1. An object that falls freely in a vacuum, close to the surface of the earth has a constant acceleration of

\[ g = 9.81 \text{ m/sec}^2 = 32 \text{ ft/sec}^2. \]

If the object is dropped from rest, find its velocity and the distance it has travelled \(t\) seconds after it was released.

Ans: Since the object dropped from rest, the initial velocity \(v_0 = 0\). Suppose the unit of time is second, then the velocity at time \(t\) is given by

\[ v_t = v_0 + gt = (0 \text{ m/sec}) + (9.81 \text{ m/sec}^2)(t \text{ sec}) = 9.81\text{m/sec}. \]

Moreover, the distance is

\[ d = v_0t + \frac{1}{2}gt^2 = \frac{1}{2}(9.81 \text{ m/sec}^2)(t \text{ sec})^2 = 4.905t \text{ m}. \]

2. Approximate the area under the parabola \(y = x^2\) from 0 to 1 using four subintervals with left endpoints.

Ans: Divide interval \([0, 1]\) into 4 intervals whose length is given by

\[ \Delta x = \frac{b - a}{n} = \frac{1 - 0}{4} = \frac{1}{4}. \]

Therefore, these four intervals are

\[ [0, \frac{1}{4}], [\frac{1}{4}, \frac{1}{2}], [\frac{1}{2}, \frac{3}{4}], \text{ and } [\frac{3}{4}, 1]. \]

The left points are

\[ x_1 = 0, x_2 = \frac{1}{4}, x_3 = \frac{1}{2}, \text{ and } x_4 = \frac{3}{4}. \]

Approximate the area under the parabola \(y = f(x) = x^2\) on the interval \([0, 1]\) using left endpoints:

\[ \text{area} \approx \Delta x (f(x_1) + f(x_2) + f(x_3) + f(x_4)) \]
\[ = \Delta x \left( f(0) + f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right) \right) \]
\[ = \frac{1}{4} \left( 0 + \frac{1}{16} + \frac{1}{4} + \frac{9}{16} \right) \]
\[ = \frac{7}{32}. \]
Solutions to problems 4 from discussion sheet #1

January 13, 2015

4. We will use that \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \) and \( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \).

- (a) \( \sum_{i=1}^{n} 9 = 9 \sum_{i=1}^{n} 1 = 9n \)

- (b) \( \sum_{i=1}^{1053} 9 = 9 \sum_{i=1}^{1053} 1 = 9 \cdot 1053 \)

- (c) \( \sum_{i=34}^{876} 9 = 9 \left( \sum_{i=1}^{876} 1 - \sum_{i=1}^{34} 1 \right) = 9 \cdot (876 - 34) = 7578 \)

- (d) \( \sum_{i=1}^{50} i(2i + 3) = 2 \sum_{i=1}^{50} i^2 + 3 \sum_{i=1}^{50} i = \frac{50 \cdot 51 \cdot 101}{3} + \frac{3 \cdot 50 \cdot 51}{2} = 89675 \)

- (e) \( \sum_{i=1}^{60} (5i - i^2) = 5 \sum_{i=1}^{60} i - \sum_{i=1}^{60} i^2 = \frac{60 \cdot 61}{2} - \frac{60 \cdot 61 \cdot 121}{6} = -64660 \)

- (f) \( \sum_{i=26}^{62} (5i - i^2) = \sum_{i=1}^{62} (5i - i^2) - \sum_{i=1}^{25} (5i - i^2) = -67710 \)

- (g) \( \sum_{i=1}^{30} (\log (i + 2) - \log (i + 1)) = \log (30 + 2) - \log (1 + 1) = \log \left( \frac{32}{2} \right) \), by the telescopic property, and the properties of the logarithm.

- (h) \( \sum_{i=1}^{4} \cos(\pi i) = \sum_{i=1}^{4} (-1)^i = 0 \)

- (i) \( \sum_{i=1}^{17} \cos(\pi i) = \sum_{i=1}^{17} (-1)^i = \sum_{i=1}^{16} (-1)^i + (-1)^{17} = -1 \)

- (j) \( \sum_{i=1}^{n} \cos(\pi i) = 0 \) if \( n \) is even, and \(-1\) if \( n \) is odd.
Solutions to problems 5 to 8 from discussion sheet #1

January 14, 2015

Problem 5

We prove this formula by induction:
first we check this formula is true when \( n = 1 \).

\[
\sum_{i=1}^{1} i^2 = 1^2 = 1
\]

then we assume this formula is true for \( n = k \) and deduce it is then true for \( n = k + 1 \)
based on this assumption.

\[
\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k + 1)^2 = \frac{k(k + 1)(2k + 1)}{6} + (k + 1)^2
\]

then we simplify the expression to get

\[
\frac{k(k + 1)(2k + 1)}{6} + (k + 1)^2 = \frac{(k + 1)(2k^2 + 7k + 6)}{6} = \frac{(k + 1)(2k + 3)(k + 2)}{6}
\]

this is the desired result as we can plug in \( n = k + 1 \) to the formula to get the same answer,
which completes the proof.

Problem 6

we are going to use \( \sum_{k=1}^{n} \frac{n(n+1)}{2} \).

\[
\text{a}
1 + 2 + 3... + 75 = \frac{75(75 + 1)}{2} = 2850
\]

\[
\text{c}
1 + 2 + 3... + 2074804 = \frac{2074804(2074804 + 1)}{2} = 2152406856610
\]

\[
\text{b}
151 + 152 + 153... + 364 = \frac{(364 - 151 + 1)(151 + 364)}{2} = 55105
\]
\[ 1.2^6 + 1.1^7 + 1.1^8\ldots + 1.1^200 = 1.2^6 + \frac{1.1^{194} - 1}{1.1 - 1} = 2088958044.051 \]

**Problem 7**

Given the definition of Riemann Integral, we first partition the interval from 0 to 1 into \( n \) pieces with the same length. So the length will be \( \frac{1}{n} \), which is exactly the \( \delta x_k \) in the definition. And we use the value of partition points, which are \( x = \frac{k}{n}, \ k = 1, 2, 3\ldots n \). Then we have the following:

\[
\int_0^1 3x^2 + 2dx = \lim_{|p|\to 0} \sum_{k=1}^{n} \frac{3(\frac{k}{n})^2 + 2}{n} = \lim_{n\to\infty} \frac{3}{n^2} \sum_{k=1}^{n} k^2 + \lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n} 1
\]

Then we use problem 5 to the first term and knowledge from 17A to get:

\[
\lim_{n\to\infty} \frac{3}{n^2} \sum_{k=1}^{n} k^2 = \lim_{n\to\infty} \frac{3n(n+1)(2n+1)}{6n^3} = 1
\]

the second term turns out to be:

\[
\lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n} 1 = \frac{2n}{n} = 2
\]

So we compute \( \int_0^1 3x^2 + 2dx = 3 \). You can check it using Fundamental Theorem of Calculus part 2.

**Problem 8**

With the same way of partition in problem 7, we need to compute the sum:

\[
\int_{-1}^{2} x^2 - 2x + 1dx = \int_{-1}^{2} (x - 1)^2dx = \lim_{n\to\infty} \frac{3}{n} \sum_{k=1}^{n} (-1 + \frac{3k}{n} - 1)^2
\]

After simplification, we have following three terms:

\[
\lim_{n\to\infty} \frac{3}{n} \sum_{k=1}^{n} \frac{9k^2}{n^3} = \lim_{n\to\infty} \frac{27n(n+1)(2n+1)}{6n^3} = 9
\]

\[
\lim_{n\to\infty} -\frac{3}{n} \sum_{k=1}^{n} \frac{12k}{n^2} = \lim_{n\to\infty} -\frac{36n(n+1)}{2n^2} = -18
\]

\[
\lim_{n\to\infty} -\frac{3}{n} \sum_{k=1}^{n} 4 = 12
\]

So we have the integral \( \int_{-1}^{2} x^2 - 2x + 1dx = 9 - 18 + 12 = 3 \). You can also check this by Fundamental Theorem of Calculus part 2.
3. Use parts a) and b) to find the area of the region bounded by the graph of \( y = x^2, y = 0 \) between \( x = 0 \) and \( x = 4 \).

(a) Approximate the area using \( n \) intervals and left endpoints.

Ans: Divide interval \([0, 4]\) into \( n \) intervals whose length is given by
\[
\Delta x = \frac{b - a}{n} = \frac{4 - 0}{n} = \frac{4}{n}.
\]

Therefore, these \( n \) intervals are
\[
\left[0, \frac{4}{n}\right], \left[\frac{4}{n}, \frac{8}{n}\right], \left[\frac{8}{n}, \frac{12}{n}\right], \ldots, \left[\frac{4n - 4}{n}, 4\right].
\]
The left points are
\[
x_1 = 0, x_2 = \frac{4}{n} = \frac{4 \cdot 1}{n}, x_3 = \frac{8}{n} = \frac{4 \cdot 2}{n}, \ldots, x_n = \frac{4n - 4}{n} = \frac{4 \cdot (n - 1)}{n}.
\]
Approximate the area under the parabola \( y = f(x) = x^2 \) on the interval \([0, 4]\) using left endpoints:
\[
\text{area} \approx \Delta x \left( f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n) \right)
\]
\[
= \Delta x \left( f(0) + f \left( \frac{4}{n} \right) + f \left( \frac{8}{n} \right) + \cdots + f \left( \frac{4n - 4}{n} \right) \right)
\]
\[
= \frac{4}{n} \left[ 0 + \left( \frac{4}{n} \right)^2 + \left( \frac{8}{n} \right)^2 + \cdots + \left( \frac{4n - 4}{n} \right)^2 \right]
\]
\[
= \frac{4}{n} \left[ 0 + \left( \frac{4 \cdot 1}{n} \right)^2 + \left( \frac{4 \cdot 2}{n} \right)^2 + \cdots + \left( \frac{4 \cdot (n - 1)}{n} \right)^2 \right]
\]
\[
= \frac{4}{n} \left( \frac{4}{n} \right)^2 \left[ 0 + 1^2 + 2^2 + \ldots + (n - 1)^2 \right]
\]
\[
= \left( \frac{4}{n} \right)^3 \frac{(n - 1)n(2n - 1)}{6}
\]
\[
= \left( \frac{32}{3} \right) \frac{(n - 1)n(2n - 1)}{n^3}.
\]

(b) Find the limit of the approximation as \( n \to \infty \) in part a) above to find the area of the region

The area of the region is
\[
\lim_{n \to \infty} \left( \frac{32}{3} \right) \frac{(n - 1)n(2n - 1)}{n^3} = \frac{32}{3} \cdot 2 = \frac{64}{3}.
\]
9. Differentiate:

(a) \( F(x) = \int_{-1}^{3x} \sqrt{1 + t^2} \, dt \)

Sol:

\[
\frac{dF(x)}{dx} = (3x)' \sqrt{1 + (3x)^2} = 3 \sqrt{1 + 9x^2}
\]

(b) \( F(x) = \int_{\tan x}^{\sec x} 5t^2 \, dt \)

Sol:

\[
\frac{dF(x)}{dx} = 5 \sec^2 x \cdot (\sec x)' - 5 \tan^2 x \cdot (\tan x)' = 5 \sec^2 x \cdot \sec x \cdot \tan x - 5 \tan^2 x \cdot \sec^2 x
\]

10. Find an equation of the line perpendicular to the graph of

(a) \( F(x) = 3 + \int_0^x 2e^{t^2} \, dt \) at \( x = 0 \)

Sol:

Note that \( \frac{dF(x)}{dx} = 2e^{x^2} \). So \( F'(0) = 2 \). Also note that \( F(0) = 3 \).

So the line perpendicular to \( F(x) \) at \( x = 0 \) should have slope \( \frac{1}{2} \) and pass the point \((0, 3)\). So the equation is \( y = \frac{1}{2} x + 3 \).

(b) \( F(x) = \int_{2x}^{x^2} \sqrt{t^2 + 5} \, dt \) at \( x = 2 \)

Sol:

Note that \( \frac{dF(x)}{dx} = 2x \sqrt{x^4 + 5} - 2 \sqrt{4x^2 + 5} \). So \( F'(2) = 2\sqrt{21} \). Also note that \( F(2) = 0 \).

So the line perpendicular to \( F(x) \) at \( x = 0 \) should have slope \( \frac{\sqrt{21}}{42} \) and pass the point \((2, 0)\). So the equation is \( y = \frac{\sqrt{21}}{42} (x - 2) \).