1. **Pr. 21.2** Consider \( f : S \rightarrow S^* \) where \((S, d)\) and \((S^*, d^*)\) are metric spaces. Show that \( f \) is continuous at \( s_0 \in S \) if and only if
   
   for every open set \( U \) in \( S^* \) containing \( f(s_0) \), there is an open set \( V \) in \( S \) containing \( s_0 \) such that \( f(V) \subset U \).

2. **Pr. 22.7** Explain why the metric space \( B \) in Exercise 13.3 can be regarded as \( C(\mathbb{N}) \). Recall that \((B, d)\) is defined as \( B = \{x = (x_1, x_2, \cdots)|x_i \in \mathbb{R}, \sup |x_i| < \infty \} \) and \( d(x, y) = \sup \{|x_i - y_i| : i = 1, 2, \cdots \} \).

3. **Pr. 22.12** Consider a subset \( \mathcal{E} \) of \( C(S), S \subset \mathbb{R} \). For this exercise, we say a function \( f_0 \) in \( \mathcal{E} \) is interior to \( \mathcal{E} \) if there exists a finite subset \( F \) of \( S \) and an \( \epsilon > 0 \) such that
   
   \[ \{ f \in C(S) : |f(x) - f_0(x)| < \epsilon, \forall x \in F \} \subset \mathcal{E}. \]

   The set \( \mathcal{E} \) is open if every function in \( \mathcal{E} \) is interior to \( \mathcal{E} \).

   (a) Reread Discussion 13.7.

   (b) Show the family of open sets defined above forms a topology for \( C(S) \).

4. **Pr. 22.4** Consider the following subset of \( \mathbb{R}^2 \):
   
   \[ E = \{(x, \sin \frac{1}{x}) : x \in (0, 1] \}; \]

   \( E \) is simply the graph of \( g(x) = \sin \frac{1}{x} \) along the interval \((0, 1] \).

   (a) Determine its closure \( E^- \).

   (b) Show \( E^- \) is connected.

   (c) Show \( E^- \) is not path-connected.
We use the notation $U_r(a) = \{ x : d(x, a) < r \}$, i.e., all points whose distance between a are less than $r$, namely the open ball with center $a$ and radius $r$.

$\Rightarrow$. A $U$ open, $f(s_0) \in U \subset S^x$, since $U$ open, $\exists \varepsilon$ s.t. $U_\varepsilon(f(s_0)) \subset U$. Since $f$ is cont. at $s_0$, we may pick $V = U_\delta(s_0)$. So, $\forall \; s \in V$, $d(s, s_0) < \delta$, so $d(f(s), f(s_0)) < \varepsilon$. So $f(s) \in U_\varepsilon(f(s_0)) \subset U$.

$\Leftarrow$. Pick $U = U_\varepsilon(f(s_0))$. By condition, $\exists \; V(\text{open}) \subset S$ w/ $s_0 \in V$ and $f(V) \subset U$. Since $V$ open, $\exists \; \delta > 0$ we have $U_\delta(s_0) \subset U$. 


2. \( \forall x = (x_1, x_2, \ldots) \in B \), identify \( x \) as
\[
f : \mathbb{N} \to \mathbb{R} \\
i \mapsto x_i
\]
Note any function \( \mathbb{N} \to \mathbb{R} \) is cont. and note \( x \) is bdd., so \( f \in C(\mathbb{N}) \)

\( \forall f \in C(\mathbb{N}) \), identify \( f \) as
\[
x = (x_1, x_2, \ldots)
\]
where \( x_i = f(i) \).

Note \( f \) is bounded, so \( x \in B \)

\[
d_B(x, y) = \sup \left\{ \| x_i - y_i \| : i = 1, 2, \ldots \right\}
\]
\[
= \sup \left\{ \| f(i) - g(i) \| : i = 1, 2, \ldots \right\}
\]
\[
= d_{C(\mathbb{N})}(f, g).
\]
3. (b) Need to show \( A = \{ \mathcal{E} \in \mathcal{C}(S) : \mathcal{E} \text{ is open} \} \) is a topology of \( \mathcal{C}(S) \).

\begin{enumerate}
\item "\( \mathcal{C}(S) \in \mathcal{A} \)" trivial
\item "\( \emptyset \in \mathcal{A} \)" trivial
\item "\( \exists i \in I, \forall i \in I \Rightarrow U \in \mathcal{E}_i \). So, \( \forall f \in \mathcal{E}_i \), \( \exists \varepsilon > 0 \) with \( \{ f \in \mathcal{C}(S) : |f(x) - f_0(x)| < \varepsilon, \forall x \in F \} \subseteq \mathcal{E}_i \)
\end{enumerate}

\( \mathcal{U} \in \mathcal{E}_i \).

So, \( \mathcal{U} \in \mathcal{E}_i \) open

\begin{enumerate}
\item "\( \exists i \in I, \forall i \in I \) finite \( \Rightarrow \mathcal{E}_i \in \mathcal{A} \)".
\end{enumerate}

\( \forall f \in \mathcal{E}_i \), \( \exists \varepsilon > 0 \) with \( \forall i \in I, \exists F_i \) finite \( \subseteq S \), \( \exists \varepsilon > 0 \) with \( \{ f \in \mathcal{C}(S) : |f(x) - f_0(x)| < \varepsilon, \forall x \in F_i \} \subset \mathcal{E}_i \)

Now pick \( F = \bigcup_{i \in I} F_i \), \( \varepsilon = \min \varepsilon_i \), then
\( \{ f \in \mathcal{C}(S) : |f(x) - f_0(x)| < \varepsilon, \forall x \in F \} \subset \mathcal{E}_i \forall i \in I \).

So, the result follows by taking \( F = \bigcup_{i \in I} F_i \), \( \varepsilon = \min \varepsilon_i \), then \( \{ f \in \mathcal{C}(S) : |f(x) - f_0(x)| < \varepsilon, \forall x \in F \} \subset \mathcal{E}_i \forall i \in I \).
We want to present the classic example of a space which is connected but not path-connected. Define
\[ S = \{ (x, y) \in \mathbb{R}^2 | y = \sin(1/x) \} \cup \{ (0) \times [-1, 1] \} \subseteq \mathbb{R}^2, \]
so \( S \) is the union of the graph of \( y = \sin(1/x) \) over \( x > 0 \), along with the interval \([-1, 1]\) in the \( y \)-axis. Geometrically, the graph of \( y = \sin(1/x) \) is a wiggly path that oscillates more and more frequently (between the lines \( y = \pm1 \)) as we get near the \( y \)-axis (more precisely, over the tiny interval \( 1/(2\pi(n + 1)) \leq x \leq 1/(2\pi n) \) the function \( \sin(1/x) \) goes through an entire wave).

We'll write \( S_+ \) and \( S_0 \) for these two parts of \( S \) (i.e., \( S_+ \) is the graph of \( y = \sin(1/x) \) over \( x > 0 \) and \( S_0 = \{ (0) \times [-1, 1] \} \)). It is clear that \( S_+ \) is path-connected (and hence connected), as is the graph of any continuous function (we use \( t \mapsto (t, \sin(1/t)) \) to define a path from \([a, b]\) to join up \((a, \sin(1/a))\) and \((b, \sin(1/b))\) for any \( 0 < a \leq b \) and then reparameterize the source variable to make our domain \([0, 1]\)). We will show that \( S \) is connected but is not path-connected. Intuitively, a path from \( S_+ \) that tries to get onto the \( y \)-axis part of \( S \) cannot get there in finite time, due to the crazy wiggling of \( S_+ \). Of course, we have to convert this idea into precise mathematics.

1. Connectedness of \( S \)

We begin with a lemma which shows how to recover \( S \) from \( S_+ \). This will enable us to show that \( S \) is connected.

**Lemma 1.1.** The closure of \( S_+ \) in \( \mathbb{R}^2 \) is equal to \( S \).

The point of the lemma is that we'll show the closure of a connected subset of a topological space is always connected, so the connectedness of \( S_+ \) and this lemma then implies the connectedness of \( S \). The fact that \( S \) turns out to not be path-connected then shows that forming closure can destroy the property of path connectedness for subsets of a topological space (even a metric space).

**Proof.** To show that \( S \) lies in the closure of \( S_+ \), we have to express each \( p \in S \) as a limit of a sequence of points in \( S_+ \). If \( p \in S_+ \) we use the constant sequence \( \{ p, p, \ldots \} \). If \( p = (0, y) \) with \( |y| \leq 1 \), we argue as follows. Certainly \( y = \sin(\theta) \) for some \( \theta \in [-\pi, \pi] \), whence \( y = \sin(\theta + 2n\pi) \) for all positive integers \( n \). Thus, for \( x_n = 1/(\theta + 2n\pi) > 0 \) we have \( \sin(1/x_n) = y \) for all \( n \). Since \( x_n \to 0 \) as \( n \to \infty \), we have \( (x_n, \sin(1/x_n)) = (x_n, y) \to (0, y) \). Geometrically, this is the infinite sequence of points where the horizontal line through \( y \) cuts the graph of \( \sin(1/x) \).

Now that we have shown that the set \( S \) containing \( S_+ \) lies inside the closure of \( S_+ \), to show that it is the closure of \( S_+ \) we just have to show that \( S \) is closed (as the closure of \( S_+ \) in \( \mathbb{R}^2 \) is the unique minimal closed subset of \( \mathbb{R}^2 \) which contains \( S_+ \)). Let \( \{(x_n, y_n)\} \) be a sequence in \( S \) with limit \( (x, y) \in \mathbb{R}^2 \). We must prove \( (x, y) \in S \). Since \( x = \lim x_n \) and \( y = \lim y_n \), we know that \( x \geq 0 \) and \( |y| = \lim |y_n| \leq 1 \). If \( x = 0 \), then clearly \( (x, y) = (0, y) \in S \) since \( |y| \leq 1 \). If \( x > 0 \), then upon dropping the first few terms of the sequence we can assume \( x_n > 0 \) for all \( n \). Then \( (x_n, y_n) \in S \) must lie on \( S_+ \), so \( y_n = \sin(1/x_n) \). Since the function \( t \mapsto \sin(1/t) \) on \((0, \infty)\) is continuous, from the condition \( x_n \to x \) we conclude
\[ y = \lim y_n = \lim \sin(1/x_n) = \sin(1/x). \]
Thus, \( (x, y) \in S_+ \subseteq S \) once again. \( \blacksquare \)

Thanks to the lemma, the connectedness of \( S \) is an immediate consequence of the following general fact (applied to the topological space \( \mathbb{R}^2 \) and the connected subset \( S_+ \)):

**Theorem 1.2.** Let \( X \) be a topological space and \( Y \) a connected subset. Then the closure \( \overline{Y} \) of \( Y \) in \( X \) is connected.
Proof. Without loss of generality, \( Y \neq \emptyset \). Suppose that \( \{U, V\} \) is a separation of \( \overline{Y} \). That is, \( U \) and \( V \) are disjoint opens of \( \overline{Y} \) with union equal to \( \overline{Y} \). We want one of them to be empty. The intersections \( U' = U \cap Y \) and \( V' = V \cap Y \) give a separation of \( Y \) (why?), so by connectedness of \( Y \) we have that one of \( U' \) or \( V' \) is empty and the other is equal to \( Y \). Without loss of generality, we may suppose \( U' = Y \) and \( V' = \emptyset \).

Since \( U \) is closed in \( \overline{Y} \), it has the form \( U = \overline{Y} \cap Z \) for some closed subset \( Z \) in \( X \). But \( Y \subseteq U \subseteq Z \), so by closedness of \( Z \) it follows that \( \overline{Y} \subseteq Z \). Then
\[
U = \overline{Y} \cap Z = \overline{Y},
\]
and by disjointness \( V \) must then be empty. Hence, \( \overline{Y} \) indeed has no non-trivial separations, so it is connected. \( \blacksquare \)

2. \( S \) is not path-connected

Now that we have proven \( S \) to be connected, we prove it is not path-connected. More specifically, we will show that there is no continuous function \( f : [0, 1] \to S \) with \( f(0) \in S_0 \) and \( f(1) \in S_0 = \{0\} \times [-1, 1] \). Assuming such an \( f \) exists, we will deduce a contradiction. Thanks to path-connectedness of \( S_0 \), we can extend our path to suppose \( f(1) = (0, 1) \). Choose \( \varepsilon = 1/2 > 0 \). By continuity, for some small \( \delta > 0 \) we have \( |f(t) - (0,1)| < 1/2 \) whenever \( 1 - \delta \leq t \leq 1 \). If you draw the picture, you'll see that the graph of \( \sin(1/x) \) keeps popping out of the disc around \((0,1)\) of radius \( 1/2 \), and that will contradict the existence of a continuous path \( f \).

To be precise, consider the image \( f([1-\delta, 1]) \), which must be connected since \( f \) is continuous and \([1-\delta, 1] \) is connected. Let \( f(1-\delta) = (x_0, y_0) \). Consider the composite of \( f : [1-\delta, 1] \to \mathbb{R}^2 \) and projection to the \( x \)-axis. Both such maps are continuous, hence so is their composite, so the image of the composite map is a connected subset of \( \mathbb{R} \) which contains \( 0 \) (the \( x \)-coordinate of \( f(1) \)) and \( x_0 \) (the \( x \)-coordinate of \( f(1-\delta) \)). But since connected subsets of \( \mathbb{R} \) must be intervals, it follows that the set of \( x \)-coordinates of points in \( f([1-\delta, 1]) \) includes the entire interval \([0, x_0]\). Thus, for all \( x_1 \in (0, x_0) \) there exists \( t \in [1-\delta, 1] \) such that \( f(t) = (x_1, \sin(1/x_1)) \).

In particular, if \( x_1 = 1/(2n\pi - \pi/2) \) for large \( n \) then \( 0 < x_1 < x_0 \) yet \( \sin(1/x_1) = \sin(-\pi/2) = -1 \). Thus, the point \( (1/(2n\pi - \pi/2), -1) \) has the form \( f(t) \) for some \( t \in [1-\delta, 1] \), and hence this point lies within a distance of \( 1/2 \) from the point \((0,1)\). But that's a contradiction, since the distance from \((1/(2n\pi - \pi/2), -1)\) to \((0,1)\) clearly at least \( 2 \) (as is the distance between any point on the line \( y = 1 \) and any other point on the line \( y = -1 \)).