1. (a) We need to check Def. 13.1. but D1 and D2 are trivial. So we check
\[ d(x, z) \leq d(x, y) + d(y, z). \]

Note \( \text{RHS} = \sup \{ |x_j - y_j| : j \in \mathbb{N} \} \]
+ \( \sup \{ |y_i - z_i| : i \in \mathbb{N} \} \)

(Ex 4.14) = \( \sup \{ |x_j - y_j| + |y_i - z_i|, i, j \in \mathbb{N} \} \)

\( A \subset B \Rightarrow \) \( (\sup A \leq \sup B) \Rightarrow \sup \{ |x_j - y_j| + |y_j - z_j|, j \in \mathbb{N} \} \)
\[ \geq \sup \{ |x_j - z_j|, j \in \mathbb{N} \} \]
= \( \text{LHS} \).

(b) No, it does not since \( d^*(x, y) \) may be infinity, e.g., take \( x = (1, 1, \ldots), y = (0, 0, \ldots) \).
2.

a) Suppose \( x \in \mathbb{Q} \). Then \( \exists a, b \in \mathbb{R} \) such that \( x \in (a, b) \) and \( (a, b) \subset \mathbb{Q} \). This is not possible since \( (a, b) \) contains (infinitely many) irr. \#s.

b) Note that the sum of lengths of intervals comprising \( F_n \) is \( \left( \frac{2}{3} \right)^n \) and \( F = \bigcap_{n=1}^{\infty} F_n \). So if \( (a, b) \subset F \) then length \( (a, b) \) < \( \left( \frac{2}{3} \right)^{n-1} \). Since \( n \) is arbitrary, we should have length \( (a, b) = 0 \). This cannot be true for an interval.
pf: We follow the hint given in Page 376.

Let $U$ be an open set on $\mathbb{R}$. Let $(q_n)$ be a sequence of rational numbers that enumerates all rational numbers in $U$, i.e., $(q_n) = U \cap \mathbb{Q}$. This is valid since $\mathbb{Q}$ is countable (thus so is $U \cap \mathbb{Q}$).

For each $n \in \mathbb{N}$, define

$$a_n = \inf \{ a \in \mathbb{R} : (a, q_n) \subseteq U \}$$

$$b_n = \sup \{ b \in \mathbb{R} : [q_n, b) \subseteq U \}$$.

• Now, we show $(a_n, b_n) \subseteq U$.

Pick an arbitrary $c \in (a_n, b_n)$. Since $c > a_n = \inf \{ a \in \mathbb{R} : (a, q_n) \subseteq U \}$, by the def. of \(\inf\),

\(\exists a' \) s.t. $a' < c$ and $(a', q_n) \subseteq U$.

Similarly, \(\exists b' \) s.t. $c < b'$ and $[q_n, b') \subseteq U$.

So,

$$c \in (a', b') = (a', q_n) \cup [q_n, b') \subseteq U$$

So, $(a_n, b_n) \subseteq U$.

• Next we show $U = \bigcup_{n=1}^{\infty} (a_n, b_n)$.

"\(\subseteq\)" was just shown. So we just show "\(\supseteq\)".
Pick an arbitrary \( u \in U \). We must show \( u \in (a_k, b_k) \) for some \( k \). Now note \( U \) is open, so \( \exists \hat{\alpha}, \hat{\beta} \) s.t. \( u \in (\hat{\alpha}, \hat{\beta}) \subset U \). By the density of \( \mathbb{R} \), \( \exists q_k \in (\hat{\alpha}, \hat{\beta}) \). Since \( (\hat{\alpha}, q_k) \subset U \), by the def. of "\( \inf \)", \( a_k \leq \hat{\alpha} \); similarly \( b_k \geq \hat{\beta} \). So, \( u \in (\hat{\alpha}, \hat{\beta}) \subset (a_k, b_k) \subseteq U \).

So, all together, \( U = \bigcup_{n=1}^{\infty} (a_n, b_n) \).

Last we show \((a_n, b_n) \cap (a_m, b_m) \neq \emptyset\) implies \((a_n, b_n) = (a_m, b_m)\).

Suppose \((a_n, b_n) \cap (a_m, b_m) \neq \emptyset\). Then \( \exists q \in \mathbb{R} \) s.t. \( q \in (a_n, b_n) \cap (a_m, b_m) \) (why?). Suppose \((a_n, b_n) \neq (a_m, b_m)\). Then W.L.O.G., we may assume \( a_n \neq a_m \).

Since \((a_n, q], (a_m, q) \subset U\), we have \( a_n \leq a_n, a_m \leq a_m \). W.L.O.G., we may assume \( a_n < a_m \) (since \( a_n \neq a_m \)).

Last note \((a_n, q_n] = (a_n, a_n] \cup (a_n, q_n] \subset U\).

This contradict to the def. of \( a_n \).

So, either there will be only finitely many distinct and disjoint intervals or else a a subseq. \( \{ (a_{n_k}, b_{n_k}) \} \)

consist of disjoint intervals for which \( U = \bigcup_{k=1}^{\infty} (a_{n_k}, b_{n_k}) \).
4.

"\Rightarrow\" E \subseteq \mathbb{R}^k, E \text{ compact}. So E closed and bounded.

So \forall (x_n) \subseteq E, by Thm 13.5, \exists (x_{n_k}) \subseteq (x_n) such \(x_{n_k}\) converges. But E is closed, so
\[
\lim x_{n_k} \to x_0 \in E
\]

"\Leftarrow\" \exists E \text{ compact then we show E must be closed and compact.}

If E is not closed, then by Prop 13.9, \exists (x_n) convergent sith \(\lim x_n = x_0 \notin E\).

Contradiction to hypothesis. If E is not bounded, then \exists (x_n) sith \(\lim x_n = \infty\).

So, every subseq. \(x_{n_k}\) \(\subseteq (x_n)\) also converges to \(\infty\), which is not a point in E.
5. We use the notation \( U_r(a) = \{ x : d(x,a) < r \} \), i.e., all points whose distance between a are less than \( r \), namely the open ball with center \( a \) and radius \( r \).

\[ \Rightarrow \] A \( U \) open, \( f(s_0) \in U \subset S^x \), since \( U \) open, \( \exists \varepsilon \) s.t. \( U_\varepsilon(f(s_0)) \subset U \). Since \( f \) is cont. at \( s_0 \), we may pick \( V = U_\delta(s_0) \). So, \( \forall s \in V \), \( d(s,s_0) < \delta \), so \( d(f(s),f(s_0)) < \delta \varepsilon \). So \( f(s) \in U_\varepsilon(f(s_0)) \subset U \).

\[ \Leftarrow \] Pick \( U = U_\varepsilon(f(s_0)) \). By condition, \( \exists V(\text{open}) \subset S \) w/ \( s_0 \in V \) and \( f(V) \subset U \). Since \( V \) open, \( \exists \delta > 0 \) we have \( U_\delta(s_0) \subset U \).