1. **Pr. 33.13** Suppose $f$ and $g$ are continuous functions on $[a, b]$ such that $\int_a^b f = \int_a^b g$. Prove there exists $x$ in $(a, b)$ such that $f(x) = g(x)$.

2. **Pr. 34.6** Let $f$ be a continuous function on $\mathbb{R}$ and define

$$G(x) = \int_0^\sin x f(t)dt \text{ for } x \in \mathbb{R}$$

Show $G$ is differentiable on $\mathbb{R}$ and compute $G'$.

3. **Pr. 34.7** Use change of variables to evaluate

$$\int_0^1 x\sqrt{1-x^2}dx.$$ 

4. **Pr. 34.8** Use integration by parts to evaluate

$$\int_0^1 x \arctan x dx.$$ 

5. UC Davis 2014 Fall Ph.D. Prelim Exam **Pr. 1** Suppose $f(x)$ is twice differentiable on $\mathbb{R}$, $|f(x)| \leq 1$ and $|f''(x)| \leq 1$ for all $x \in \mathbb{R}$. Prove or disprove: $|f'(x)| \leq 2$ for all $x \in \mathbb{R}$. 


Discussion 7 Solution (Sketch)

1. Applying Thm 33.9 to \( f - g \).

Or, consider \( h(x) = \int_a^x f - g \). So, \( h(a) = h(b) = 0 \).

So, by MVT, \( \exists x_0 \in (a, b) \) s.t.

\[
    h'(x_0) = \frac{h(b) - h(a)}{b - a} = 0.
\]

Note \( h'(x) = f(x) - g(x) \).

2. Note

\[
    G(x) = F(u(x))
\]

where \( F(u) = \int_u^0 f(t) \, dt \), \( u(x) = \sin x \). So,

Since \( F \) is diff. w.r.t. \( u \), \( u(x) \) is diff. w.r.t. \( x \),

their composition is diff. w.r.t. \( x \), i.e., \( G \) is differentiable.

\[
    G'(x) = \frac{dF}{dx} = \frac{dF}{du(x)} \cdot \frac{du(x)}{dx} = f(u(x)) \cdot u'(x)
\]

\[
    = f(\sin x) \cdot \cos x.
\]
3. Let \( u = 1 - x^2 \). Then

\[
\int x \sqrt{1 - x^2} \, dx
\]

\[
= -\int \frac{1}{2} \sqrt{u} \, du
\]

\[
= -\frac{1}{3} u^{3/2}
\]

\[
= -\frac{1}{3} (1 - x^2)^{3/2}
\]

So,
\[
\int_0^1 x \sqrt{1 - x^2} \, dx = -\frac{1}{3} (1 - x^2)^{3/2} \bigg|_0^1 = \frac{1}{3}
\]

4. \( \int_0^1 x \arctan x \, dx \)

\[
= \int_0^1 \arctan x \, d\left(\frac{1}{2} x^2\right)
\]

\[
= \frac{1}{2} x^2 \arctan x \bigg|_0^1 - \frac{1}{2} \int_0^1 x \, d\arctan x
\]

\[
= \frac{1}{2} x^2 \arctan x \bigg|_0^1 - \frac{1}{2} \int_0^1 \frac{x^2}{x^2 + 1} \, dx
\]

\[
= \frac{1}{2} x^2 \arctan x \bigg|_0^1 - \frac{1}{2} \left[ x - \arctan x \right]_0^1
\]

\[
= \frac{3}{4} - \frac{1}{2}
\]
5. \( \forall a \in \mathbb{R}, \) we show \( |f'(a)| \leq 2. \)

By Taylor’s Thm, \( \forall x > a, \) we have

\[
f(x) = f(a) + f'(a)(x-a) + \frac{f''(b)(x-a)^2}{2}, \quad \text{for some } b \in (a, x).
\]

Now, pick \( x = a+2, \) we have

\[
f(a+2) = f(a) + 2f'(a) + 2f''(b), \quad \text{for some } b \in (a, a+2)
\]

So,

\[
f'(a) = \frac{f(a+2) - f(a)}{2} - f''(b)
\]

So

\[
|f'(a)| \leq \frac{|f(a+2)| + |f(a)|}{2} + |f''(b)| = 2.
\]