

1.

(a) \[ \lim_{x \to 4} (g(x) + 3) = \lim_{x \to 4} g(x) + \lim_{x \to 4} 3 = -3 + 3 = 0 \quad [\text{Sum Rule}] \]

(b) \[ \lim_{x \to 4} x f(x) = \lim_{x \to 4} x \cdot \lim_{x \to 4} f(x) = 4 \cdot 0 = 0 \quad [\text{Product Rule}] \]

(c) \[ \lim_{x \to 4} (g(x))^2 = [\lim_{x \to 4} g(x)]^2 = [-3]^2 = 9 \quad [\text{Power Rule}] \]

(d) \[ \lim_{x \to 4} \frac{g(x)}{f(x) - 1} = \frac{\lim_{x \to 4} g(x)}{\lim_{x \to 4} [f(x) - 1]} \quad [\text{Product Rule}] \]

\[ = \frac{\lim_{x \to 4} g(x)}{\lim_{x \to 4} f(x) - \lim_{x \to 4} 1} \quad [\text{Difference Rule}] \]

\[ = \frac{-3}{0 - 1} = 3 \]

2.

\[ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{h} - \frac{1}{x} = \lim_{h \to 0} \frac{x - (x+h)}{x(x+h)h} \]

\[ = \lim_{h \to 0} \frac{-h}{x+h} \cdot \frac{1}{h} = \lim_{h \to 0} \left( \frac{-1}{x(x+h)} \right) \]

\[ = -\lim_{h \to 0} \frac{1}{\lim_{h \to 0} [x(x+h)]]} = -\frac{1}{\lim_{h \to 0} x \cdot \lim_{h \to 0} (x+h)} \]

\[ = -\frac{1}{x \cdot x} = -\frac{1}{x^2} \]

If \( x = -2 \), then the limit is \( -\frac{1}{4} \).
3.

(a) \( \lim_{{x \to 0}} \left( \frac{1}{2} - \frac{x^2}{24} \right) = \frac{1}{2} - 0 = \frac{1}{2} \)

\( \lim_{{x \to 0}} \frac{1}{2} = \frac{1}{2} \).

So, by the Sandwich Theorem, \( \lim_{{x \to 0}} \frac{1 - \cos x}{x^2} = \frac{1}{2} \).

(b)

4.

The Sandwich Theorem allows us to conclude that \( \lim_{{x \to 2}} f(x) = 5 \). However, we cannot conclude anything about \( f(2) \), \( g(2) \) or \( h(2) \) since the limit of \( f, g \) at 2 and \( h \) has nothing to do with the value \( f(2) \), \( g(2) \) and \( h(2) \).
5.
- To satisfy \( |f(x) - 4| < \varepsilon \), i.e., \( |(2x - 2) - (-6)| < 0.02 \),
we need \( |2x + 4| < 0.02 \), i.e., \(-0.02 < 2x + 4 < 0.02\),
i.e., \(-2.01 < x < -1.99\), i.e., \(-0.01 < x < 0.01\) (*)
- We need to find a \( \delta \) such that if \( |x - (-2)| < \delta \),
then (*) is satisfied. In fact, \( |x - (-2)| < \delta \)
is equivalent to \(-\delta < x + 2 < \delta \). So, we just need \( \delta = 0.01 \),
in order to satisfy (*)

6.
- To satisfy \( |\sqrt{x} - 7 - 4| < 1 \), we need \(-1 < \sqrt{x} - 7 - 4 < 1\)
i.e., \(9 < x - 7 < 25\), i.e., \(6.16 < x < 32\) (*)
- We need to find a \( \delta \) such that if \( |x - 23| < \delta \),
then (*) is satisfied. In fact, \( |x - 23| < \delta \)
is equivalent to \(-\delta < x - 23 < \delta \), i.e., \(-\delta + 23 < x < \delta + 23\). To satisfy (*), we need
\[
\begin{align*}
\delta - \delta + 23 & \geq 16 \\
\delta + 23 & \leq 32,
\end{align*}
\]
which gives \( \delta \leq 7 \) and \( \delta \geq 9 \). So, we may take \( \delta = 7 \).
7. \( \forall \varepsilon > 0 \), we need to find a \( \delta \), such that as long as \( |x - (-2)| < \delta \), we have \( |f(x) - 4| < \varepsilon \).

To satisfy \( |f(x) - 4| < \varepsilon \), we need \( |x^2 - 4| < \varepsilon \), i.e., \(-\varepsilon < x^2 - 4 < \varepsilon \), i.e., \(-\sqrt{4 + \varepsilon} < x < \sqrt{4 - \varepsilon} \) \( (*) \).

Then we need to find a \( \delta \) such that if \( |x - (-2)| < \delta \) then \( (*) \) is satisfied. In fact, \( |x - (-2)| < \delta \) is equivalent to \(-\delta - 2 < x < \delta - 2 \). To satisfy \( (*) \), we need

\[
\begin{align*}
\delta & \leq \sqrt{4 + \varepsilon} - 2 \\
\delta & \leq -\sqrt{4 - \varepsilon} + 2.
\end{align*}
\]

i.e.,

\[
\begin{align*}
\delta & \leq \sqrt{4 + \varepsilon} - 2 \\
\delta & \leq -\sqrt{4 - \varepsilon} + 2.
\end{align*}
\]

To satisfy these two inequalities at the same time, we just need

\[
\delta = \min \left\{ -\sqrt{4 - \varepsilon} + 2, \sqrt{4 + \varepsilon} - 2 \right\}.
\]
8. To show \( \lim_{x \to c} f(x) = L \), we need to show \( \forall \varepsilon > 0, \exists \delta > 0 \) such that \( \forall x \) with \( |x - c| < \delta \), then \( |f(x) - L| < \varepsilon \). So, to show \( \lim_{x \to c} h(x) \neq L \), we need to show \( \exists \varepsilon > 0, \forall \delta > 0 \) such that \( \exists x \) with \( |x - c| < \delta \), such that \( |f(x) - L| \geq \varepsilon \).

a. To show \( \lim_{x \to 2} h(x) \neq 4 \), pick \( \varepsilon = 1 \) (actually you may pick \( \varepsilon \) to be any number less than 1.

Then \( \forall \delta > 0 \), \( \exists x = 2 + \frac{\delta}{2} \) such that \( |x - 2| = \frac{\delta}{2} < \delta \) but \( |f(2 + \frac{\delta}{2}) - 4| = |2 - 4| = 2 > \varepsilon = 1 \).

b. Pick \( \varepsilon = \frac{1}{2} \). \( \forall \delta > 0 \), \( \exists x = 2 + \frac{\delta}{2} \) such that \( |x - 2| = \frac{\delta}{2} < \delta \), but \( |f(2 + \frac{\delta}{2}) - 3| = |2 - 3| = 1 > \frac{1}{2} \).

c. Pick \( \varepsilon = 1 \), \( \forall \delta > 0 \), \( \exists x = 2 - \frac{\delta}{2} \) such that \( |x - 2| = \frac{\delta}{2} < \delta \), but \( |f(2 - \frac{\delta}{2}) - 2| > \varepsilon = 1 \).

9.

a. T, b. F, \( \lim_{x \to 2} f(x) = 1 \), e. F, d. T

e. T, f. T, g. T, h. T

i. F, f has no def. if \( x \leq 1 \)

k. T.
10.

a. \( \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x}{2} = 1. \)

\[ \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 3-x = 1 \]

So, \( \lim_{x \to 2} f(x) = 1. \) But \( f(2) = 2. \)

b. See a.

c. \( \lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} 3-x = 4 = \lim_{x \to -1^+} f(x) \)

So, \( \lim_{x \to -1} f(x) = 4. \)

d. See c.