§ 3.2 Linear Homogeneous Eq.

In this section, we consider the general case of linear 2nd order diff. eq. (not necessarily with constant coefficient):

\[ y'' + p(t)y' + q(t)y = 0 \quad \alpha < t < \beta \]  

(1)

We will have following conclusions:

- If we can find two sol. of (1) \( y_1(t) \) & \( y_2(t) \) on \( \alpha < t < \beta \), and we can verify \( y_1(t) \) & \( y_2(t) \) satisfy certain condition (Wronskian ≠ 0 at \( t_0 \)). Then

\[ y(t) = C_1 y_1(t) + C_2 y_2(t) \]

is the general sol. of (1).

- \( C_1 \) & \( C_2 \) can be determined by initial conditions.

Introduction of a notation. For fixed \( p(t), q(t) \), given an arbitrary twice differentiable function \( \phi(t) \), define

\[ L[\phi](t) = \phi''(t)\sigma + p(t)\phi'(t) + q(t)\phi(t) \]

So, (1) can be rewritten as

\[ L[y](t) = 0 \quad \alpha < t < \beta \]

or

\[ L[y] = 0 \quad \alpha < t < \beta \]  

So, IVP that we will focus on can be written as

\[ \begin{cases} L[y](t) = 0 & (2) \\ y(t_0) = y_0 \\ y'(t_0) = y'_0 \end{cases} \] 

where \( t_0 \) is any point in \( I = (\alpha, \beta) \).
Recall: Thm 2.4.1. For 1st order linear diff. eq.
\[ y' + p(t)y = q(t), \quad y(t_0) = y_0 \] (1)

If there exists an I on which \( p(t) \) & \( q(t) \) are continuous and to \( \in I \), then there exists a unique \( y(t) \) that solves IVP (1).

An analogous Thm exists for 2nd order linear Eq:

Thm 5.2.1. (Existence & Uniqueness for 2nd order linear Eq.)
Consider IVP
\[ L[y] = y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0 \] (4)

If there exists an I on which \( p(t) \) & \( q(t) \) & \( g(t) \) are continuous and to \( \in I \), then there exists a unique \( y(t) \) that solves IVP (4).

Ex 1. Find the longest interval in which the sol. of IVP:
\[
\begin{cases}
(t^2 - 3t^2)y'' + ty' - (t + 3)y = 0 \\
y(1) = 2 \\
y'(1) = 1.
\end{cases}
\]

Rewrite the ODE as
\[ y'' + \frac{t}{t^2 - 3t} y' - \frac{t + 3}{t^2 - 3t} y = 0. \]

\( p(t) = t / t^2 - 3t \), \( q(t) = -(t + 3) / (t^2 - 3t) \). They are both cont. on \((-\infty, 0) \cup (0, 3) \cup (3, \infty)\).

So, \((3, \infty)\) is the longest interval containing \( t_0 = 1 \) and on which \( p(t) \), \( q(t) \) are continuous.
Thm 3.2.2 (Principle of Superposition): Consider ODE
\[ L[y] = y'' + p(t)y' + q(t)y = 0. \quad (2) \]

If \( y_1 \) and \( y_2 \) are solution of (2), i.e., \( L[y_1] = 0 \), \( L[y_2] = 0 \), then so is any linear combination of \( y_1 \) and \( y_2 \), say
\[ y = c_1 y_1 + c_2 y_2. \quad (3) \]

PF: It suffices to verify \( L[y] = 0 \).

\[ L[y] = L[c_1 y_1 + c_2 y_2] = (c_1 y_1' + c_2 y_2)' + p(t) (c_1 y_1' + c_2 y_2) + q(t) (c_1 y_1 + c_2 y_2) \]
\[ = c_1 (y_1'' + p(t)y_1' + q(t)y_1) + c_2 (y_2'' + p(t)y_2' + q(t)y_2) \]
\[ = c_1 L[y_1] + c_2 L[y_2] = 0. \quad \Box \]

Q: Can we find \( c_1 \) and \( c_2 \) in (3) to solve IVP
\[ \begin{align*}
(2) 
\{ y(t_0) &= y_0, \\
y'(t_0) &= y'_0
\end{align*} \]

Q: Does (3) include every solution of (2)?

To answer Q, we have Thm 3.2.3.

If \( y_1, y_2 \) are sol. of (2), by Thm 3.2.2, we know (3),
\[ y = c_1 y_1 + c_2 y_2 \]
gives infinitely many sol. of (2). To satisfy initial condition we need
\[ \begin{align*}
y(t_0) &= c_1 y_1(t_0) + c_2 y_2(t_0) = y_0, \\
y'(t_0) &= c_1 y_1'(t_0) + c_2 y_2'(t_0) = y'_0
\end{align*} \]
So
\[
C = \frac{ \begin{vmatrix} y_0 & y_1(t_0) \\ y_0' & y_1'(t_0) \end{vmatrix} } { \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} }
\]
and
\[
C_2 = \frac{ \begin{vmatrix} y_2(t_0) & y_0 \\ y_2'(t_0) & y_0' \end{vmatrix} } { \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} }
\]
as long as
\[
W(y_1, y_2)(t_0) := \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \neq 0,
\]
where \( W(y_1, y_2)(t_0) \) is defined as the Wronskian Determinant or just Wronskian.

So, this argument leads to:

**Thm 3.2.3:** Consider IVP
\[
\begin{cases}
L[y] = y'' + p(t)y' + q(t)y = 0 \quad (2) \\
y(t_0) = y_0 \\
y'(t_0) = y_0'
\end{cases}
\]

Suppose \( y_1(t), y_2(t) \) are solutions of ODE (2) (not IVP), then for \( y_0, y_0', \exists c_1, c_2 \) such that \( y = c_1 y_1(t) + c_2 y_2(t) \) solves IVP (2), (4) if and only if
\[
W(y_1, y_2)(t_0) \neq 0.
\]

**Ex 3.** \( y_1(t) = e^{-2t}, \ y_2(t) = e^{-3t}. \) Compute \( W(y_1, y_2)(t) \).
To answer (2), we have

Thm 3.2.3. Consider ODE (2), \( L[y] = y'' + p(t)y' + q(t)y = 0 \). Suppose \( y_1(t), y_2(t) \) are two solutions of (2). Then the family of solutions

\[ y = c_1 y_1(t) + c_2 y_2(t) \]

includes every solution of (2) if and only if

\[ W(y_1, y_2)(t_0) \neq 0, \text{ for some } t_0. \]

\[ \iff \text{ First we show that if } W(y_1, y_2)(t_0) \neq 0, \text{ for some } t_0, \text{ then } y = c_1 y_1(t) + c_2 y_2(t) \text{ includes every solution of } (2), \text{ i.e., } \]

if \( \phi(t) \) is a sol. of (2), then we can adjust \( c_1, c_2 \)

\[ \phi(t) = c_1 y_1(t) + c_2 y_2(t). \]

To prove, we need Thm 3.2.2. Let \( y_0 = \phi(t_0), y_0' = \phi'(t_0) \)

and consider IVP (5):

\[ L[y] = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0'. \]

Then, obviously, \( \phi(t) \) \text{ solves IVP (5)}.

Also, we can find another sol. of (5): Since \( y_1, y_2 \) solves \( L[y] = 0 \) and \( W(y_1, y_2)(t_0) \neq 0 \), so by Thm 3.2.2, \( \exists C_1, C_2 \)

\[ y = C_1 y_1 + C_2 y_2 \] \text{ solves (5).}

Now, we have two sol. of (5): \( \phi(t) \) and \( y = c_1 y_1 + c_2 y_2 \).

But by 3.2.1, the sol. of (5) is unique, so

\[ \phi(t) = C_1 y_1(t) + C_2 y_2(t). \]
\[ \Rightarrow \text{ To prove } W(y_1, y_2) \neq 0 \text{ for some } t_0 \text{ is necessary, we show by contradiction.} \]

Suppose \( W(y_1, y_2)(t_0) = 0 \) for all \( t_0 \), for contradiction. Then pick an arbitrary \( t_0 \). By \( W(y_1, y_2)(t_0) = 0 \) and Thm 3.2.2, it is NOT ALWAYS possible to find \( c_1, c_2 \) s.t. \( y = c_1 y_1 + c_2 y_2 \) solves IVP (5). This means we can find \( y_0, y_0' \) such that we are not able to find \( c_1, c_2 \) s.t. \( y = c_1 y_1 + c_2 y_2 \) solves (5):

\[
L[y] = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0'.
\]

But, by Thm 3.2.1, Sol. of (5) does exist, say \( \varphi(t) \). So in this case \( \varphi(t) \neq c_1 y_1 + c_2 y_2 \) for any \( c_1, c_2 \).

**Remark:**
- For ODE \( L[y] = y'' + p(t)y' + q(t)y = 0 \), if we are able to find two Sol. \( y_1, y_2 \) and \( W(y_1, y_2)(t_0) \neq 0 \) for some \( t_0 \), then \( y = c_1 y_1(t) + c_2 y_2(t) \) gives all solutions, which is called the general solution of \( L[y] = 0 \).
- \( y_1(t), y_2(t) \) is called a fundamental set of solution.
- Fundamental set of solution always exists, thus does the general solution.

**Thm 3.2.5.** Consider ODE (2): \( L[y] = y'' + p(t)y' + q(t)y = 0 \), where \( p(t), q(t) \) are continuous. Then Fund. Set of Sol. of \( L[y] \) exists.

**Pf:** Consider (2) together with \( y(t_0) = 1, y'(t_0) = 0 \). By Thm 3.2.1, \( \exists y_1(t) \) solves this IVP.

Consider (2) together with \( y(t_0) = 0, y'(t_0) = 1 \). By Thm 3.2.1, \( \exists y_2(t) \) solves this IVP.

Now \( W(y_1, y_2)(t_0) = \frac{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}{1} = 1 \neq 0 \)

So, by Thm 3.2.4, \( y_1(t), y_2(t) \) is a Fund. Set of Sol.
Thm 3.2.1: Existence & Uniqueness

Thm 3.2.: Superposition of Nonhomogeneous

Solution of Main Theorems in § 3.2.

Thm 3.2.3: \( [y] = 0 \), [De] = 0, \( W(y, r_2) = 0 \)

Thm 3.2.4: Structure of ODE (2): \( l[y] = 0 \)

Every set of \( l[y] = 0 \) has the form:

Thm 3.2.5: Fundamental Set of Solutions

\( y_p + C_1 y_1 + C_2 y_2 \), where \( y_p, y_1, y_2 \) is a

always exists

Set of \( y \), \( \omega \) is a
Some additional theorems

**Thm 3.2.6**: Consider ODE \((2)\) \(L[y] = y'' + p(t)y' + q(t)y = 0\), if \(y = u(t) + iv(t)\) is a complex valued sol. of \((2)\), then so are the real part \(u(t)\) and image part \(v(t)\).

Verify directly by definition.

**Thm 3.2.7**: Suppose \(y_1(t), y_2(t)\) solve ODE \((2)\): \(L[y] = 0\).

Then
\[
W(y_1, y_2)(t) = c \exp \left[- \int p(t) \, dt \right],
\]
where \(c\) is a constant depends on \(y_1\) & \(y_2\).

Remark: Structure of \(W(y_1, y_2)(t)\) tells us it is either constantly zero or never zero.

**HW**: § 3.2. Pr 4, 6, 10, 11, 15, 16 (Hint: Read Example 2).
20, 25, 27, 31, 32, 37.
Add. 1. Prove 3.2.6.
Add. 2. Read the Pf of 3.2.4 and try to prove by yourself.
Add. 3. Read all examples of in § 3.2.