1. Pr. 17.9(b) Prove \( f(x) = \sqrt{x} \) is continuous at \( x_0 = 0 \) by verifying the \( \epsilon - \delta \) property.

2. Pr. 17.10(b) Prove \( g(x) = \sin \frac{1}{x} \) for \( x \neq 0 \) and \( g(0) = 0 \) is discontinuous at \( x_0 = 0 \).

3. Pr. 17.11 Let \( f \) be a real-valued function with \( \text{dom}(f) \subseteq \mathbb{R} \). Prove \( f \) is continuous at \( x_0 \) if and only if, for every monotonic sequence \( (x_n) \) in \( \text{dom}(f) \) converging to \( x_0 \), we have \( \lim f(x_n) = f(x_0) \). Hint: Use the theorem 11.4: Every sequence has a monotonic subsequence.

4. Pr. 17.15 Let \( f \) be a real-valued function whose domain is a subset of \( \mathbb{R} \). Show \( f \) is continuous at \( x_0 \) in \( \text{dom}(f) \) if and only if, for every sequence \( (x_n) \) in \( \text{dom}(f) \{ x_0 \} \) converging to \( x_0 \), we have \( \lim f(x_n) = f(x_0) \).

1. \( \text{Dom } f = (0, \infty) \).

Now, \( \forall \epsilon > 0 \), \( \exists \delta = \epsilon^2 \), such that \( \forall x \in (0, \infty) \) and \( |x - a| = x < \delta = \epsilon^2 \), we have \( |\sqrt{x} - \sqrt{a}| = \sqrt{x} < \epsilon \).

2. \( g(x) = \begin{cases} \sin \left( \frac{1}{x} \right), & x \neq 0 \\ 0, & x = 0 \end{cases} \)

By def. we want to find a seq. \( (x_n) \) with \( x_n \to x_0 = 0 \) but \( g(x_n) \not\to g(x_0) = 0 \). Pick \( x_n = \frac{1}{(2n+\frac{1}{2})\pi} \to 0 \). Then \( g(x_n) = \sin \left( \frac{1}{(2n+\frac{1}{2})\pi} \right) = 1 \not\to g(x_0) = g(0) = 0 \).

3. \( \Rightarrow \) f cont. at \( x_0 \) if \( \forall (x_n) \in \text{dom } f, \ x_n \to x_0 \), we have \( f(x_n) \to f(x_0) \). So, particularly for a monotonic seq. \( (x_n) \in \text{dom } f \) with \( x_n \to x_0 \), we have \( f(x_n) \to f(x_0) \).

\( \Leftarrow \) Suppose for every monotonic seq. \( (x_n) \) in dom \( f \) converging to \( x_0 \), we have \( \lim f(x_n) = f(x_0) \), then \( f \) cont. at \( x_0 \). Suppose, for the sake of contradiction, \( f \) is not cont. at \( x_0 \). Then \( \exists \epsilon > 0, \forall \delta > 0 \), \( \exists x \in \text{dom } f \), \( |x - x_0| < \delta \), we have \( |f(x) - f(x_0)| > \epsilon \). New Pick \( \sigma = 1/n \). Then \( \exists \epsilon > 0 \), for \( \sigma_n = 1/n \), \( \exists (x_n) \) w/ \( |x_n - x_0| < \sigma_n \) (So \( x_n \to x_0 \)), we have \( |f(x_n) - f(x)| > \epsilon \). For this \( (x_n) \).

\( \exists (x_{n_k}) \subset (x_n), (x_{n_k}) \) monotonic, but \( |f(x_{n_k}) - f(x_0)| > \epsilon \). Contr.!
4. "\Rightarrow" Just like Pr. 3 "\Rightarrow" part. Trivial.

"\Leftarrow": If \( f \) is not cont. at \( x_0 \), \( \exists \ x_n \rightarrow x_0 \), such that \( f(x_n) \not\rightarrow f(x_0) \). Now delete all \( x_0 \) in \( (x_n) \) so that what remains is a subseq \( (x_{n_k}) \subset (x_n) \). Note \( (x_{n_k}) \subset \text{dom } f \setminus \{x_0\} \) and \( f(x_{n_k}) \rightarrow f(x_0) \). Contr!.