HW 8 Solution (Sketch).

1.
(a). Integrability follows from Thm 33.6.
\[ \int_a^b f \, dx = \sum_{j=1}^m c_j (u_j - u_{j-1}) \]

(b). \( \int_0^4 p(x) \, dx = 4A + 6B \)

2. \[ |\int_{-2\pi}^{2\pi} x^2 \sin^8(e^x) \, dx| \leq \int_{-2\pi}^{2\pi} |x^2| |\sin^8(e^x)| \, dx \leq \int_{-2\pi}^{2\pi} x^2 \, dx = \frac{16\pi^3}{3} \]

3.
(a). Follows from the Hint. and Thm 33.3
(b). Follows from Ex. 17.8 and Thm 33.3, 33.5.
4. First part follows from Thm 33.4 (ii). We now show the second part. It suffices to show \( \int_a^b f(x) \, dx = 0 \).

Note

\[
\int_a^b f^2(x) \, dx = \int_a^b f^2(x) \, dx - \int_a^b f(x) \, p(x) \, dx \\
= \int_a^b f(x) \left[ f(x) - p(x) \right] \, dx
\]

A polynomial \( p(x) \), since \( \int_a^b f(x) \cdot x^n \, dx = 0 \), \( \forall n = 0, 1, \ldots \).

Now, it suffices to evaluate the last integral.

Note: \( |f(x)| < M \) for some \( M \), \( \forall x \in [a, b] \), since \( f \) is cont. on \( [a, b] \) thus unif. cont. on \( [a, b] \) thus bounded. Also by Weierstrass Approx. Thm., \( \forall \varepsilon > 0 \), \( \exists p_n(x) \) polynomial S.t. \( \sup \{ |f(x) - p_n(x)| : x \in [a, b] \} < \frac{\varepsilon}{M(b-a)} \). So,

\[
\left| \int_a^b f(x) \left[ f(x) - p_n(x) \right] \, dx \right| \leq \int_a^b |f(x)| |f(x) - p_n(x)| \, dx \\
\leq \int_a^b M \cdot \frac{\varepsilon}{M(b-a)} \, dx \\
= \varepsilon.
\]

The result follows since \( \varepsilon \) is arbitrary.