Lecture 12
§ 7.1. Introduction.

Why learn 1st order diff. eq. SYSTEM?

- Can always transform higher order diff. eq. to an equivalent 1st order system.

- Although finding the explicit sol. may require the same labor, it is more convenient to find numerical sol. for 1st order systems.

Ex 1. Rewrite the ODE

\[ u^{(3)} + 2t u'' + t^3 u' - 4t^4 u = 0 \]  

as a 1st order diff. eq. system.

Let \( x_1 = u \), \( x_2 = u' \), \( x_3 = u'' \). Then \((1) \iff x'_2 + 2tx_3 + t^3 x_2 - 4tx_1 = 0 \). So we may rewrite \((1)\) as

\[
\begin{align*}
    x'_1 &= x_2 \\
    x'_2 &= x_3 \\
    x'_3 &= 4tx_1 - t^3 x_2 - 2tx_3
\end{align*}
\]

Generally, consider a \( n^{th} \) order diff. eq.

\[ u^{(n)} = F(t, u, u', \ldots, u^{(n-1)}) \]  

To rewrite \((2)\) as a 1st order system, let

\[ x_1 = u, \ x_2 = u', \ldots, \ x_n = u^{(n-1)} \]

Then \((2) \iff x'_n = F(t, x_1, x_2, \ldots, x_n) \). So \((2)\) can be rewritten as
\[
\begin{align*}
\begin{cases}
X'_1 &= X_2 \\
X'_2 &= X_3 \\
&\vdots \\
X'_{n-1} &= X_n \\
X'_n &= F(t, X_1, \ldots, X_n)
\end{cases}
\end{align*}
\tag{3}.
\]

(3) can be regarded as a special case of a 1st order diff. eq. system with general form

\[
\begin{align*}
\begin{cases}
X'_1 &= F_1(t, X_1, \ldots, X_n) \\
&\vdots \\
X'_n &= F_n(t, X_1, \ldots, X_n)
\end{cases}
\end{align*}
\tag{4}.
\]

A sol. of (4) on the interval \(I = (\alpha, \beta)\) is a vector function (or a set of functions)

\[
\bar{X}(t) = \begin{bmatrix} \phi_1(t) \\ \vdots \\ \phi_n(t) \end{bmatrix},
\tag{5}
\]

such that

\[
\begin{align*}
\begin{cases}
\phi'_1(t) &= F_1(t, \phi_1(t), \ldots, \phi_n(t)) \\
&\vdots \\
\phi'_n(t) &= F_n(t, \phi_1(t), \ldots, \phi_n(t))
\end{cases}
\end{align*}
\]

Remark: In (5), \(\bar{X}(t)\), the vector function, is considered as ONE sol., instead of \(n\) sol.
Another special case of (4), (which we will mainly discuss), is that each $F_j$ is linear. In this case we say this system is linear:

\[
\begin{align*}
X_1' &= P_{11}(t)X_1 + \cdots + P_{1n}(t)X_n + f_1(t) \\
& \quad \vdots \\
X_n' &= P_{n1}(t)X_1 + \cdots + P_{nn}(t)X_n + f_n(t)
\end{align*}
\]

(6)

or, in a much more convenient way,

\[
\begin{align*}
\dot{X}(t) &= P(t)\cdot \dot{X} \quad + \quad \mathbf{g}(t) \\
(6')
\end{align*}
\]

where $P(t) = (P_{ij}(t))_{n\times n}$.

If, in addition, $g_j(t) \equiv 0 \quad \forall j=1, \ldots, n$, (6') becomes

\[
\begin{align*}
\dot{X}(t) &= P(t)\cdot \dot{X} \\
(7)
\end{align*}
\]

In this case we say the eq. system is homogeneous.

An IVP for (6) (or (6')) has the form

\[
\begin{align*}
\begin{cases}
\dot{X}(t) &= P(t)\cdot \dot{X} \quad + \quad \mathbf{g}(t) \\
X_1(t) &= X_1, \quad \ldots, \quad X_n(t) = X_n.
\end{cases}
\end{align*}
\]

(8)

For (8), we have the following

Thm 7.1.2 (Existence & Uniqueness of Sol.) For (8), if $P_{ij}$ - the entries of $P(t)$, $g_j$ - the entries of $\mathbf{g}(t)$ are cont. on an open interval of $t$, say $I = (a, b)$. Then there exists a unique vector $\mathbf{X}(t)$ that solves (8).
Remark:
- $\alpha$ can be $-\infty$ and $\beta$ can be $+\infty$.
- Meaning of uniqueness: if
  \[
  \begin{bmatrix}
  y_1(t) \\
  \vdots \\
  y_n(t)
  \end{bmatrix},
  \begin{bmatrix}
  x_1(t) \\
  \vdots \\
  x_n(t)
  \end{bmatrix}
  \]
  are both sol. of (8), then $\begin{bmatrix} y_1(t) \\
  \vdots \\
  y_n(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\
  \vdots \\
  x_n(t) \end{bmatrix}$, i.e., $\phi_j(t) = y_j(t)$,
  $\forall j = 1, \ldots, n, \quad \forall t \in I$.
- See Thm 3.2.1 for the analogous theorem for second order eq.

HW: 3, 4, 5, 6, 7.
§ 7.4. Basic Theory.

In this section, we consider the following system
\[ \vec{X}' = P_{n \times n}(t) \vec{X} + \vec{f}(t), \]  \hspace{1cm} (1) \]
or the corresponding homogeneous system
\[ \vec{X}' = P_{n \times n}(t) \vec{X} \] \hspace{1cm} (2).

Thm: (Principle of superposition). If \( \vec{X}^{(1)} \) and \( \vec{X}^{(2)} \) solve (2), then the linear combination \( C_1 \vec{X}^{(1)} + C_2 \vec{X}^{(2)} \) also solves (2).

Pf: By condition
\[ \vec{X}^{(1)'} = P(t) \vec{X}^{(1)}, \hspace{0.5cm} \vec{X}^{(2)'} = P(t) \vec{X}^{(2)}. \]
So \[ [C_1 \vec{X}^{(1)} + C_2 \vec{X}^{(2)}]' = C_1 \vec{X}^{(1)'} + C_2 \vec{X}^{(2)'} \] \hspace{1cm} [add. rule for matrix fn.]
\[ = C_1 P(t) \vec{X}^{(1)} + C_2 P(t) \vec{X}^{(2)} \] \hspace{1cm} [assumption]
\[ = P(t) [C_1 \vec{X}^{(1)} + C_2 \vec{X}^{(2)}] \]
So, \( C_1 \vec{X}^{(1)} + C_2 \vec{X}^{(2)} \) solves (2).

Remark: By induction, if \( \vec{X}^{(1)}, \ldots, \vec{X}^{(m)} \) solve (2), then the linear combination
\[ \vec{X} = C_1 \vec{X}^{(1)} + \cdots + C_m \vec{X}^{(m)} \] \hspace{1cm} (3)
also solves (2).

Q: Whether all sol. of (2) have this form?
Recall for 2nd order linear diff. eq. \( y'' + p(t)y' + q(t)y = 0 \), if \( \{ y_1, y_2 \} \) is a fund. set of sol. (\( W(y_1, y_2)(t_0) = 0 \) for some \( t_0 \)), then every sol. has the form \( C_1 y_1 + C_2 y_2 \).

Similar situation happens to (2):

Let \( \bar{x}^{(1)}, \ldots, \bar{x}^{(n)} \) be \( n \) sol. of (2), consider the matrix \( X_{nn}(t) \) whose columns are \( \bar{x}^{(1)}, \ldots, \bar{x}^{(n)} \):

\[
X_{nn}(t) = [\bar{x}^{(1)}(t), \ldots, \bar{x}^{(n)}(t)]
\]

\[
= \begin{bmatrix}
    x_{11}(t) & \cdots & x_{1n}(t) \\
    \vdots & & \vdots \\
    x_{n1}(t) & \cdots & x_{nn}(t)
\end{bmatrix}
\]

Note \( \{ \bar{x}^{(1)}, \ldots, \bar{x}^{(n)} \} \) are linearly independent if and only if

\[
\det X_{nn}(t) \neq 0.
\]

So, we define the Wronskian determinant of \( \{ \bar{x}^{(1)}, \ldots, \bar{x}^{(n)} \} \) as

\[
W[\bar{x}^{(1)}, \ldots, \bar{x}^{(n)}](t) = \det X(t).
\]

If \( W[\bar{x}^{(1)}, \ldots, \bar{x}^{(n)}](t) \neq 0 \), we actually define \( \{ \bar{x}^{(1)}, \ldots, \bar{x}^{(n)} \} \) are \( n \) linearly independent sol.

Compare to Thm. 3.2.4 for second order, we have:

Thm 7.4.2 Consider (2): \( \bar{x}' = P(t) \bar{x} \). If \( \bar{x}^{(1)}, \ldots, \bar{x}^{(n)} \) solve (2) on \( I = (\alpha, \beta) \). Then

\[
\bar{y}(t) = C_1 \bar{x}^{(1)}(t) + \cdots + C_n \bar{x}^{(n)}(t)
\]

includes all sol. of (2) iff \( W[\bar{x}^{(1)}, \ldots, \bar{x}^{(n)}] \neq 0 \), \( \forall t \in I \).
Remark: - In this case, i.e., if \( W[\dot{x}^{(1)}, \ldots, \dot{x}^{(n)}] \neq 0 \), we say \([\dot{x}^{(1)}, \ldots, \dot{x}^{(n)}]\) is a fundamental solution set of (2).

- Since, in this case, (6) includes all sol. of (2), we say (6) is the general sol. of (2).

pf: Suppose \( \tilde{\phi}(t_0) \) is a sol. of (2). Then define

\[
\tilde{\phi}'(t) = \tilde{\phi}''(t_0),
\]

where \( t_0 \) is an arbitrary point in \( I \). Obviously, \( \tilde{\phi}(t) \) solves the IVP

\[
\begin{align*}
\dot{\tilde{x}} &= \text{P}(t) \tilde{x} \\
\tilde{\phi}(t_0) &= \tilde{\phi}(t_0)
\end{align*}
\]

(7).

But we can find another sol. of (7):

- it already solves the ODE in (7).

- We can show for some \( C_1, \ldots, C_n \), \( \text{det} X(t_0) \neq 0 \) by assumption.

\[
C_1 \tilde{x}(t_0) + \cdots + C_n \tilde{x}(t_0) = \frac{\tilde{x}}{3}.
\]

It has sol. since \( \text{det} X(t_0) \neq 0 \) by assumption.

\( \Rightarrow \) Now suppose \( \exists \bar{t} \in I \) s.t. \( W[\dot{x}^{(1)}, \ldots, \dot{x}^{(n)}] \neq 0 \).

Then

\[
C_1 \dot{x}^{(1)} + \cdots + C_n \dot{x}^{(n)} = \frac{\tilde{x}}{3}.
\]

doesn't have sol. for some \( \tilde{\phi}(t_0) \).

Let \( \phi(t) \) be the sol. of IVP

\[
\begin{align*}
\dot{\phi}' &= \text{P}(t) \phi \\
\phi(t_0) &= \frac{\tilde{\phi}(t_0)}{3}.
\end{align*}
\]

The existence of \( \phi(t) \) is ensured by Thm 7.1.2. But in this case

\[
\phi(t_0) \neq C_1 \dot{x}^{(1)} + \cdots + C_n \dot{x}^{(n)},
\]

for any \( C_1, \ldots, C_n \).
Thm 7.4.4. (Existence of Fund. set of Sol.). Consider (2):
\[ \overrightarrow{X}' = P(t) \overrightarrow{X}, \]
where \( P_{ij}(t) \) are continuous on an interval \( I = (\alpha, \beta) \), for \( i, j = 1, \ldots, n \). Then there exists a fund. set of sol. of (2).

Pf: Consider IVP
\[ \begin{align*}
\overrightarrow{X}' &= P(t) \overrightarrow{X}, \\
\overrightarrow{X}(t_0) &= \overrightarrow{e}^{(i)},
\end{align*} \tag{8} \]
where \( t_0 \) is a point in \( (\alpha, \beta) \), \( \overrightarrow{e}^{(i)} = (1, 0, \ldots, 0)^T \).

Then by Thm 7.1.2, Sol. of (8) exists, denoted by \( \overrightarrow{X}^{(i)} \).

Generally, consider IVP
\[ \begin{align*}
\overrightarrow{X}' &= P(t) \overrightarrow{X}, \\
\overrightarrow{X}(t_0) &= \overrightarrow{e}^{(i)},
\end{align*} \tag{9} \]
where \( t_0 \) is a point in \( (\alpha, \beta) \), \( \overrightarrow{e}^{(i)} = (0, \ldots, 1, \ldots, 0)^T \)

Then by Thm 7.1.2, Sol. of (9) exists, denoted by \( \overrightarrow{X}^{(i)} \).

Then it is easy to see \( W[\overrightarrow{X}^{(1)}, \ldots, \overrightarrow{X}^{(i)}](t_0) \neq 0 \).
So, \( \{ \overrightarrow{X}^{(i)} : i = 1, \ldots, n \} \) is a fund. set of sol.

Thm 7.4.3. (Structure of Wronskian). Consider (2): \( \overrightarrow{X}' = P(t) \overrightarrow{X} \).
If \( \overrightarrow{X}^{(1)}, \ldots, \overrightarrow{X}^{(n)} \) is a fund. set if \( \overrightarrow{X}^{(1)}, \ldots, \overrightarrow{X}^{(n)} \)
are sol. of (2) Then
\[ W[\overrightarrow{X}^{(1)}, \ldots, \overrightarrow{X}^{(n)}] = c \exp \left\{ \int [p_{11}(t) + \ldots + p_{nn}(t)] \, dt \right\} \]
\[ = c \exp \left\{ \int \text{tr} P_0(t) \, dt \right\} \]
So, Wronskian is either constantly zero or never zero.
Thm 7.4.5. Consider (2): \( \dot{\mathbf{x}} = P(t)\mathbf{x} \), where \( P(t) \) is a real-valued matrix function. If \( \mathbf{x} = \mathbf{u}(t) + i\mathbf{v}(t) \) is a complex valued sol. of (2), then its real part \( \mathbf{u}(t) \) and imaginary part \( \mathbf{v}(t) \) are also sol. of (2).