Math 21A, Fall 2016.

Discussion Problems 5 (Thu., Nov. 3)

1. A 6 ft. tall woman is walking towards a 15 ft. streetlight. At one instant, she is 10 ft. from the base of the streetlight and is walking at the speed of 2 ft./sec. How fast is the length of her shadow changing at that instant?

2. Car B is 30 miles directly East of Car A, and begins moving West at 90 mph. At the same time, Car A begins moving North at 60 mph. At what rate is the distance between them changing after $t = \frac{1}{3}$ hours? After $t = \frac{1}{2}$ hours?

3. A rectangular reservoir has a 10m × 10m base and height 3m. A reservoir in the shape of inverted cone has height 10m and radius of the base 5m. Water is flowing from the square reservoir into the conical one. Initially, the square reservoir is full and the conical empty. At one point, the level of water in the conical reservoir is measured to be 2m and is increasing at the rate of 1m/sec. Find the rate at which the water is decreasing in the rectangular reservoir.

4. Use linearization to approximate: (a) $\sqrt{27}$, (b) $\sqrt[3]{26}$, (c) $(9900)^{1/4}$.

5. Find the linearization of $f(x) = \frac{4e^x}{e^x + 1}$ at $x = 0$.

6. Assume that a baseball is projected directly upward from the ground with an initial velocity of 112 ft/s. Assuming acceleration due to gravity is $-32$ ft/sec$^2$, derive the equation for the height of the ball above ground after $t$ seconds. In how many seconds does the ball strike the ground?
Let $s(t)$ be the shadow's length w.r.t time $t$. Let $y(t)$ be the distance between the woman and the light. We know $dy/dt = -2$, we want to find $ds/dt$.

All we know about $s(t)$ and $y(t)$ is:

$$\frac{6}{15} = \frac{s(t)}{s(t)+y(t)}.$$  

So, $s(t) = \frac{2}{3} y(t)$. 

So, 

$$\frac{ds}{dt} = \frac{d}{dt} \frac{2}{3} y = \frac{2}{3} \frac{dy}{dt} = -\frac{4}{3}.$$  

So, the shadow decreases at a rate of $\frac{4}{3}$ ft/sec constantly. Here, the condition "loft from the light" is redundant.
Let $y(t)$ be the distance between $A$ and $B$. We want to find $\frac{dy}{dt}$.

All we know about $y(t)$ is:

$$y(t) = \sqrt{a^2(t) + b^2(t)}$$

and $a(t) = 60t, b(t) = 30 - 90t$. So

$$y(t) = \sqrt{(60t)^2 + (30 - 90t)^2}$$

$$= 30 \sqrt{13t^2 - 6t + 1}$$

So,

$$\frac{dy(t)}{dt} = 30 \cdot \frac{13t - 3}{\sqrt{13t^2 - 6t + 1}}$$

So,

$$\left. \frac{dy(t)}{dt} \right|_{t=\frac{1}{3}} = -21.2 \text{ mph}$$

$$\left. \frac{dy(t)}{dt} \right|_{t=\frac{1}{5}} = 60.0 \text{ mph.$$}
Let \( h_r(t) \) be the height of water in the rectangular reservoir and \( h_c(t) \) be the one in the conical reservoir. We know when \( h_c(t) = 2 \), \( \frac{dh_c(t)}{dt} = 1 \), we want to find \( \frac{dh_r(t)}{dt} \) at this instant.

We know the equality of the water:

\[
10 \times 10 \times \frac{1}{3} (3 - h_r(t)) = \frac{1}{3} \pi (3 \cdot h_c(t)) h_c(t) \quad (\star)
\]

and \( \frac{r_c(t)}{5} = h_c(t)/10 \Rightarrow r_c(t) = \frac{1}{2} h_c(t) \).

So, by \( (\star) \):

\[
h_r(t) = 3 - \frac{1}{1200} \cdot h_c^3(t)
\]

So,

\[
\frac{dh_r(t)}{dt} = -\frac{1}{1200} \cdot 3 \cdot h_c^2(t) \cdot \frac{dh_c(t)}{dt}
\]

when \( h_c(t) = 2 \), \( \frac{dh_c(t)}{dt} = 1 \), we have

\[
\frac{dh_r(t)}{dt} = -\frac{1}{1200} \cdot 3 \cdot 2^2 \cdot 1 = -\frac{1}{100} \text{ m/s}
\]
Linearization:

Goal: We want to approximate a function $f(x)$ by a linear function around a certain point $x_0$.

Solution: This linear function should satisfy:

i. it passes $(x_0, f(x_0))$
ii. its slope $= f'(x_0)$.

So, the linearization is

$$y - f(x_0) = f'(x_0)(x - x_0).$$

i.e.,

$$y = f'(x_0)(x - x_0) + f(x_0).$$

So, around $x_0$, we may think $f(x)$ (which may be complicated) as

$$y = f'(x_0)(x - x_0) + f(x_0),$$

which is a (simple) linear function.
4.

(a) Consider \( f(x) = \sqrt{x} \). Its linearization around \( x_0 = 25 \) is

\[
y = f'(25) (x-25) + f(25)
\]
i.e.,

\[
y = \frac{1}{2 \cdot \sqrt{25} } \cdot (x-25) + \sqrt{25}
\]

So, when \( x = 27 \),

\[
f(27) = \sqrt{27} \approx y(27) = \frac{1}{2 \cdot 5} \cdot (27-25) + 5 = 5.2.
\]

(b) Consider \( f(x) = \frac{1}{\sqrt{x}} \). Around \( x_0 = 267 \) we use:

\[
y = f'(27) (x-27) + f(27)
\]
i.e.,

\[
y = \frac{1}{27} (x-27) + f(27)
\]
to approximate. So

\[
f(26) = \frac{1}{\sqrt{26}} \approx y(26) = \frac{1}{27} \cdot (-1) + 3 \approx 2.9630.
\]
(true value \( \approx 2.9625 \))

(c) Consider \( f(x) = x^{1/4} \). Around \( x_0 = 10000 \), we use:

\[
y = f'(10000) (x-10000) + f(10000)
\]
i.e.,

\[
y = \frac{1}{4 \cdot 10000} (x-10000) + 10
\]
to approximate. So

\[
f(9900) \approx y(9900) = \frac{10000}{4 \cdot 10000} + 10 = 9.975.
\]
(true value \( \approx 9.9749 \))

(5)
5.

\[ f'(x) = \frac{4e^x(e^x + 1) - 4e^x(e^x)}{(e^x + 1)^2} \]

\[ = \frac{4e^x}{(e^x + 1)^2}. \]

So, \( f'(0) = 1 \).

\[ f(0) = \frac{4e^0}{e^0 + 1} = 2. \]

So, linearization around \( x = 0 \) is

\[ y - 2 = 1 (x - 0) \]

i.e.,

\[ y = x + 2. \]
\[ a(t) = -32. \]

Since \( v'(t) = a(t) \), we have
\[ v(t) = -32t + v_0. \]

Since \( v(0) = 112 = -32 \cdot 0 + v_0 \), we know \( v_0 = 112 \).

So,
\[ v(t) = -32t + 112. \]

Since \( h'(t) = v(t) \), we have
\[ h(t) = -16t^2 + 112t + h_0. \]

Since \( h(0) = 0 = -16 \cdot 0^2 + 112 \cdot 0 + h_0 \), we know \( h_0 = 0 \).

So,
\[ h(t) = -16t^2 + 112t. \]

To find time that the ball strike the ground, we need \( h(t) = 0 \), i.e.,
\[ -16t^2 + 112t = 0. \]

So, \( t = 0 \) (when the ball is projected) or \( t = 7 \) (when the ball strikes the ground).