MAT 22B: KEY. Name (PRINT): ______________________
Differential Equations ID: ______________________
Summer Session 1, 2015
Midterm 2
7/17/2015
Time Limit: 60 Minutes

Instruction

1. This exam contains 14 pages (including this cover page) and 6 questions.
2. No notes, books, or calculators allowed.
3. Show all your work. Unsupported answers will receive zero credits.
4. You can remove the last page as the scratch paper. Raise your hand if you need more.
5. Total score is 110. Your midterm 2 grade = min(100, your score on this exam).

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1. (10 points) Consider the ODE
\[ \frac{d^2 y}{dt^2} - 8 \frac{dy}{dt} + 25y = 0 \]  
(1)
(a) (3 points) Determine the order of ODE (1). Say if (1) is linear, non-linear or undecided. Say if (1) is homogeneous, non-homogeneous or undecided.
(b) (7 points) Find the real-valued general solution of (1).

(a) 2nd order. Linear. Homogeneous.

(b) The char. eq. is
\[ r^2 - 8r + 25 = 0, \]
with roots
\[ r_{1,2} = \frac{8 \pm 6i}{2} = 4 \pm 3i. \]
So, the general solution of (1) is
\[ y(t) = C_1 e^{4t} \cos 3t + C_2 e^{4t} \sin 3t, \]
where \( C_1 \) & \( C_2 \) are two constants.
(Additional Solution Area for Problem 1)
2. (15 points) Consider the ODE
\[ y'' - y - 1 = 0. \] \hspace{1cm} (2)

(a) (3 points) Determine the order of ODE (2). Say if (2) is linear, non-linear or undecided. Say if (2) is homogeneous, non-homogeneous or undecided.

(b) (7 points) Solve (2) with initial condition
\[ y(0) = 0, \]
\[ y'(0) = a, \]
where a is a real constant.

(c) (5 points) Suppose \( y(t) \) is the solution of the IVP you found in (a). For what value of \( a \) does \( y(t) \) approach a constant finite limit as \( t \to \infty \)? What is the solution in that case?

(a) 2\textsuperscript{nd} order. Linear. Non-homogeneous.

(b) Rewrite (2) as
\[ y'' - y = 1 \] \hspace{1cm} (2')

A particular sol. of (2') has the form
\[ y(t) = A. \]

So, we need
\[ y'' - y = 0 - A = 1 \]

So, \[ y(t) = -1 \] is a sol. of (2').

Then we need to solve
\[ y'' - y = 0, \] \hspace{1cm} (2'')
the corresponding homogeneous eq. of (2'). Note the char. eq. of (2'') is \( r^2 - 1 = 0 \), which gives \( r_{1,2} = \pm 1 \). So, the general sol. of (2'') is
\[ y = c_1 e^t + c_2 e^{-t} \]

So, the general sol. of (2) (or (2')) is
\[ y = c_1 e^t + c_2 e^{-t} - 1 \]
To solve the IVP, we need
\[
\begin{align*}
    y(0) &= c_1 + c_2 - 1 = 0 \\
    y'(0) &= c_1 - c_2 = a
\end{align*}
\]
This gives
\[
\begin{align*}
    c_1 &= \frac{1+a}{2} \\
    c_2 &= \frac{1-a}{2}
\end{align*}
\]
So, the sol. of the IVP is
\[
y = \frac{1+a}{2} e^t + \frac{1-a}{2} e^{-t} - 1.
\]

(b) \( \lim_{t \to \infty} y(t) < \infty \) if and only if \( \frac{1+a}{2} = 0 \), i.e., \( a = -1 \).

In this case
\[
y = e^{-t} - 1.
\]
3. (15 points) Consider the ODE

\[(\ln |t|)y'' + e^t y' + \frac{t^2 - 9}{t - 4} y = 0, \quad t \neq \pm 1.\]  \hfill (3)

(a) (4 points) Suppose \(y_1(t)\) is a solution of (3) with initial condition

\[y_1(2) = 1, \quad y_1'(2) = 1.\]

Then on what interval of \(t\) is \(y_1\) defined?

(b) (4 points) Suppose \(y_2(t), y_3(t)\) are solutions of (3) on the same interval \(I\) you found in (a) and they satisfy the initial conditions:

\[y_2(2) = 0, \quad y_2'(2) = 1,\]
\[y_3(2) = 1, \quad y_3'(2) = -1,\]

Verify that \(\{y_2(t), y_3(t)\}\) is a fundamental set of solution.

(c) (7 points) \(y_1(t)\) can be written as a linear combination of \(y_2(t)\) and \(y_3(t)\), i.e., for some \(c_2\) and \(c_3\), \(y_1(t) = c_2 y_2(t) + c_3 y_3(t)\), for all \(t \in I\). Why? Find this linear combination explicitly.

(a) \(3) \iff y'' + \frac{e^t}{\ln |t|} y' + \frac{t^2 - 9}{|t| (t - 4)} y = 0.\)

\[\begin{align*}
\text{Notice:} & \quad \frac{e^t}{\ln |t|} \text{ is continuous on } & \mathbb{R} \setminus \{1, -1\} \\
\text{Notice:} & \quad \frac{t^2 - 9}{|t| (t - 4)} \text{ is continuous on } & (-\infty, -1) \cup (-1, 1) \cup (1, \infty)
\end{align*}\]

So, \((1, 4)\) is an interval on which \(p(t), q(t)\) are continuous and contains 2.

So, \(y_1(t)\) is defined on \([1, 4]\) = \(I\).

(b) \(W(y_2, y_3)(2) = \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} = 1 \neq 0\)

So, \(\{y_2, y_3\}\) is a fundamental set of solutions of (3).
(c). Since \( \{y_2, y_3\} \) is a fund. set of sol. every
sol. of (3), particularly \( y(t) \), can be written
as a linear combination of \( y_2 \) & \( y_3 \), i.e.,
\[
y(t) = C_2 y_2(t) + C_3 y_3(t),
\]
(*)
for all \( t \in (1, 4) \), for some \( C_2 \) and \( C_3 \).

To satisfy (*) for all \( t \in (1, 4) \), we need
(*) to be satisfied particularly at \( t = 2 \). So
we need
\[
\begin{align*}
1 &= y(2) = C_2 y_2(2) + C_3 y_3(2) = C_2 \cdot 0 + C_3 \cdot 1 \\
1 &= y'(2) = C_2 y_2'(2) + C_3 y_3'(2) = C_2 \cdot 1 + C_3 (-1)
\end{align*}
\]
So \( C_2 = 2 \), \( C_3 = 1 \)

To see \( y(t) = 2y_2(t) + y_3(t) \) for all \( t \in I \),
let \( y_4(t) = 2y_2(t) + \ast y_3(t) \). Then
\[
y_4(2) = 1, \quad y_4'(2) = 1.
\]
Since the sol. of IVP
\[
\begin{align*}
(3) \\
y(2) &= 1 \\
y'(2) &= 1
\end{align*}
\]
\( \ast \) is unique, so \( y(t) = y_4(t) = 2y_2(t) + y_3(t) \) for
all \( t \in (1, 4) \).

So, \[ y(t) = 2y_2(t) + y_3(t) \]
4. (20 points) Consider the ODEs

\[ y'' + 2y' + y = 0, \quad (4) \]
\[ y'' + 2y' + y = \sin 2t, \quad (5) \]
\[ y'' + 2y' + y = e^{-t}. \quad (6) \]

(a) (5 points) Find the general solution of (4).

(b) (5 points) Find a particular solution of (5).

(c) (5 points) Find a particular solution of (6). (Hint: You may need to observe the solution of (4) you found in (a), before you decide what the form of a particular solution of (6) might be.)

(d) (5 points) Find the general solution of

\[ y'' + 2y' + y = 3\sin 2t - e^{-t}. \quad (7) \]

(a). The char. eq. of (4) is

\[ r^2 + 2r + 1 = 0, \]

with roots \( r_{1,2} = -1 \). So the general sol. of (4) is

\[ y = C_1 e^{-t} + C_2 te^{-t}. \]

(b). Consider a sol. with form

\[ Y(t) = A \cos 2t + B \sin 2t. \]

Then \( Y'(t) = -2A \sin 2t + 2B \cos 2t, \)
\[ Y''(t) = -4A \cos 2t - 4B \sin 2t. \]

So we need

\[ \begin{vmatrix} -4A \cos 2t - 4B \sin 2t \\ -2A \sin 2t + 2B \cos 2t \end{vmatrix} + 2 \begin{vmatrix} -2A \sin 2t + 2B \cos 2t \\ A \cos 2t + B \sin 2t \end{vmatrix} - \sin 2t. \]

i.e.,

\[ \begin{cases} -4A - 3B = 1 \\ -3A + 4B = 0 \end{cases} \Rightarrow \begin{cases} A = -\frac{4}{25} \\ B = -\frac{3}{25} \end{cases} \]

So, \[ Y(t) = -\frac{4}{25} \cos 2t - \frac{3}{25} \sin 2t \] is a sol. of (5).
(c). Note $Ce^{-t}$ and $Cte^{-t}$ solve (4). So they cannot solve (6). So, to solve (6), consider a sol. with form

$$Y(t) = Ae^{-t}.$$ 

So, $Y'(t) = 2Ae^{-t} - At^2e^{-t}$. 

$$Y''(t) = 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}.$$ 

So, we need 

$$[2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}] + 2[2Ae^{-t} - At^2e^{-t}] + At^2e^{-t} = e^{-t}.$$ 

i.e., $2Ae^{-t} = e^{-t}$. 

So, $A = \frac{1}{2}$. So $Y(t) = \frac{1}{2}t^2e^{-t}$ is a sol. of (6).

(d). Let $L[y] = y'' + 2y' + y$. Then $L[\cdot]$ is a linear operator. So, by (b) $L[Y_1] = \sin 2t$; by (c) $L[Y_2] = e^{-t}$. So 

$$L[3Y_1 - Y_2] = 3L[Y_1] - L[Y_2] = 3\sin 2t - e^{-t}.$$ 

So, $3Y_1 - Y_2 = Y_3 = -\frac{12}{25}\cos 2t - \frac{9}{25}\sin 2t - \frac{1}{2}t^2e^{-t}$ is a particular sol. of $Y$ (7).

So, general sol. of (7) is 

$$y = C_1e^{-t} + C_2te^{-t} - \frac{12}{25}\cos 2t - \frac{9}{25}\sin 2t - \frac{1}{2}t^2e^{-t}.$$
5. (20 points) Consider the ODE

\[ y' = -2 + y, \quad t > 0. \tag{8} \]

(a) (2 points) Determine the order of ODE (8). Say if (8) is linear, non-linear or undecided.

(b) (5 points) Solve (8) with the initial condition \( y(1) = e + 2 \).

(c) (8 points) Use Euler's method to find the approximate values of the solution to initial value problem in (c) at \( t = 1, 2, 3 \).

(d) (5 points) Sketch the exact solution you found in (b) and the approximation you found in (c) in one \( t - y \) plane. You may need to use \( e \approx 2.7 \).

(a) 1st order. Linear.

(b) \( (8) \iff y' - y = -2 \).

So, consider integrating factor (mult) \( e^{\int -1\, dt} = e^{-t} \).

\( \iff e^{-t} y' - e^{-t} y = -2 e^{-t} \)

\( \iff d \left[ e^{-t} y \right] = -2 e^{-t} \, dt \)

\( \iff \int d e^{-t} y = \int -2 e^{-t} \, dt \)

\( \iff y = 2 + ce^{t} \)

We need \( y(1) = 2 + c \cdot e = 2 + e \). So \( c = 1 \).

So, \( \boxed{y = 2 + e^{t}} \) is the sol.

(c) \( P_1 = (1, e + 2) \). \( \therefore f_1 = -2 + y_1 = -2 + e + 2 = e \).

So \( y_2 = y_1 + f_1 \cdot (t_2 - t_1) = e + 2 + e \cdot (2-1) = 2e + 2 \).

So \( P_2 = (2, 2e + 2) \). \( \therefore f_2 = -2 + y_2 = -2 + 2e + 2 = 2e \).

So \( y_3 = y_2 + f_2 \cdot (t_3 - t_2) = 2e + 2 + 2e \cdot (3-2) = 4e + 2 \).

So, \( \boxed{y_1 = e + 2, \quad y_2 = 2e + 2, \quad y_3 = 4e + 2} \) are the approximate values.
d.

\[ e^3 + 2 \]

\[ 4e^2 + 2 \]

\[ e^2 + 2 \]

\[ 2e + 2 \]

\[ e + 2 \]

\[ 0 \quad 1 \quad 2 \quad 3 \]

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- exact sol.
- approx. sol.
6. (30 points) Consider the ODE

\[ 2t^2y'' + 3ty' - y = 0, \quad t > 0. \tag{9} \]

(a) (3 points) Determine the order of ODE (9). Say if (9) is linear, non-linear or undecided. Say if (9) is homogeneous, non-homogeneous or undecided.

(b) (3 points) Can we use a characteristic equation to solve (9)? Explain the reason.

(c) (4 points) Verify that \( y_1(t) = t^{-1} \) is a solution of (9).

(d) (10 points) Apply the technique of reduction of order to find another solution \( y_2(t) \) of (9).

(e) (5 points) Verify that \( \{y_1(t), y_2(t)\} \) is a fundamental set of solutions of (9).

(f) (5 points) Write down the general solution of (9).

\[ \text{(a) \hspace{1cm} 2nd order \hspace{1cm} Linear \hspace{1cm} Homogeneous.} \]

\[ \text{(b) \hspace{1cm} No. \hspace{1cm} Not with constant coefficients.} \]

\[ \text{(c) \hspace{1cm} } \]

\[ y'_1 = -t^{-2}, \quad y''_1 = 2t^{-3} \]

\[ \text{L.H.S.} = 2t^2 \cdot 2t^{-3} + 3t(-t^{-2}) - t^{-1} \]

\[ = 0 = \text{R.H.S.} \]

\[ \text{(d) \hspace{1cm} Set } y = u(t) \cdot t^{-1}. \text{ Then} \]

\[ y' = u't^{-1} - ut^{-2}, \quad y'' = u''t^{-1} - 2u't^{-2} + 2ut^{-3} \]

\[ \text{So, we need} \]

\[ 2t^2[u''t^{-1} - 2u't^{-2} + 2ut^{-3}] + 3t[u't^{-1} - ut^{-2}] - ut^{-1} \]

\[ = 2tu'' - u' = 0. \quad \text{(**) \hspace{1cm}} \]

\[ \text{Let } w = u'. \text{ Then } (**) \iff 2tw' - w = 0. \]

\[ \text{So } w = Ct^{1/2}. \text{ So } u(t) = \frac{3}{2} Ct^{3/2} + C_2. \]

\[ \text{Just take } C = \frac{3}{2}, C_2 = 0. \text{ Then } u(t) = t^{3/2}. \]

\[ \text{So } y(t) = t^{3/2} \text{ is another sol.} \]
(6).

\[ W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{-1} & t^{1/2} \\ -t^{-2} & \frac{1}{2}t^{-1/2} \end{vmatrix} = \frac{3}{2} t^{-3/2} \]

\[ \neq 0 \text{ for } t > 0. \]

So, \( \{y_1(t), y_2(t)\} \) is a fund. set of sol.

(f). So, the general sol. of (9) is

\[ y = C_1 t^{-1} + C_2 t^{1/2} \]