1. [20 pts.] (a) Solve the following initial value problem for $y(t)$:

\[ y' + 2ty = e^{-t^2}, \quad y(0) = y_0. \]

(b) For what initial-value $y_0$ is $y(2) = 0$?

Solution.

• (a) This is a first-order, linear ODE, so we can solve it by the integrating-factor method.

• Multiplication of the ODE by the integrating factor

\[ \mu(t) = e^{\int 2t \, dt} = e^{t^2} \]

gives

\[ \left( e^{t^2} y \right)' = 1. \]

Integration of this equation and imposition of the initial condition gives

\[ y(t) = (t + y_0) e^{-t^2}. \]

• (b) We have $y(2) = 0$ if $y_0 = -2$. 
2. [20 pts.] (a) Solve the initial value problem

\[ yy' + 1 = t, \quad y(6) = 3. \]

(b) For what \( t \)-interval is the solution defined?

Solution.

- (a) The equation is separable. Separating variables, we get
  \[ y \, dy = (t - 1) \, dt. \]

- Integrating this equation, and multiplying the result by 2, we get
  \[ y^2 = (t - 1)^2 + c. \]

- The initial condition implies that \( c = -16 \). After solving for \( y \), we find that the solution is
  \[ y(t) = \sqrt{(t - 1)^2 - 16}. \]

- (b) The solution is well-defined and differentiable provided that the quantity inside the square-root is positive, meaning that \( 5 < t < \infty \).
3. [20 pts.] (a) Find the equilibrium solutions of the equation

\[ y' = y (y - 2)^3. \]

(b) Sketch the phase line of the equation, and determine the stability of the equilibria you found in (a).

(c) How does the solution with \( y(0) = -1 \) behave as \( t \to +\infty \)? How does the solution with \( y(0) = 1 \) behave as \( t \to -\infty \)?

Solution.

• (a) The equilibria are \( y = 0 \) and \( y = 2 \).

• (b) The function \( f(y) = y (y - 2)^3 \) is positive if \( y < 0 \) or \( y > 2 \), and negative if \( 0 < y < 2 \). Hence, the flow on the phase line is to the right with increasing \( t \) if \( y < 0 \) or \( y > 2 \), and to the left if \( 0 < y < 2 \). The equilibrium \( y = 0 \) is asymptotically stable, and the equilibrium \( y = 2 \) is unstable. (A sketch of the phase line is omitted.)

• (c) If \( y(0) = -1 \), then \( y(t) \to 0 \) as \( t \to +\infty \). If \( y(0) = 1 \), then \( y(t) \to 2 \) as \( t \to -\infty \).
4. [20 pts.] Suppose that $y_1(t)$, $y_2(t)$, $y_3(t)$ are solutions of the following initial value problems:

\[ e^t y_1'' + y_1 = \frac{1}{1 + t}, \quad y_1(0) = 1, \quad y_1'(0) = 0, \]
\[ e^t y_2'' + y_2 = \frac{1}{1 - t}, \quad y_2(0) = 0, \quad y_2'(0) = 1, \]
\[ e^t y_3'' + y_3 = \frac{2}{1 - t^2}, \quad y_3(0) = 1, \quad y_3'(0) = 1. \]

(a) According to general theorems, for what $t$-intervals are $y_1(t)$, $y_2(t)$, $y_3(t)$ uniquely defined?

(b) Express $y_3(t)$ in terms of $y_1(t)$, $y_2(t)$, and justify your answer.

**Solution.**

- (a) Since $e^t$ is never zero, the continuity of the coefficient functions

\[ p(t) = 0, \quad q(t) = e^{-t}, \quad g_1(t) = \frac{e^{-t}}{1 + t}, \quad g_2(t) = \frac{e^{-t}}{1 - t}, \quad g_3(t) = \frac{2e^{-t}}{1 - t^2} \]

implies that $y_1(t)$ is defined for $-1 < t < \infty$, $y_2(t)$ is defined for $-\infty < y < 1$, and $y_3(t)$ is defined for $-1 < t < 1$.

- Let $y(t) = y_1(t) + y_2(t)$. Then, by linearity,

\[ e^t y'' + y = e^t y_1'' + y_1 + e^t y_2'' + y_2 = \frac{1}{1 + t} + \frac{1}{1 - t} = \frac{2}{1 - t^2}, \]

and

\[ y(0) = y_1(0) + y_2(0) = 1, \]
\[ y'(0) = y_1'(0) + y_2'(0) = 1. \]

It follows that $y(t)$ satisfies the same IVP as $y_3(t)$, so $y(t) = y_3(t)$ by the existence-uniqueness theorem, and

\[ y_3(t) = y_1(t) + y_2(t) \quad \text{for} \quad -1 < t < 1. \]
5. [20 pts.] (a) Find the general solution of the equation
\[ y'' - 4y' + 3y = 0. \]
(b) Find the general solution of the equation
\[ 2y'' + 2y' + 5y = 0. \]

Solution.
• (a) The characteristic equation is
\[ r^2 - 4r + 3 = 0, \]
with solutions
\[ r = 1, 3. \]
The general solution is
\[ y(t) = c_1 e^t + c_2 e^{3t}, \]
where \( c_1, c_2 \) are arbitrary constants.
• (b) The characteristic equation is
\[ 2r^2 + 2r + 5 = 0, \]
with solutions
\[ r = -\frac{1}{2} \pm \frac{3}{2} i. \]
The general solution is
\[ y(t) = c_1 e^{-t/2} \cos \left( \frac{3t}{2} \right) + c_2 e^{-t/2} \sin \left( \frac{3t}{2} \right), \]
where \( c_1, c_2 \) are arbitrary constants.
6. [20 pts.] Suppose that $\omega_0$, $\omega$, $F_0$ are nonzero constants, and consider the ordinary differential equation

$$y'' + \omega_0^2 y = F_0 \cos(\omega t).$$

(a) Find a particular solution for $y(t)$ if $\omega \neq \omega_0$.

(b) Find a particular solution for $y(t)$ if $\omega = \omega_0$.

(c) Give a brief physical interpretation of these solutions for oscillators.

**Solution.**

- (a) If $\omega \neq \omega_0$ (and also assuming $\omega \neq -\omega_0$), then, using the method of undetermined coefficients, we look for a solution of the form

$$y(t) = A \cos \omega t.$$ 

In that case,

$$y'' + \omega_0^2 y = (-\omega^2 + \omega_0^2) A \cos \omega t.$$ 

We obtain a particular solution if $(-\omega^2 + \omega_0^2) A = F_0$, meaning that

$$y(t) = \frac{F_0}{\omega_0^2 - \omega^2} \cos \omega t.$$ 

(b) If $\omega = \omega_0$, then a solution of the form assumed in (a) does not work. Instead, we look for a particular solution of the form

$$y(t) = At \sin \omega_0 t.$$ 

Then we compute that

$$y'' + \omega_0^2 y = 2A \omega_0 \cos \omega_0 t.$$ 

We obtain a particular solution if $2A \omega_0 = F_0$, meaning that

$$y(t) = \frac{F_0}{2\omega_0} t \sin \omega_0 t.$$ 

(c) These solutions illustrate the phenomenon of resonance. If an undamped simple harmonic oscillator is forced sinusoidally by an external force whose frequency $\omega$ is different from the natural frequency $\omega_0$ of the oscillator, then the response of the oscillator is also sinusoidal, with large amplitude if $\omega$ is close to $\omega_0$. If the frequency of the external force is equal to the natural frequency of the oscillator ($\omega = \omega_0$), then the amplitude of the oscillator grows linearly in time.
7. [20 pts.] Find the general solution of the following $2 \times 2$ system, and express your answer in terms of real-valued functions:

$$
\vec{x}'(t) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \vec{x}(t).
$$

Solution.

- The characteristic polynomial of the coefficient matrix is

$$
\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda + 1
$$

Solving the equation $\lambda^2 - \lambda + 1 = 0$, we find that the eigenvalues are

$$
\lambda = \frac{1}{2} \pm i \frac{\sqrt{3}}{2}.
$$

- For a system with real coefficients, it is sufficient to consider only one of the eigenvalues in a complex conjugate pair, say $\lambda = 1/2 + i\sqrt{3}/2$. Then

$$
A - \lambda I = \begin{pmatrix} 1/2 - i\sqrt{3}/2 & 1 \\ -1 & -1/2 - i\sqrt{3}/2 \end{pmatrix}.
$$

An eigenvector is

$$
\vec{\xi} = \begin{pmatrix} 1 \\ -1/2 + i\sqrt{3}/2 \end{pmatrix}.
$$

- The corresponding solution is

$$
e^{\lambda t} \vec{\xi} = e^{(1/2+i\sqrt{3}/2)t} \begin{pmatrix} 1 \\ -1/2 + i\sqrt{3}/2 \end{pmatrix}
$$

$$
= e^{t/2} \left[ \cos \left( \frac{\sqrt{3}}{2} t \right) + i \sin \left( \frac{\sqrt{3}}{2} t \right) \right] \left[ \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} + i \begin{pmatrix} 0 \\ \sqrt{3}/2 \end{pmatrix} \right]
$$

$$
= e^{t/2} \cos \left( \frac{\sqrt{3}}{2} t \right) \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} - e^{t/2} \sin \left( \frac{\sqrt{3}}{2} t \right) \begin{pmatrix} 0 \\ \sqrt{3}/2 \end{pmatrix}
$$

$$
+i \left\{ e^{t/2} \sin \left( \frac{\sqrt{3}}{2} t \right) \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} + e^{t/2} \cos \left( \frac{\sqrt{3}}{2} t \right) \begin{pmatrix} 0 \\ \sqrt{3}/2 \end{pmatrix} \right\}
$$
The real and imaginary parts of this complex solution form a fundamental pair of real-valued solutions for the ODE. The general solution is

\[
\ddot{x}(t) = c_1 e^{t/2} \left\{ \cos \left( \frac{\sqrt{3}}{2} t \right) \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} - \sin \left( \frac{\sqrt{3}}{2} t \right) \begin{pmatrix} 0 \\ \sqrt{3}/2 \end{pmatrix} \right\} \\
+ c_2 e^{t/2} \left\{ \sin \left( \frac{\sqrt{3}}{2} t \right) \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} + \cos \left( \frac{\sqrt{3}}{2} t \right) \begin{pmatrix} 0 \\ \sqrt{3}/2 \end{pmatrix} \right\},
\]

where \( c_1 \) and \( c_2 \) are arbitrary constants.
8. [20 pts.] Suppose that a $2 \times 2$ matrix $A$ has the following eigenvalues and eigenvectors:

$$r_1 = -2, \quad \vec{\xi}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}; \quad r_2 = 1, \quad \vec{\xi}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$ 

(a) Classify the equilibrium $\vec{x} = 0$ (node, saddle, spiral, center). Is it stable or unstable?

(b) Sketch the trajectories of the system $\vec{x}' = A\vec{x}$, where $\vec{x} = (x_1, x_2)^T$, in the phase plane below.

(c) On the next page, sketch the graphs of $x_1(t)$ and $x_2(t)$ versus $t$ for the solution that satisfies the initial condition $x_1(0) = 1, x_2(0) = 1$.

Solution.

- (a) The equilibrium is a saddle point (two real eigenvalues with opposite signs), which is unstable.

- Sketches for (b), (c) are omitted.
9. [20 pts.] Recall that
\[ e^{tA} = I + tA + \frac{1}{2!} t^2 A^2 + \ldots + \frac{1}{k!} t^k A^k + \ldots \]

(a) Compute \( e^{tA} \) explicitly if
\[ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

**HINT.** Recall the expansions:
\[ e^t = 1 + t + \frac{1}{2!} t^2 + \frac{1}{3!} t^3 + \ldots + \frac{1}{k!} t^k + \ldots; \]
\[ e^{-t} = 1 - t + \frac{1}{2!} t^2 - \frac{1}{3!} t^3 + \ldots + \frac{(-1)^k}{k!} t^k + \ldots \]

(b) Use your result from (a) to write out the solution \( \bar{x}(t) = (x_1(t), x_2(t))^T \) of the initial value problem:
\[ \bar{x}' = A \bar{x}, \quad \bar{x}(0) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \]

For what initial values \((c_1, c_2)^T\) is it true that \( \bar{x}(t) \to 0 \) as \( t \to +\infty \)?

**Solution.**

- (a) We compute that
  \[ A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I. \]
  It follows that \( A^3 = A^2 \cdot A = I \cdot A = A, \ A^4 = A^2 \cdot A^2 = I \cdot I = I, \) and so on. In general,
  \[ A^k = I \quad \text{if } k \text{ is even}, \quad A^k = A \quad \text{if } k \text{ is odd}. \]

- The series for \( e^{tA} \) therefore becomes
  \[ e^{tA} = \left( 1 + \frac{1}{2!} t^2 + \frac{1}{4!} t^4 + \ldots \right) I + \left( t + \frac{1}{3!} t^3 + \frac{1}{5!} t^5 + \ldots \right) A. \]
  Adding and subtracting the power series expansions of \( e^t \) and \( e^{-t} \), we find that
  \[ \frac{1}{2} (e^t + e^{-t}) = 1 + \frac{1}{2!} t^2 + \frac{1}{4!} t^4 + \ldots; \]
  \[ \frac{1}{2} (e^t - e^{-t}) = t + \frac{1}{3!} t^3 + \frac{1}{5!} t^5 + \ldots. \]
It follows that
\[
e^{tA} = \frac{1}{2} (e^t + e^{-t}) I + \frac{1}{2} (e^t - e^{-t}) A
\]
\[
= \frac{1}{2} (e^t + e^{-t}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2} (e^t - e^{-t}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
\[
= \frac{1}{2} e^t \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2} e^{-t} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.
\]

• (b) The solution is \( \vec{x}(t) = e^{tA} \vec{x}(0) \), which gives
\[
\vec{x}(t) = \frac{1}{2} e^t \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \frac{1}{2} e^{-t} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}
\]
\[
= \frac{1}{2} e^t (c_1 + c_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} e^{-t} (c_1 - c_2) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]

• The solution approaches 0 as \( t \to +\infty \) if the coefficient of term proportional to \( e^t \) is 0. This is the case when
\[
c_1 + c_2 = 0,
\]
meaning that
\[
\vec{x}(0) = c \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]
for some scalar \( c \).

Remark. We can write these results in more a compact form by use of the hyperbolic cosine and sine,
\[
cosh t = \frac{e^t + e^{-t}}{2}, \quad \sinh t = \frac{e^t - e^{-t}}{2}.
\]
Then
\[
e^{tA} = \cosh t I + \sinh t A
\]
\[
= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix},
\]
and the solution of the ODE is
\[
\vec{x}(t) = c_1 \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + c_2 \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix}.
\]
Note the analogy between these expressions for the exponential of the symmetric matrix $A$ with real eigenvalues, and the exponential of the antisymmetric matrix

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with imaginary eigenvalues that we considered in class. We showed there that

$$e^{tB} = \cos t I + \sin t B = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$
10. [20 pts.] (a) Use the method of variation of parameters to solve the ODE

\[ y'' - y = g(t), \]

where \( g(t) \) is a given continuous function. That is, write

\[ y(t) = e^t u_1(t) + e^{-t} u_2(t), \quad e^t u'_1(t) + e^{-t} u'_2(t) = 0. \]

Solve for \( u'_1(t) \), \( u'_2(t) \), and find the general solution for \( y(t) \).

(b) Show that you can write the solution of the initial value problem

\[ y'' - y = g(t), \quad y(0) = 0, \quad y'(0) = 0 \]

as a convolution

\[ y(t) = \int_0^t K(t - s) g(s) \, ds, \]

and determine the function \( K(t) \).

Solution.

- (a) Using the product rule and the equation satisfied by \( u'_1 \), \( u'_2 \), we compute that

\[
\begin{align*}
y' &= e^t u'_1 + e^{-t} u'_2 + e^t u_1 - e^{-t} u_2 \\
e' u_1 - e^{-t} u_2.
\end{align*}
\]

Differentiating this equation, we get

\[
y'' = e^t u'_1 - e^{-t} u'_2 + e^t u_1 + e^{-t} u_2.
\]

It follows that

\[
y'' - y = e^t u'_1 - e^{-t} u'_2.
\]

Thus, \( y \) satisfies the ODE if \( u_1 \), \( u_2 \) satisfy

\[
\begin{align*}
e^t u'_1 - e^{-t} u'_2 &= g(t), \\
e^t u'_1 + e^{-t} u'_2 &= 0.
\end{align*}
\]
Solving this pair of linear equations for $u'_1, u'_2$, we get

$$u'_1(t) = \frac{1}{2} e^{-t} g(t),$$

$$u'_2(t) = -\frac{1}{2} e^{t} g(t).$$

Integrating these equations, we get

$$u_1(t) = \frac{1}{2} \int_0^t e^{-s} g(s) \, ds + c_1,$$

$$u_2(t) = -\frac{1}{2} \int_0^t e^{s} g(s) \, ds + c_2,$$

where $c_1, c_2$ are constants of integration.

Using these expressions in the equation $y = e^t u_1 + e^{-t} u_2$, we find that the general solution for $y$ is

$$y(t) = \frac{1}{2} e^t \int_0^t e^{-s} g(s) \, ds - \frac{1}{2} e^{-t} \int_0^t e^{s} g(s) \, ds + c_1 e^t + c_2 e^{-t}.$$

(b) From the expressions

$$y = e^t u_1 + e^{-t} u_2, \quad y' = e^t u_1 - e^{-t} u_2,$$

we see that $y(0) = y'(0) = 0$ if and only if $u_1(0) = u_2(0) = 0$ which implies that $c_1 = c_2 = 0$. The solution of the initial value problem for $y(t)$ is then

$$y(t) = \frac{1}{2} e^t \int_0^t e^{-s} g(s) \, ds - \frac{1}{2} e^{-t} \int_0^t e^{s} g(s) \, ds$$

$$= \frac{1}{2} \int_0^t e^{t-s} g(s) \, ds - \frac{1}{2} \int_0^t e^{-t+s} g(s) \, ds$$

$$= \int_0^t \left[ \frac{e^{t-s} - e^{-(t-s)}}{2} \right] g(s) \, ds$$

$$= \int_0^t \sinh(t-s) g(s) \, ds.$$
This solution has the required convolution form, with

\[
K(t) = \frac{e^t - e^{-t}}{2} = \sinh t.
\]

**Remark.** Note the analogy with the problem

\[
\begin{align*}
y'' + y &= g(t), \\
y(0) &= y'(0) = 0,
\end{align*}
\]

in which the homogeneous ODE has \(\cos t\), \(\sin t\) as solutions instead of \(e^t\), \(e^{-t}\). As we showed in class, by the use of variation of parameters, the solution is

\[
y(t) = \int_0^t \sin(t - s)g(s)\,ds.
\]