MAT 22B
Differential Equations
Summer Session 1, 2015
Midterm 1
7/1/2015
Time Limit: 60 Minutes

Name (PRINT): KEY
ID: __________________

Instruction

1. This exam contains 12 pages (including this cover page) and 5 questions.
2. No notes, books, or calculators allowed.
3. Show all your work. Unsupported answers will receive zero credits.
4. You can remove the last page as the scratch paper. Raise your hand if you need more.

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</table>
1. (15 points) Consider the ODE \[ \frac{dy}{dt} + y^2 = 0, \quad t > 0. \] (1)

(a) (2 points) Determine the order of ODE (1). Say if (1) is linear, non-linear or undecided.

(b) (8 points) Solve (1) explicitly with the initial condition \[ y(1) = 1. \]

(c) (5 points) What is the largest \( t \)-interval in which the solution exists?

\[(a) \quad 1^{st} \text{ order, non-linear.} \]

\[(b) \]

\[1 \implies -\int \frac{dy}{y^2} = \int \frac{dt}{t} \]

\[\Rightarrow \frac{1}{y} = \ln t + C \]

\[\Rightarrow y(t) = \frac{1}{\ln t + C} \]

To satisfy \( y(1) = 1 \), we need

\[\frac{1}{\ln 1 + C} = 1 \implies \frac{1}{C} = 1 \implies C = 1 \]

So \( y(t) = \frac{1}{\ln t + 1} \)

\[(c) \quad \text{The solution exists if and only if} \]

\[\ln t + 1 \neq 0, \quad \text{i.e.,} \quad \ln t \neq -1, \quad \text{i.e.,} \]

\[t \neq \frac{1}{e}. \quad \text{So the sol. exists on} \]

\[(0, \frac{1}{e}) \quad \text{and} \quad (\frac{1}{e}, \infty) \]

So, \((\frac{1}{e}, \infty)\) is the largest one.
(Solution Area for Problem 1)
2. (20 points) Consider the ODE
\[ \frac{dy}{dx} + ay = f(x), \] (2)
where \( a \) is a constant. Suppose \( f(x) \) is continuous on \([0, +\infty)\), \( f(x) > b \) for all \( x > 0 \) and \( \lim_{x \to +\infty} f(x) = b \), where \( b > 0 \) is a constant.

(a) (2 points) Determine the order of ODE (2). Say if (2) is linear, non-linear or undecided.

(b) (10 points) Show that if \( y = y(x) \) is a solution of (2) and \( y(x) \) satisfies \( y(x_0) = y_0 \),
then
\[ y(x) = e^{-a(x-x_0)}y_0 + \int_{x_0}^{x} f(s)e^{a(s-x_0)}ds. \] (3)

(c) (8 points) If \( a > 0 \), show that \( \lim_{x \to +\infty} y(x) = \frac{b}{a} \), where \( y(x) \) is given in (3).

(a) 1st order, linear.

(b) \( y(x) \) is a sol. of (2) \( \Rightarrow \)
\[ \frac{dy(x)}{dx} + ay(x) = f(x). \] (*

Consider integrating factors
\[ \mu(x) = \exp\{\int adx\} = e^{ax} + c, \]
specifically, to get (3). Consider
\[ c = -ax_0. \]

So, (*) \( \iff \)
\[ \frac{d[e^{ax-x_0} y(x)]}{dx} = f(x) e^{ax-x_0} \]
\[ \Rightarrow e^{ax-x_0} y(x) = \int f(x) e^{ax-x_0} dx + C_1. \]
\[ = \int_{x_0}^{x} f(s) e^{a(s-x_0)} ds + C_1 \]
\[ \Rightarrow y(x) = e^{-a(x-x_0)} \left[ C + \int_{x_0}^{x} f(s) e^{a(s-x_0)} ds \right] \]
To find $c_1$, note $y(x_0) = y_0$.

So, $A = y_0$.

So, $y(x) = e^{-a(x-x_0)} \left[ y_0 + \int_{x_0}^{x} f(s) e^{a(s-x_0)} \, ds \right]$.

**(C)**

\[
\lim_{x \to \infty} y(x) = \lim_{x \to \infty} \frac{y_0 + \int_{x_0}^{x} f(s) e^{a(s-x_0)} \, ds}{e^{a(x-x_0)}}
\]

\[
= \lim_{x \to \infty} \frac{y_0}{e^{a(x-x_0)}} + \lim_{x \to \infty} \frac{\int_{x_0}^{x} f(s) e^{a(s-x_0)} \, ds}{e^{a(x-x_0)}}
\]

\[
= A + B.
\]

**For $A$:** $e^{a(x-x_0)} \to \infty$ as $x \to \infty$. So $A = 0$.

**For $B$:** $f(s) e^{a(s-x_0)} \to b e^{a(s-x_0)}$ since $f(s) > b$.

So $\int_{x_0}^{x} f(s) e^{a(s-x_0)} \, ds \to \int_{x_0}^{x} b e^{a(s-x_0)} \, ds = \infty$.

So, it is $\frac{\infty}{\infty}$ type.

So, apply L'Hopital

\[
B = \lim_{x \to \infty} \frac{f(x) e^{a(x-x_0)}}{a e^{a(x-x_0)}} = \lim_{x \to \infty} \frac{f(x)}{a} = \frac{b}{a}.
\]
3. (20 points) Let \( f(x, y) \) and \( \frac{\partial f}{\partial y} \) be continuous. Show that equation

\[
\frac{dy}{dx} = f(x, y)
\]

is linear if and only if it has an integrating factor that only depends on \( x \).

"Linear \implies \text{integrating factor}".

Suppose \( \frac{dy}{dx} = f(x, y) \) linear, then

\[
f(x, y) = p(x)y + q(x). \quad \text{So}
\]

\[
\frac{dy}{dx} = p(x)y + q(x).
\]

i.e., \( p(x)y + q(x) + \left( -\frac{dy}{dx} \right) = 0 \)

So, consider a function \( \mu(x) \), such

\[
\mu(x) \left[ p(x)y + q(x) \right] + \left( -\mu(x) \frac{dy}{dx} \right) = 0. \quad (*)
\]

it is exact iff

\[
\frac{\partial}{\partial y} (\mu p(x)y + q(x)) = \frac{\partial}{\partial x} (-\mu)
\]

iff \( \mu(x)p(x) = -\frac{d\mu(x)}{dx} \)

So, \( -\frac{d\mu(x)}{\mu(x)} = p(x) \, dx \)

So \( \int \frac{d\mu(x)}{\mu(x)} = -\int p(x) \, dx \)

So \( \mu(x) = e^{-\int p(x) \, dx} \).

Since \( \frac{\partial f}{\partial y} = p(x) \) continuous, so \( \mu(x) = e^{-\int p(x) \, dx} \) exists, which makes \( (*) \) exact.

So, \( \mu(x) \) is an integrating factor.
Suppose \( \mu(x) \) is a integrating factor for \( \frac{dy}{dx} = f(x, y) \).

So, \( \mu f - \mu \frac{dy}{dx} = 0 \) exact.

So, \( \frac{\partial (\mu f)}{\partial y} = \frac{\partial (-\mu)}{\partial x} \)

So, \( \frac{\partial (\mu f)}{\partial y} \cdot f(x, y) + \frac{\partial f(x, y)}{\partial y} \cdot \mu(x) = - \frac{du(x)}{dx} \)

So, \( \frac{\partial f(x, y)}{\partial y} \cdot \mu(x) = - \frac{du(x)}{dx} \)

So, \( \frac{\partial f(x, y)}{\partial y} = - \frac{1}{\mu(x)} \cdot \frac{du(x)}{dx} \).

Let \( - \frac{1}{\mu(x)} \cdot \frac{du(x)}{dx} = g(x) \), so

\( f(x, y) = g(x) y + h(x) \).

That is, \( \frac{dy}{dx} = g(x) y + h(x) \),

which is linear.
4. (20 points) Consider the ODE

\[(\ln |t|)y' + t^2 - 9 = \frac{t^2 - 9}{t - 4} . \tag{4}\]

(a) (4 points) Determine the order of ODE (4). Say if (4) is linear, non-linear or undecided.

By use of the existence and uniqueness theorems given in class, without solving (4), say what you can conclude about the existence, uniqueness, and t-interval of definition of a solution \(y(t)\) of the ODE (4) with the initial conditions

(b) (8 points) \(y(2) = 5;\)

(c) (8 points) \(y(5) = 2.\)

\[\textbf{a). 1st order, linear.}\]

\[\textbf{b). } (4) \iff y' + \frac{e^t}{\ln|t|} y = \frac{t^2 - 9}{(t - 4) \ln|t|}
\]

\[p(t) = \frac{e^t}{\ln|t|} \text{ is continuous on } (-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, \infty).
\]

\[q(t) = \frac{t^2 - 9}{(t - 4) \ln|t|} \text{ is continuous on } (-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, 4) \cup (4, \infty).
\]

\((1, 4) \text{ is an interval on which } p(t) \& q(t) \text{ are continuous and contains } 2. \text{ So on } (1, 4),\)

there is a solution \(y = \phi(t)\) that solves (4) \& \(y(2) = 5.\)

\[\textbf{c). } (4, \infty) \text{ is an interval on which } p(t) \& q(t) \text{ are continuous and contain } 5. \text{ So, on } (4, \infty),\)

there is a solution \(y = \phi(t)\) that solves (4) \& \(y(5) = 2.\)
(Solution Area for Problem 4)
5. (25 points) Consider the Logistic Model with ODE

\[
\frac{dy}{dt} = 0.5y(1 - y/100), \quad t \geq 0
\]  
(5)

(a) (2 points) Determine the order of ODE (5). Say if (5) is linear, non-linear or undecided.

(b) (1 point) Is (5) autonomous?

(c) (2 points) What is the intrinsic growth rate? What is the environmental carrying capacity?

(d) (10 points) Solve (5) explicitly with initial condition \(y(0) = 50\).

(e) (4 points) Find the equilibrium solutions.

(f) (6 points) Draw the phase line. Find whether the equilibrium solutions in (e) are stable or unstable.

\[
\begin{align*}
(a) & \quad 1^{st} \text{ order. Linear} \\
(b) & \quad \text{Yes} \\
(c) & \quad r = 0.5, \quad k = 100 \\
(d) & \quad (5) \iff (\frac{1}{y} + \frac{1/100}{1-y/100}) dy = 0.5 \, dt \\
& \iff \int \frac{1}{y} + \frac{1}{100-y} \, dy = \int 0.5 \, dt \\
& \Rightarrow \ln y - \ln (100-y) = 0.5t \\
& \Rightarrow \frac{y}{100-y} = ce^{0.5t} \\
& \Rightarrow y = \frac{100ce^{0.5t}}{1 + ce^{0.5t}} \\
& y(0) = 50 \Rightarrow c = 1 \\
& \text{So} \quad y = \frac{100e^{0.5t}}{1 + e^{0.5t}} = \frac{100}{e^{-0.5t} + 1}
\end{align*}
\]
(e) equilibrium sol. are
\[ \hat{y}_1 = 0 \] \text{ and } \hat{y}_2 = 100.

(f) 

\[ \begin{array}{ccc}
0 & \rightarrow & 100 \\
\rightarrow & \rightarrow & \leftarrow
\end{array} \]

So, \( \hat{y}_1 = 0 \) is unstable.
\( \hat{y}_2 = 100 \) is stable.