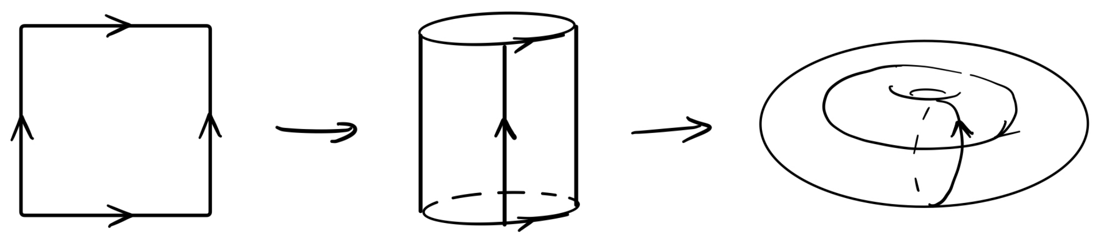


FIGURE 2. Fundamental Domain and Polygonal Representation for T^2

We can also take a product of manifolds to obtain a different manifold:

Lemma 1.8. *The product of two manifolds M and N, of dimensions m and n respectively, is a manifold of dimension $m + n$.*

Proof. Suppose for every $p \in M$, there exists an open neighborhood $U \subset M$ with a chart $\varphi\colon U \longrightarrow \varphi(U) \subset \mathbb{R}^m$. Similarly, for every $q \in N$, there exists an open neighborhood $V \subset N$ with a chart $\psi\colon V \longrightarrow \psi(V) \subset \mathbb{R}^n$. Since p and q lie in the domains of these charts, it follows that $(p, q) \in U \times V \subset M \times N$. Then consider the map

$$\varphi \times \psi\colon U \times V \longrightarrow \varphi(U) \times \psi(V) \subset \mathbb{R}^{m+n}$$

It is straightforward to check that $\varphi \times \psi$ is well-defined, bijective, and continuous, and its inverse is also continuous, thus it is a homeomorphism. Therefore, as (p, q) varies over $M \times N$, we collect charts of the form $(U \times V,\ \varphi \times \psi)$, forming an atlas for $M \times N$. Hence $M \times N$ is a manifold of dimension $m + n$. $\square$

Example 1.9. Since $T^n = S^1 \times \cdots \times S^1$ and S^1 is a manifold, it follows that T^n can be shown to be a manifold by our lemma.

Thus far, we have considered manifolds in the sense of Definition 0.4, which implicitly assumes that each point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$. We can broaden this idea by allowing neighborhoods homeomorphic to open subsets of the half-space

$$\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}.$$

This then gives us manifolds with boundary. Formally:

Definition 1.10. An *n-manifold with boundary* is a second countable, Hausdorff topological space M such that for every point $p \in M$, there is a neighborhood of p homeomorphic either to an open subset of $\mathbb{R}^n$ (in which case p is called an *interior point*) or to an open subset of $\mathbb{H}^n$ (in which case p is a *boundary point*, landing on $x_n = 0$).

Additionally, the boundary of an n-manifold with boundary turns out to be an $(n-1)$-dimensional manifold *with no boundary.* In particular, if M is compact and $\partial M = \varnothing$, then we recover our usual notion of a boundaryless manifold and call it a *closed manifold.*

Remark. The product of two manifolds without boundary is a manifold without boundary.

Example 1.11. The 2-dimensional disk:

$$D^2 \;=\; \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\},$$

shown in Figure 3, is a manifold with boundary. Its boundary is $\partial D^2 = S^1$. Moreover, S^2 and T^2 are examples of *closed* manifolds, since they are both compact and have no boundary.

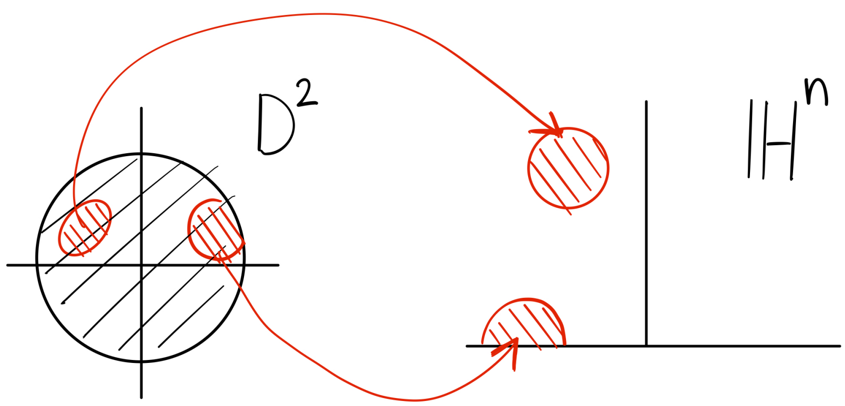


FIGURE 3

We know 1-manifolds are fully classified, and in particular that there is exactly one connected, closed 1-manifold: S^1.

As for 2-manifolds, they are also fully classified, and thus far we have only seen T^2 and S^2, both of which are closed and orientable. To consider more types of surfaces, we introduce the *connect sum* operation and the topological invariant of *orientability.*

Definition 1.12. We denote by

$$M\#N \;=\; \big(M\setminus D^n\big)\cup_\varphi \big(N\setminus D^n\big)$$

the *connect sum* of two n-manifolds M and N, where φ is a homeomorphism identifying the boundary spheres of $M\setminus D^n$ and $N\setminus D^n$. In other words, we remove $\mathbb{R}^2$ neighborhoods from each surface, then glue the resulting surfaces together. A nice visual for $T^2\#T^2$ is given in Figure 4.

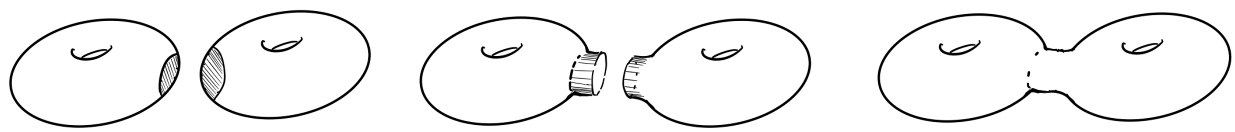


FIGURE 4

Note that S^2 is the identity element in the connect sum operation. In fact, if an n-manifold M can be written as $M_1\#S^1$, then this connected sum is trivial, and we call M *prime*. Formally:

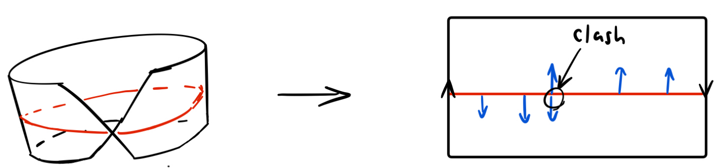
Definition 1.13. An n-manifold M is called *prime* if whenever M can be written as a connected sum

$$M \;=\; M_1\#M_2,$$

then one of M_1 or M_2 must be the n-sphere S^2. Equivalently, any separating 2-sphere in M bounds a 3-ball. This is because $M = M_1\#M_2$ uses an S^2 that cuts (or separates) M into two pieces, then caps off each piece with a 3-ball.

To understand what it means for a surface to be orientable or non-orientable, we consider the following example.

Example 1.14. The *Möbius strip* has an orientation reversing closed path (shown in red).



By gluing a disk to the boundary of the Möbius strip, one obtains $\mathbb{RP}^2$, an 2-dimensional manifold that can also be viewed as the sphere S^2 with antipodal points identified. Since $\mathbb{RP}^2$ contains a Möbius strip in its polygonal representation, it has an orientation reversing loop and is thus non-orientable. In the following theorem, we see $\mathbb{RP}^2$ serves as a building block for non-orientable surfaces.

Definition 1.15. A surface is orientable if it contains no orientation-reversing closed path.

Theorem 1.16 (Classification of Surfaces)**.** *Every closed, connected 2-manifold is homeomorphic to exactly one of the following types:*

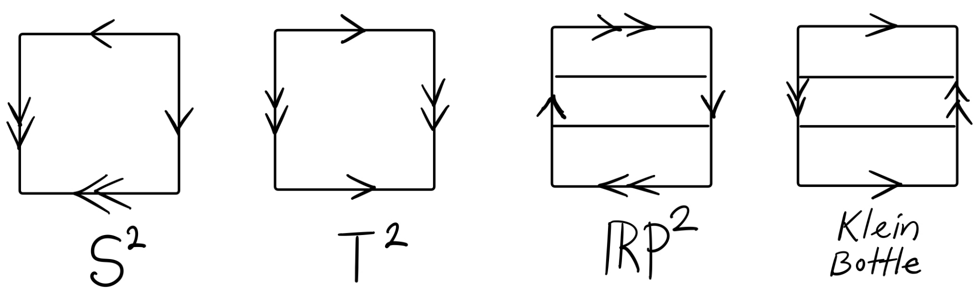
$$S^2, \quad T^2, \quad T^2\#T^2, \quad \dots \quad (\textit{orientable surfaces of genus } g),$$

$$\mathbb{RP}^2, \quad \mathbb{RP}^2\#\mathbb{RP}^2, \quad \mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2, \quad \dots \quad (\textit{non-orientable surfaces of genus } g).$$

Here, the *genus* of a surface is defined either by the number of tori T^2 or the number of projective planes $\mathbb{RP}^2$ in the connected sum. We may denote a genus g orientable surface by $\#^g T^2$ and a genus g non-orientable surface by $\#^g \mathbb{RP}^2$.

Note also that there is a classification of 2-manifolds with boundary.

Example 1.17. A standard way to visualize these surfaces is via polygonal representations with identified edges.



Remark. As for 3-manifolds, we can list all possible orientable, closed, connected 3-manifolds using constructions like Heegaard splittings or surgery. Determining whether two such 3-manifolds are homeomorphic relies on several deep results, such as Kneser's theorem, where we decompose a 3-manifold as a connect sum of prime 3-manifolds. Then using JSJ decomposition we can further decompose it along tori into atoroidal or Seifert fibered spaces. Each piece would admit one of Thurston's eight geometries (see Perelman [10]). However, that is all beyond the scope of this thesis.

For our purposes, we now return to some of the simplest 3-manifolds we can build: product manifolds. Let us see some examples of all the different types of product 3-manifolds without boundary that are possible.

Example 1.18. We have:

$$S^3 \times S^1, \quad T^2 \times S^1, \quad K \times S^1, \quad \mathbb{RP}^2 \times S^1,$$

where K is the Klein bottle. More generally, we can form

$$(\text{orientable or non-orientable surface of genus } g) \times S^1$$

for any $g \geq 0$.

These examples all come from taking the product of a surface with S^1. However, they do not account for the ways a surface can *twist* as it goes around the circle. Indeed, product manifolds are the simplest examples of fiber bundles over S^1, in the sense that they admit *no twist* (i.e., zero monodromy).

2. Twisting, Fibration, and Monodromy

In this chapter, we introduce concepts such as *Dehn twists, monodromy,* and the *mapping class group*—all of which formalize "twists" and help classify surface bundles over S^1. We begin by formally introducing the bundle structure:

Definition 2.1. A *fiber bundle* is a continuous surjective map

$$\pi : E \longrightarrow B$$

from a topological space E (the *total space*) onto a topological space B (the *base*), together with a fiber F satisfying the following local triviality condition: For every $b \in B$, there is an open neighborhood U of b and a homeomorphism

$$\varphi : \pi^{-1}(U) \longrightarrow U \times F$$

making the following diagram commute:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ \downarrow \pi & & \downarrow \pi_{proj} \\ U & \xrightarrow{\text{id}} & U \end{array}$$

where π_{proj} is the projection onto the first factor. That is, φ restricts on each fiber $\pi^{-1}(u)$ for $u \in U$, to a homeomorphism onto $\{u\} \times F$. The bundle is *locally trivial* over each U.

Remark. In Homotopy Theory, a *fibration* is a continuous map $p : E \to B$ satisfying the *Homotopy Lifting Property* (HLP). Because local triviality implies HLP, every fiber bundle is automatically a fibration; the converse is not always true. When the fiber is 0-dimensional, the fiber bundle is simply a *covering space*.

We now focus on bundles whose fiber is a surface, and whose base is a circle S^1.

Definition 2.2. A *surface bundle over* S^1 is a pair (M, π) consisting of a 3–manifold M and a fiber–bundle projection

$$\pi : M \longrightarrow S^1,$$

whose fiber is a closed, connected, orientable surface Σ_g. Equivalently, every surface bundle over S^1 can be realized as a *mapping torus*

$$MT(\varphi) \;=\; \frac{\Sigma_g \;\times\; [0,1]}{(x,1) \;\sim\; \big(\varphi(x), 0\big)}$$

of some self–homeomorphism $\varphi\colon \Sigma_g \to \Sigma_g$, called the *monodromy* of the bundle.

Example 2.3. One of the major breakthroughs in modern 3-manifold theory is Ian Agol's proof of Thurston's Virtual Fibration Conjecture, which states that any closed hyperbolic 3-manifold admits a finite covering space N that is a surface bundle over S^1.

Definition 2.4. Given Σ_g, the *mapping torus* of an orientation-preserving homeomorphism $\varphi : \Sigma_g \to \Sigma_g$, is the quotient space

$$MT(\varphi) = \frac{\Sigma_g \times [0,1]}{\sim}$$

where $\sim$ is the equivalence relation given by $(\varphi(x), 0) \sim (x, 1) \quad \text{for all } x \in \Sigma_g.$

Example 2.5. Let $M = [-1, 1]$ and $f = \text{id}$. Define

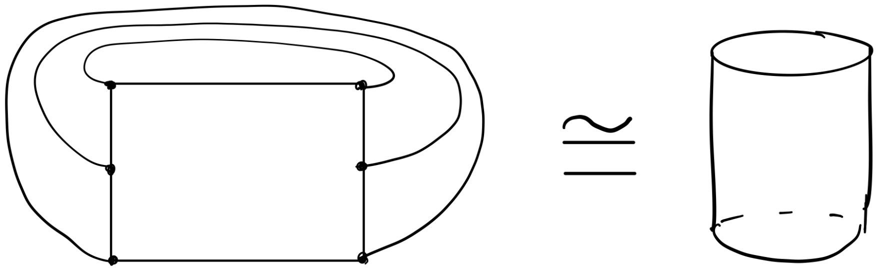
$$MT(\text{id}) \;=\; \frac{M \times [0,1]}{\sim}$$

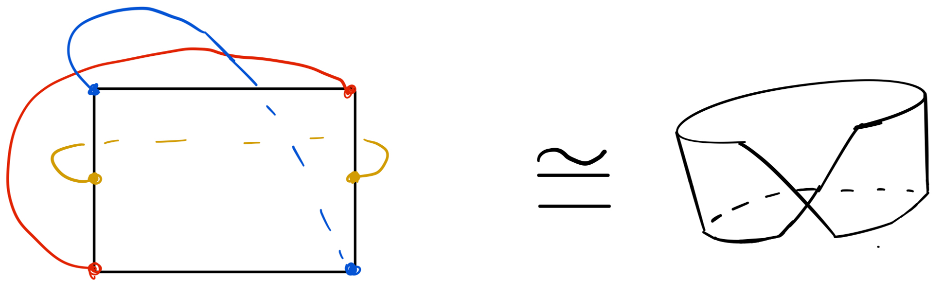
where $(x, 0) \sim (x, 1)$. This constructs a cylinder.

Now suppose $g(x) = -x$. Then

$$MT(g) \;=\; \frac{[-1,1] \times [0,1]}{\sim}$$

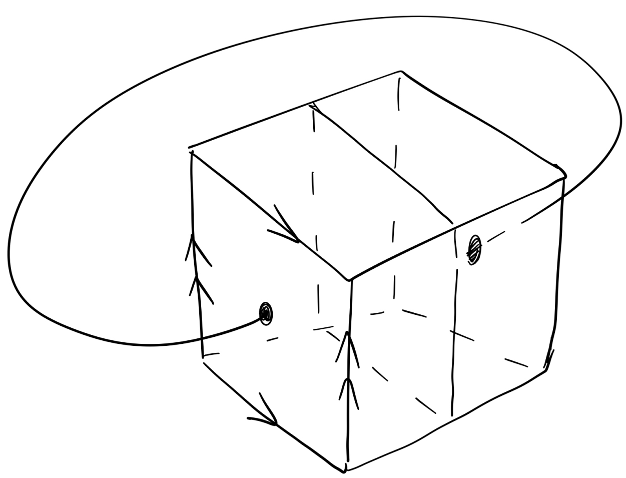
where $(x, 0) \sim (-x, 1)$. This is the usual Möbius band.





Example 2.6. Let $M = T^2$ (the 2-torus) and $f = \text{id}$. Then

$$MT(\text{id}) \;=\; \frac{T^2 \times [0,1]}{\sim} \;\cong\; T^2 \times S^1 \;\cong\; T^3.$$



Proposition 2.7. *Any surface bundle over S^1 can be realized with the mapping torus construction. See Martelli [8] for a proof.*

The homeomorphism φ used in the mapping torus construction lives in the *mapping class group*, the group of all orientation-preserving homeomorphisms of a surface modulo isotopy. Formally:

Definition 2.8. Let Σ be a closed, orientable surface. The *mapping class group* of Σ is defined as:

$$\mathrm{Mod}(\Sigma) = \frac{\mathrm{Homeo}^+(\Sigma)}{\mathrm{Homeo}_0(\Sigma)}$$

where $\mathrm{Homeo}^+(\Sigma)$ is the group of orientation-preserving homeomorphisms of Σ, and $\mathrm{Homeo}_0(\Sigma) = \{f \in \mathrm{Homeo}^+(\Sigma) : f \text{ is isotopic to } \mathrm{id}_\Sigma\}$.

Example 2.9. Let $M = [-1, 1]$. Consider the self-homeomorphism

$$f : M \to M, \quad f(x) = -x.$$

Then f reflects the interval $[-1, 1]$ about 0. The drawing below shows $\mathrm{id} \in \mathrm{Mod}(M)$ and $f \notin \mathrm{Mod}(M)$.

One important element of the mapping class group are *Dehn twists*.

Definition 2.10. Let S be an orientable surface, and let $c \subset S$ be a simple (does not intersect itself) closed curve. A *Dehn twist* about c is a homeomorphism

$$T_c : S \to S$$

such that:

- There exists a closed subset $A \subset S$ that is an annulus, i.e. $A \cong S^1 \times [0, 1]$, where c is the middle circle of A.
- $T_c|_{S \setminus A}$ is the identity on $S \setminus A$.
- $T_c|_A$ is given by
$$T_c(z, t) \ = \ \big(z \cdot e^{2\pi i\, t},\ t\big),$$
for all $(z, t) \in S^1 \times [0, 1]$.

Figure 5 shows a Dehn twist $T_c(d)$ and its mapping torus $MT(\mathrm{T}_c) \ = \ \frac{T^2 \times [0,1]}{\sim}$

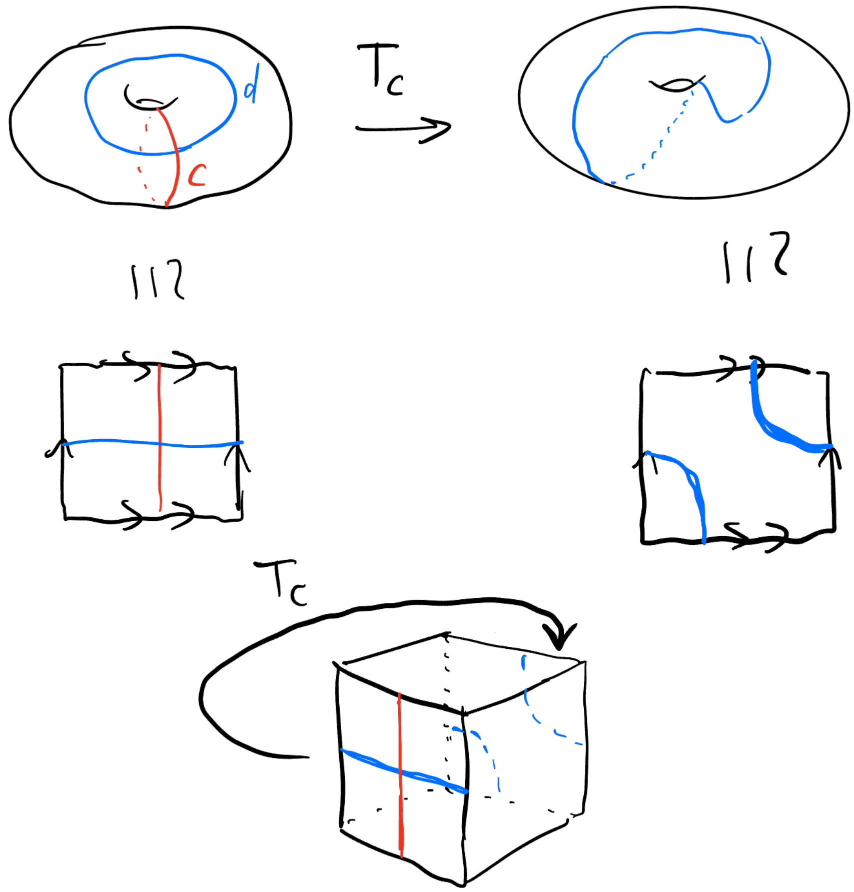


FIGURE 5. A Dehn twist $T_c(d)$ and its mapping torus $MT(T_c)$

Theorem 2.11 (Lickorish [7])**.** *The mapping class group of a compact, orientable surface can be generated by finitely many Dehn twists. That is, every element of the mapping class group can be expressed as a product of Dehn twists.*

Surface bundles over S^1 come with a *monodromy representation* that describes how the fiber surface "twists" as we go around the circle. Following the exposition of Salter–Tshishiku [11], suppose we have a surface Σ_g with homeomorphism $\varphi : \Sigma_g \to \Sigma_g$ forming a mapping. Then formally, a *monodromy representation* is a group homomorphism

$$\rho : \pi_1(S^1) \to \text{Mod}(\Sigma_g).$$

Because $\pi_1(S^1) \cong \mathbb{Z}$ is generated by a single loop γ, ρ must satisfy

$$\rho(\gamma^n) = \rho(\gamma)^n, \quad \text{for } n \in \mathbb{Z},$$

once we fix the image $\rho(\gamma) = [\varphi] \in \mathrm{Mod}(\Sigma_g)$, we have fixed the entire representation. That is, the "twisting" is precisely encoded by $[\varphi]$. Hence $MT(\varphi)$ is characterized by its isotopy class of φ in $\mathrm{Mod}(\Sigma_g)$.

3. From Triangulations to Handlebodies

There is one last type of manifold we must bring up before we arrive at Heegaard splittings: a *triangulated manifold*, built by gluing simplices together. Ultimately, in this chapter, we want to get to the point where we take a triangulated manifold and consider its skeleton. We will then thicken that skeleton to form a handlebody, which will make up a third of our Heegaard Splitting.

Definition 3.1. A *simplicial complex* is an ordered pair $K = (V, F)$, where V is the set of vertices, and F is a set of simplices (subsets of V) satisfying:

- If $v \in V$, then $\{v\} \in F$. (This ensures every vertex is itself a 0-dimensional simplex.)
- If $A \in F$ and $B \subset A$, then $B \in F$. (This ensures that any face of a simplex is also part of the complex.)

Definition 3.2. Let $K = (V, F)$ be a simplicial complex. A *subcomplex* is given by $K' = (V', F')$, where $V' \subset V$ and $F' \subseteq F$.

Additionally, The *n-skeleton* of K, denoted K^n, is the subcomplex consisting of those simplices whose dimension is at most n. Formally,

$$K^n = \{\sigma \in F \mid \dim(\sigma) \leq n\}.$$

Note that the 1-skeleton K^1 is a graph, i.e., a 1-dimensional CW complex.

Example 3.3. A simplex containing $k+1$ vertices is said to have *dimension* k. Hence:

0-simplices are vertices, 1-simplices are edges, 2-simplices are triangles, ...

Now we turn this simplicial complex into a topological space by assigning each vertex to a point in Euclidean space. To do this we define the *support* of a point (which vertices are involved in that point's coordinate description), and realization(the subspace of $\mathbb{R}^n$ spanned by these vertices). Formally:

Definition 3.4. Let $x = (x_1, x_2, \ldots, x_N)$ be a point in $\mathbb{R}^N$. The *support* of x is

$$\mathrm{supp}(x) \;=\; \{\, i \;\mid\; x_i \neq 0\}.$$

Suppose $K = (V, F)$ is a simplicial complex where $V = \{1, 2, \ldots, N\}$. We identify each vertex $i \in V$ with the standard basis vector e_i in $\mathbb{R}^N$.

Then the *realization* of K, denoted $|K|$, is the set of all points

$$x \;=\; (x_1, x_2, \ldots, x_N) \;\in\; \mathbb{R}^N$$

such that:

$$x_i \geq 0 \quad \text{for all } i, \quad \sum_{i=1}^{N} x_i = 1, \quad \text{and} \quad \mathrm{supp}(x) \in F.$$

Example 3.5. Let $T = (V, F)$ be a simplicial complex where

$$V = \{a, b, c\}, \quad F = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}\}.$$

We identify a, b, c with the basis vectors e_a, e_b, e_c in $\mathbb{R}^3$. Since $\{a, b, c\} \in F$, a point (x_a, x_b, x_c) lies in $|T|$ precisely when $x_a,\, x_b,\, x_c \;\geq\; 0 \quad \text{and} \quad x_a + x_b + x_c = 1$. Hence the realization $|T|$ is just a filled triangle in $\mathbb{R}^3$.

With this, we can now formally define a triangulation:

Definition 3.6. Let X be a topological space. A *triangulation* of X is a pair (T, ϕ) where T is a simplicial complex and $\phi : |T| \longrightarrow X$ is a homeomorphism from the realization of T to X. We say that "T triangulates X."

The following theorem by Moise allows us to black-box triangulations. For our purposes, we only need to know whether a triangulation exists, and fortunately Moise's theorem gives us just that. So we can go around the details of explicitly triangulating by support and realizations.

Theorem 3.7. *(Moise [9]) Every compact 3-manifold admits a triangulation. That is, if $\mathcal{M}$ is a compact 3-manifold, then there exists a simplicial complex T and a homeomorphism $\phi\colon |T| \longrightarrow \mathcal{M}$.*

We say that a manifold that admits a triangulation is *piecewise-linear* (PL for short). Additionally, a subset $A \subset M$ is called *piecewise-linear* if it arises as the image of a subcomplex of K under the same homeomorphism ϕ. Formally, if L is a subcomplex of K, then A is piecewise-linear if $A \;=\; \phi\big(|L|\big)$. Finally, a map between two simplicial complexes

$\phi\colon |K| \longrightarrow |K'|$. is a *piecewise-linear map* if $\phi|_\sigma$ is linear for some simplex σ in K. From here on, we assume all manifolds and maps to be piecewise-linear.

The proof for Theorem 3.7 may be viewed in Moise's work [9]. We will, however, prove some related theorems, here, but we require a few more tools. We will need to be able to refine triangulations via *barycentric subdivision*. Also for the previously mentioned "thickening" of a skeleton, we will formally define it as taking a *regular neighborhood* of the skeleton.

Definition 3.8. Let $K = (V, F)$ be a simplicial complex. The *first barycentric subdivision* of K, denoted $K' = (V', F')$, is constructed as follows:

(1) For every simplex $\sigma \in F$, introduce a new vertex v_σ. Hence

$$V' = \{ v_\sigma \mid \sigma \in F \}.$$

(2) A set $\{ v_{\sigma_0}, v_{\sigma_1}, \ldots, v_{\sigma_m} \}$ is a simplex in K' if and only if

$$\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_m, \quad \text{where each } \sigma_i \in F.$$

That is, each σ_i is a face of σ_{i+1}.

Note that there exists a canonical homeomorphism $\psi : |K'| \longrightarrow |K|$ induced by this subdivision.

Example 3.9. Figure 6 shows the first and second barycentric subdivisions K' and K'' along with a regular neighborhood of the 1-simplices of K. From here we can see how K' satisfies the barycentric subdivision construction. For (1), consider its faces: the entire 2-simplex triangle $\sigma_{(a,b,c)}$, three 1-simplex edges $\sigma_{(a,b)}$, $\sigma_{(b,c)}, \sigma_{(c,a)}$, and three 0-simplex vertices $\sigma_{(a)}$, $\sigma_{(b)}, \sigma_{(c)}$. We can see how for each face we add a new vertex v_σ and we get a new vertex set $V' = \{ v_{\sigma_{(a,b,c)}}, v_{\sigma_{(a,b)}}, v_{\sigma_{(b,c)}}, v_{\sigma_{(c,a)}}, v_{\sigma_{(a)}}, v_{\sigma_{(b)}}, v_{\sigma_{(c)}} \}$.

For (2), we see each new triangle 2-simplex corresponds to a a chain of faces in the original triangle. That is, $\sigma_{(a)} \subset \sigma_{(a,b)} \subset \sigma_{(a,b,c)}$.

Definition 3.10. Let $|L|$ be the realization of a subcomplex $L \subseteq K$ inside a 3-manifold M. A *regular neighborhood* of $|L|$ can be constructed as follows: Take the second barycentric subdivision K'' of K, and let $\psi : |K''| \longrightarrow |K|$ be its canonical map. Then there is a unique subcomplex $L' \subseteq K''$ such that $\psi(|L'|) = |L|$. We define $\mathcal{N} = \{\sigma \in F'' \mid \sigma \cap L' \neq \varnothing\}$, i.e. all simplices in K'' that intersect L' in at least one point. The *closed regular neighborhood*

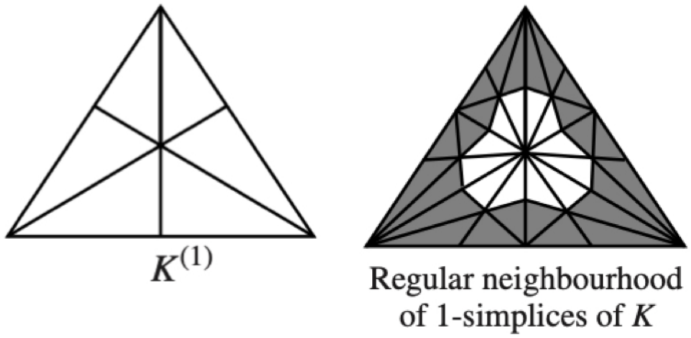


FIGURE 6. From Lackenby [6]

of $|L|$ is then

$$N(L) \;=\; \Psi\Big(\bigcup_{\sigma\in\mathcal{N}}\sigma\Big).$$

We can similarly define an *open* regular neighborhood by taking interiors of the simplices $\sigma \in \mathcal{N}$

While the shape of a regular neighborhood can depend on how we choose our subdivision, the following theorem tells us that this choice is topologically unique.

Theorem 3.11. *Let M be a manifold arising from a simplicial complex K, so we have $\psi : |K| \longrightarrow M$. Then let $|L| \subset M$ be the image of a subcomplex $L \subset K$, that is $\phi\big(|L|\big) \subset M$. If N and N' are two closed regular neighborhoods of $|L|$ in M, then there is an ambient isotopy of M taking N onto N'.*

We now introduce handles as an additional structure we can place on our piecewise-linear manifolds. Attaching handles is done by taking a k-handle and attaching it to our manifold M via a homeomorphism ϕ (gluing map) from $\partial D^k \times D^{n-k}$ to part of the manifold's boundary,

$$\phi : \partial D^k \times D^{n-k} \to M.$$

Definition 3.12. A k-handle is an n-ball parametrized as $D^k \times D^{n-k}$.

That is, the k in k-handle refers to the way it is attached, for example, in dimension 3, for a 1-handle and 2-handle, h_1 and h_2 respectively, we have $h_1 = D^1 \times D^2$ and $h_2 = D^2 \times D^1$. h_1 attaches D^1 to M by the two end point boundaries of D^1, and h_2 attaches D^2 to M by S^1.

After attaching a handle, we have a new manifold M'. But suppose we didn't start with M, but rather the empty set, then we attach 0-handles (not actually being attached, since there is no boundary yet, appearing disjointly as 3-balls) to make M_0. Then suppose 1-handles are attached to the boundary of M_0(which now exists thanks to the 0-handle) to make M_1. We can continue to attach handles in order of increasing dimension to build the final compact manifold M'.

Definition 3.13. The above iterative process is known as a *handle decomposition* for M'. Formally defined as the following sequence

$$M' = \emptyset \cup M_0 \cup M_1 \cup \ldots \cup M_n$$

This process can be reversed, i.e., breaking down M' into its constituent handles.

As stated before, this handle structure can be placed on piecewise linear manifolds. Formally:

Theorem 3.14. *There exists a handle decomposition for every piecewise-linear manifold.*

Remark. Since Morse functions exist in a 3-manifold, we can sweep through the range of a Morse function, attaching handles at each critical point, the union of these handles is the handle decomposition of the manifold. This allows for the proof of Theorem 3.14 in the case of smooth manifolds.

As we will be focusing on 3-manifolds, we consider a simple case of handle decomposition where only 0 and 1-handles are used, ie. handlebodies.

Definition 3.15. A *handlebody of genus* $g \geq 0$, denoted H_g, is a compact, connected, orientable 3–manifold with boundary, that admits a handle decomposition consisting of a single 0–handle and g 1–handles. The genus g of a handlebody, corresponds to the genus of the boundary of the handlebody.

Note. Please refer to Johnson [5] for proofs of the following lemmas.

Lemma 3.16. *Let M be a 3-manifold with boundary, and let $D_1, \ldots, D_m$ be a collection of properly embedded disks in M. Suppose $N = \text{int}(M)$, and each component of $N \Big\backslash \bigcup_{i=1}^{m} D_i$ is homeomorphic to an open 3-ball. Then M is a handlebody with genus $m + 1 - n$, where n is the number of those components.*

Lemma 3.17. *Let M be an orientable 3-manifold, and let $K \subset M$ be a 1-dimensional piecewise-linear subcomplex with n vertices and m edges. Consider a regular neighborhood $N(K)$ of K. Then*

$$\overline{N(K)} \;\cong\; H_{m-n+1},$$

where $\overline{N(K)}$ *is the closure of* $N(K)$*, and* H_{m-n+1} *is a handlebody of genus* $m-n+1$.

4. Arriving at Heegaard Splittings

We now go from handlebodies to Heegaard Splittings. In this chapter we formally introduce Heegaard Splittings, then, using the Lemmas from the previous chapter we show that every compact, orientable 3-manifold can be expressed as a Heegaard Splitting.

Definition 4.1. Let M be a connected, closed, orientable 3-manifold. A *Heegaard splitting* of M is an ordered triple (Σ, V, W) satisfying the following conditions:

- $\Sigma \subset M$ is a closed, connected, orientable surface;
- V and W are handlebodies embedded in M;
- $V \cup W = M$ and $V \cap W = \Sigma$;

The surface Σ is called the *Heegaard surface* (or *splitting surface*) of the decomposition. The *genus* of the splitting (Σ, V, W) is the genus of Σ. Moreover, two Heegaard splittings (Σ, V, W) and (Σ', V', W') of M are said to be *equivalent* if there is an ambient isotopy of M carrying Σ onto Σ'.

Theorem 4.2. *Every compact, orientable, connected, closed 3-manifold admits a Heegaard splitting.*

Example 4.3. The genus 0 Heegaard Splitting of S^3 was shown in the intro Example 0.1. We now consider its genus 1 Heegaard Splitting.

$$\begin{aligned} S^3 &= \partial B^4 \\ &= \partial\big(B^2 \times B^2\big) \\ &= \big(\partial D^2 \times D^2\big) \cup \big(D^2 \times \partial D^2\big) \\ &= H_1 \cup H_2 \end{aligned}$$

Thus, we have a decomposition of two solid tori, ie., handlebodies, glued along their boundary T^2 using a homeomorphism gluing map g, where

$$\begin{aligned} g : (S^1 \times \partial D^2) &\xrightarrow{\text{gluing map}} (S^1 \times \partial D^2) \\ (\theta, \phi) &\longmapsto (\phi, \theta) \end{aligned}$$

This identification is orientation reversing such that the meridional circles of one solid torus map to the longitudinal circles of the other. Figure 1 shows the two handlebodies

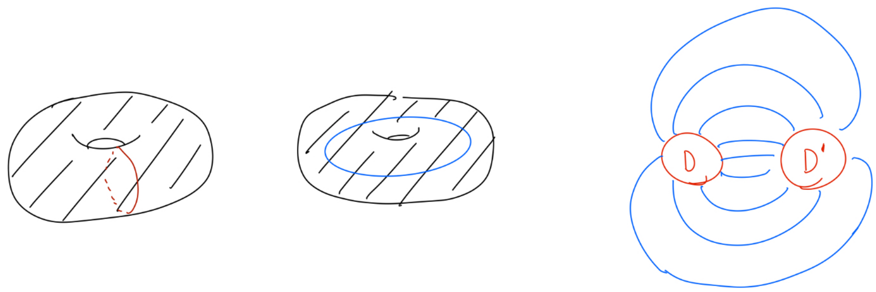


FIGURE 7. Genus 1 Heegaard Splitting of S^3

with their simple closed meridional and longitudinal curves specified, and a picture of the Heegaard Splitting with disks of H_1 seen as arcs from D to D', where D and D' are disks in H_2. Additionally, from the point of view of handle attachment, we can think of the genus–1 splitting as arising from the genus–0 splitting by starting with a 0-handle B_1, adding a 1-handle, then adding a cancelling 2-handle and then finally adding a 3-handle.

Example 4.4. Here we look at the genus–1 Heegaard splitting of $S^1 \times S^2$. Consider the sphere factor of $S^1 \times S^2$ with its equator S^1_{eq}:

$$S^2 = D^2_+ \cup D^2_-, \qquad D^2_+ \cap D^2_- = S^1_{\text{eq}}.$$

Crossing everything with S^1 gives

$$S^1 \times S^2 \;=\; (D^2_+ \times S^1) \cup (D^2_- \times S^1),$$

$$(D^2_+ \times S^1) \cap (D^2_- \times S^1) \;=\; S^1_{\text{eq}} \times S^1 \;=\; T^2.$$

Where $H_1 = D^2_+ \times S^1$, $H_2 = D^2_- \times S^1$, are solid tori handlebodies and T^2 is the Heegaard surface. Since the two handlebodies are the same product, we take the gluing homeomorphism $\varphi\colon \partial H_1 \longrightarrow \partial H_2$ to be the identity, sending meridians of one solid torus to meridians of the other. Hence

$$S^1 \times S^2 \;=\; H_1 \cup_{T^2} H_2,$$

is a genus-1 Heegaard splitting.

Example 4.5. We can consider T^3 as a surface bundle over S^1 through the mapping torus construction.

$$MT(f) = \big(T^2 \times [0,1]\big) \,/\sim \;\text{ where } (f(x),0) \sim (x,1)$$

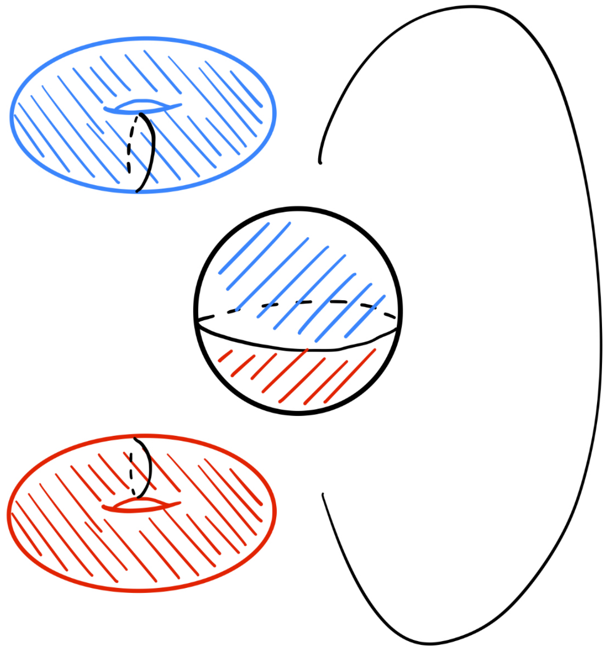


FIGURE 8. Genus 1 Heegaard Splitting of $S^1 \times S^2$.

Here we simply consider f as the identity transformation $f_{\text{id}}(x) = x, \forall x \in T^2$, ending up with the standard 3-torus with a trivial monodromy, essentially the product structure

$$MT(f) = T^2 \times S^1$$

We know T^2 as a quotient space which allows us to view T^2 as a square with opposite edges identified. Likewise, we can view T^3 as a cube with opposite faces identified. This identification process can formally be described with $\phi : C \to T^3$, where C is the cube – a triangulation of T^3.

We see ϕ maps all 8 vertices of the cube to one vertex in T^3, all parallel edges to one edge, and all opposite faces are glued together. Denote the vertex, together with the 3 edges by K. Then take N to be the regular neighborhood of K, its closure H_1 is then a handlebody by Lemma 3.17. We let H_2 be the closure of the complement of N. Since the faces of the cube chop H_2 into an open ball, by Lemma 3.16 it is also a handlebody. Since the "spine" K of the "cubulation" has three edges and one vertex, the splitting has genus-3. Hence, taking $\Sigma = H_1 \cap H_2$, the triple (Σ, H_1, H_2) is a Heegaard splitting for T^3.

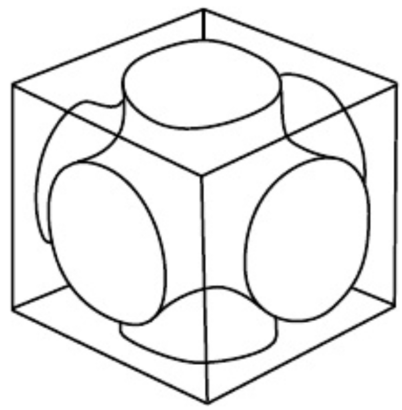


FIGURE 9. Genus 3 Heegaard Splitting of T^3. From Johnson [5]

5. Heegard Splittings of Surface Bundles over S^1

Finally, we now combine what we've learned in the previous chapters, and consider a nice construction by Bachman and Schleimer [1] of a Heegaard Splitting for all $MT(\phi) = (\Sigma_g \times I)/\sim$, where the identification is given by $(\phi(x), 0) \sim (x, 1)$.

Definition 5.1. Suppose we have a surface bundle with a natural projection map

$$f : \Sigma \to MT(\phi) \to S^1,$$

where Σ_g is a closed orientable surface of genus $g \geq 1$, and ϕ preserves orientation. Choose a surface as a fiber of the bundle $MT(\phi)$. Choose surfaces $\Sigma_N = f^{-1}(N)$ and $\Sigma_S = f^{-1}(S)$ as fibers of the bundle $MT(\phi)$. Then pick points $a, b \in S$ with $a \neq b$ and $a \neq \phi(b)$, and let A and B be disjoint closures of regular neighborhoods of $a \times [0, \pi]$ and $b \times [\pi, 2\pi]$. Let $D_N \subset \Sigma_N$ and $D_S \subset \Sigma_S$ be disks. Then we have handlebodies:

$$H_1 = (\Sigma_N / D_N \times [0, \pi]) \cup (D_S \times [\pi, 2\pi]),$$

$$H_2 = (\Sigma_S / D_S \times [\pi, 2\pi]) \cup (D_N \times [0, \pi]),$$

with genus $2g + 1$. The Heegaard surface is

$$H = \partial H_1 = \partial H_2.$$

We call (H, H_1, H_2) the *standard Heegaard Splitting* of $MT(\phi)$ (see Figure 10 and 11).

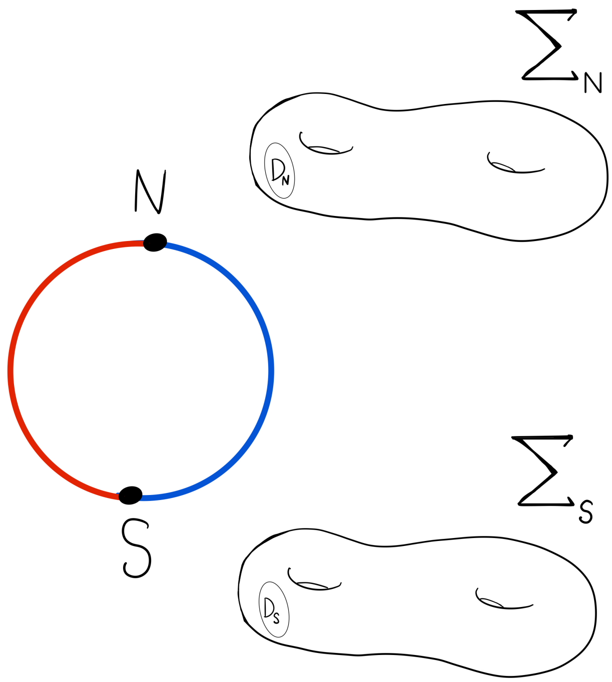


FIGURE 10. Sketch from Gabriel Islambouli's notes.[3]

Since Σ_g/D^2 gives a wedge of $2g$ circles as a "spine," by definition of a regular neighborhood, crossing the spine with an interval thickens it to its handlebody. Thus, we get the fact $\Sigma_g/D^2 \cong H_{2g}$.

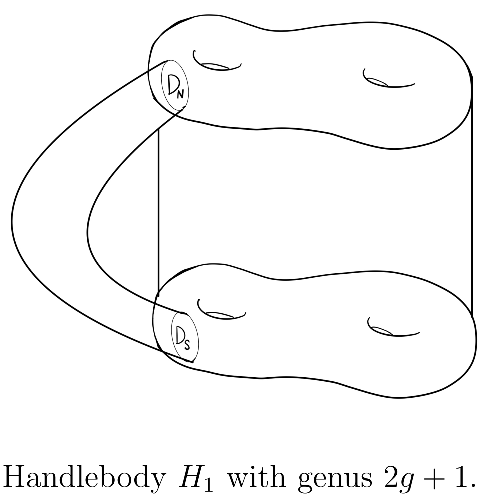


FIGURE 11. Handlebody H_1 with genus $2g+1$. H_2 is similar.

Acknowledgements

We miss you Gabe! Thank you for holding my hand throughout this! And thank you Professor Schultens! For carrying me at the end.

References

[1] David Bachman and Saul Schleimer. Surface bundles versus Heegaard splittings. *Comm. Anal. Geom.*, 13(5):903–928, 2005. ISSN 1019-8385,1944-9992. doi: 10.4310/cag.2005.v13.n5.a3. URL https://doi.org/10.4310/cag.2005.v13.n5.a3.

[2] John Hempel. *3-manifolds*. AMS Chelsea Publishing, Providence, RI, 2004. ISBN 0-8218-3695-1. doi: 10.1090/chel/349. URL https://doi.org/10.1090/chel/349. Reprint of the 1976 original.

[3] Gabriel Islambouli. Heegaard splittings. Unpublished lecture notes, shared privately with the author, 2023.

[4] Klaus Jänich. *Topologie*. Springer-Verlag, Berlin, eighth edition, 2005. ISBN 978-3-540-21393-2; 3-540-21393-7.

[5] Jesse Johnson. Notes on Heegaard Splittings. https://www.scribd.com/document/75183305/Jesse-Johnson-Notes-on-Heegaard-Splittings. Preprint.

[6] Marc Lackenby. Three-dimensional manifolds. Graduate course notes, Michaelmas Term, University of Oxford, 1999. URL https://people.maths.ox.ac.uk/lackenby/3-manifolds.html.

[7] W. B. R. Lickorish. A representation of orientable combinatorial 3-manifolds. *Ann. of Math. (2)*, 76:531–540, 1962. ISSN 0003-486X. doi: 10.2307/1970373. URL https://doi.org/10.2307/1970373.

[8] Bruno Martelli. *An Introduction to Geometric Topology*. Independently published, 3rd edition, 2023. ISBN 979-8858790709. Freely available under a Creative Commons BY-NC-SA licence at https://people.dm.unipi.it/martelli/geometric_topology.html.

[9] Edwin E. Moise. *Geometric topology in dimensions* 2 *and* 3, volume Vol. 47 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Heidelberg, 1977.

[10] Grisha Perelman. Ricci flow with surgery on three-manifolds. *arXiv preprint math/0303109*, 2003.

[11] Nick Salter and Bena Tshishiku. Surface bundles in topology, algebraic geometry, and group theory. *Notices Amer. Math. Soc.*, 67(2):146–154, 2020. ISSN 0002-9920,1088-9477.

[12] Jennifer Schultens. *Introduction to 3-manifolds*, volume 151 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2014. ISBN 978-1-4704-1020-9. doi: 10.1090/gsm/151. URL https://doi.org/10.1090/gsm/151.