Discussion Problems 7.

1. It suffices to show

\[ f(x) = \arctan \frac{1}{x} + \arctan x - \frac{\pi}{2} \]

is constantly zero when \( x > 0 \).

- Note \( f'(x) = \frac{1}{1 + (\frac{1}{x})^2} \cdot (-\frac{1}{x^2}) + \frac{1}{1+x^2} = 0 \).

- Also \( f(1) = \arctan 1 + \arctan 1 - \frac{\pi}{2} = 0 \).

So, \( f(x) = 0 \) when \( x > 0 \).

To see the case that \( x < 0 \), we note above \( f(x) \) has \( f'(x) = 0 \) no matter \( x > 0 \) or \( x < 0 \). The only changed thing can be seen when we plug in \( x = -1 \): \( f(-1) = \arctan(-1) = -\frac{\pi}{4} \). So, when \( x < 0 \), we should change \( -\frac{\pi}{2} \) to \( \frac{\pi}{2} \). That is, the formula became

\[ \arctan \frac{1}{x} = -\frac{\pi}{2} - \arctan x \]
2.

\[ f'(x) = 1 + 5 \cdot \frac{1}{1 + \left(\frac{1}{x}\right)^2} \cdot (-\frac{1}{x^2}) \]

\[ = 1 - 5 \cdot \frac{1}{1 + x^2} \]

So, \( f'(x) = 0 \implies x = \pm 2 \).

So, the critical pt. on \((0, \infty)\) is \(x = 2\).

So, we have the sign chart

\[
\begin{array}{c|c|c|c|c}
 x & 1 & (1, 2) & 2 & (2, 5) \\
 f'(x) & 0 & + & 0 & - \\
 f(x) & f(1) & \downarrow & f(2) & f(5) \\
\end{array}
\]

So, \( f(2) \) is the global (also local) minimum, which is \( 2 + 5 \cdot \arctan(\frac{1}{2}) \).

To see the global max, we need to compare \( f(1) \) and \( f(5) \). \( f(1) = 1 + \frac{5}{4} \approx 5 \).

To compute \( f(\frac{1}{5}) \), consider the linearization of \( \arctan x \) at \( x = 0 \). \( \frac{1}{1 + x^2} \approx (\frac{1}{5} - 0) + \arctan 0 \approx 0.2 \).

So, \( \arctan(\frac{1}{5}) \approx \frac{1}{1 + 0^2} \cdot (\frac{1}{5} - 0) + \arctan 0 \approx 0.2 \).

So, \( f(5) \approx 6 > f(1) \). So \( f(5) = 6 \) is the global max.
3.

- \( f(x) = \sqrt{x^2(1-x^2)} \). So, we need \( x^2(1-x^2) \geq 0 \). So, \( x \in [-1, 1] \) is the domain.

- Let \( g(x) = x^2(1-x^2) \). Then \( f(x) = \sqrt{g(x)} \).

So, let discuss about \( g(x) \) first:

\[ g'(x) = 2x - 4x^3 = 0 \Rightarrow x = 0, \pm \sqrt{\frac{1}{2}}. \]

So, sign chart of \( g(x) \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-1)</th>
<th>(-\sqrt{\frac{1}{2}})</th>
<th>0</th>
<th>(\frac{\sqrt{2}}{2})</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(x) )</td>
<td>NA</td>
<td>+</td>
<td>0</td>
<td>-</td>
<td>0</td>
</tr>
</tbody>
</table>

So, range of \( g(x) \) is \([0, \frac{1}{2}]\). So, range of \( f(x) \) is \([0, \sqrt{\frac{1}{2}}]\).

NA: Not apply since \( g(x) \) only has right derivative at \(-1 \) (left at 1).
4.

(a). Impossible. By Mean Value Theorem, \( \exists \xi \in (2, 1) \) such that
\[
f'\left(\xi\right) = \frac{f(2) - f(1)}{2 - 1} = \frac{3 - 1}{1} = 2.
\]

(b). By hint, let \( f(x) = ax^2 + bx + c \).
So, \( f'(x) = 2ax + b \). So we need
\[
\begin{align*}
f'(0) &= b = 1 \implies \ a = -0.5, \ b = 1. \\
f'(1) &= 2a + b = 0
\end{align*}
\]
So, \( f(x) = -0.5x^2 + x \) satisfies the condition. (You may check \( f(0) < f(1) \)).

(c). \( y = -\frac{1}{2} x^2 + 5 \) satisfies the conditions.

(d). \( f'(x) = 4 + 2 \sin 2x > 0 \ \forall x \in \mathbb{R} \).

- By Rolle's Theorem, there is at most one \( x \) such that \( f(x) = 0 \).
- By IVT, \( f(0) = -1, \ f(\frac{\pi}{4}) = \pi \).
So, \( \exists x \in (0, \frac{\pi}{4}) \).

So only one \( x \)-intercept.