1.

(a) \( \liminf_{n \to \infty} a_n = \lim_{n \to \infty} \inf \{a_k : k \geq n\} \)

\( \limsup_{n \to \infty} a_n = \lim_{n \to \infty} \sup \{a_k : k \geq n\} \)

(b) \( a_n = (-1)^n \).

\( a_n \) is bounded by 1.

But \( \liminf a_n = -1 \neq 1 = \limsup a_n \)

(c) NO.

If \( (a_n) \) is increasing and bounded, then \( \lim a_n \) exists, which means \( \liminf a_n = \limsup a_n \)

2.

(a) \( (x_n) \) is increasing.

Note \( x_{n+1}^2 = 1 + x_n^2 \). So, \( x_{n+1}^2 - x_n^2 = 1 \).

So \( (x_{n+1} - x_n)(x_{n+1} + x_n) = 1 > 0 \).

As \( x_n > 0, \forall n \in \mathbb{N} \). So \( x_{n+1} - x_n > 0 \). That is \( x_{n+1} > x_n \).

So, \( (x_n) \) is increasing.

\( x_n \to \infty \).

We show \( (x_n) \) is not bounded, thus not convergent.

Suppose it is bounded.

Then as it is also increasing, so it has a limit, say \( \alpha \). Also note \( \lim x_{n+1} = \lim x_n = \alpha \).

Then we have

\( \alpha = \sqrt{1 + \alpha^2} \),

which gives \( \alpha = 1 \). Contradiction.
(b). "(\(x_n\)) is increasing".

**Base Step:** \( x_1 = \sqrt{1 + \frac{1}{5} x^2} = \sqrt{6/5} > x_0 \). TRUE.

**Inductive Step:**

Suppose \( x_n > x_{n-1} \). We show \( x_{n+1} > x_n \).

\[
  x_{n+1} = \sqrt{1 + \frac{1}{5} x^2} > \sqrt{1 + \frac{1}{5} x^2_{n-1}} = x_n \text{. TRUE.}
\]

So. \((x_n)\) is increasing.

"Bounded".

We show \( |x_n| < \frac{\sqrt{5}}{2} \) \( \forall n \in \mathbb{N} \), by induction.

**Base Step:** \( |x_0| = 1 < \frac{\sqrt{5}}{2} \). TRUE.

**Inductive Step:**

Suppose \( |x_{n-1}| < \frac{\sqrt{5}}{2} \), we show \( |x_{n}| < \frac{\sqrt{5}}{2} \).

\[
  |x_{n}| = \sqrt{1 + \frac{1}{5} x_{n-1}^2} < \sqrt{1 + \frac{1}{5} (\frac{\sqrt{5}}{2})^2} = \frac{\sqrt{5}}{2} \text{. TRUE.}
\]

So, \((x_n)\) is bounded by \( \frac{\sqrt{5}}{2} \).

"Limit".

As \((x_n)\) is increasing and bounded, the limit exists.

Also note \( \lim x_n = \lim x_{n+1} \). Denote \( \alpha = \lim x_n \). Then

\[
  \alpha = \sqrt{1 + \frac{1}{5} \alpha^2},
\]

which gives \( \alpha = \pm \frac{\sqrt{5}}{2} \).

As \( x_n > 0, \forall n \in \mathbb{N} \). \( \alpha = \frac{\sqrt{5}}{2} \).
3.

(a) \( \lim a_n = \infty \Rightarrow \lim \frac{1}{a_n} = 0 \).

As \( \lim a_n = \infty \), \( \forall M > 0, \exists N \in \mathbb{N}, \text{s.t.} \ n > N \Rightarrow a_n > M \).

Now, \( \forall \varepsilon > 0 \), define \( M = \frac{1}{\varepsilon} \). Then for this \( M \), \( \exists N \in \mathbb{N} \) \( \text{s.t.} \ n > N \Rightarrow a_n > \frac{1}{\varepsilon} \), i.e., \( \frac{1}{a_n} < \varepsilon \). That is \( |\frac{1}{a_n} - 0| < \varepsilon \). So, \( \lim \frac{1}{a_n} = 0 \).

\( \lim a_n = \infty \iff \lim \frac{1}{a_n} = 0 \).

As \( \lim \frac{1}{a_n} = 0 \), \( \forall \varepsilon > 0, \exists N \in \mathbb{N} \) \( \text{s.t.} \ n > N \Rightarrow a_n > \frac{1}{\varepsilon} \). That is \( a_n > \frac{1}{\varepsilon} \).

Now, \( \forall M > 0 \), define \( \varepsilon = \frac{1}{M} \). Then for this \( \varepsilon \), \( \exists N \in \mathbb{N} \) \( \text{s.t.} \ n > N \Rightarrow a_n > \frac{1}{\varepsilon} = M \). That is \( a_n \to \infty \).

(b) Without the assumption that \( a_n > 0 \), we still have the statement \( \lim a_n = \infty \Rightarrow \lim \frac{1}{a_n} = 0 \) to be true. But the converse is NOT necessarily to be true.

Counter example 1:
\[ a_n = (-1)^n \frac{1}{n} \] Then \( a_n \to 0 \) as \( n \to \infty \).
But \( \frac{1}{a_n} = (-1)^n n \), which does not go to \( \infty \).
This is because, \( \exists M = 1, \forall N \in \mathbb{N}, \exists n > N \) \( \text{s.t.} \ a_n < 0 < M \).

Counter example 2:
\[ a_n = -n \]
(a) \( \sum a_n \) converges absolutely means \( \sum |a_n| \) converges.

\( \sum a_n \) converges conditionally means \( \sum |a_n| \) does not converge, but \( \sum a_n \) converges.

(b) For any \( k \in \mathbb{N} \), \( 1 + \frac{1}{k!} > 1 \), so \( (1 + \frac{1}{k!})^k > 1 \).

So \( \sum_{k=1}^{\infty} (1 + \frac{1}{k!})^k > \sum 1 \).

So, it diverges.

(c) Note \( \frac{k}{k^3 + 1} < \frac{k}{k^3} = \frac{1}{k^2} \).

So, \( \sum \frac{k}{k^3 + 1} < \sum \frac{1}{k^2} \), which converges.

So, \( \sum (-1)^k \frac{k}{k^3 + 1} \) converges absolutely.

(d) Note \( \frac{1}{k^{-\frac{1}{k}}} > \frac{1}{k} \).

So, \( \sum \frac{1}{k^{-\frac{1}{k}}} > \sum \frac{1}{k} \), which diverges.

So, \( \sum (-1)^k \frac{1}{k^{-\frac{1}{k}}} \) does not converge absolutely.

But \( \frac{1}{k^{-\frac{1}{k}}} \) is decreasingly approaching zero, so by alternating series test theorem, \( \sum (-1)^k \frac{1}{k^{-\frac{1}{k}}} \) converges conditionally.

\[
\left[ \frac{1}{k} \downarrow \Rightarrow -\frac{1}{k} \uparrow \Rightarrow k^{-\frac{1}{k}} \uparrow \Rightarrow \frac{1}{k^{-\frac{1}{k}}} \downarrow \right]
\]
(e). Note \( \frac{2k+3}{3k+2} < \frac{2k+3}{3k} \)
\[ \begin{align*}
&< \frac{2k+\frac{1}{2}k}{3k} \\
&\quad \text{if } 3 < \frac{1}{2}k, \text{ i.e., } 6 < k
\end{align*} \]
\[ = \frac{5}{6} \)

So, if \( k > 6 \), \( \left( \frac{2k+3}{3k+2} \right)^k < (\frac{5}{6})^k \).

So, \( \sum_{k=7}^{\infty} \left( \frac{2k+3}{3k+2} \right)^k < \frac{18}{5} \left( \frac{5}{6} \right)^k < \infty \).

So, \( \sum_{k=1}^{\infty} \left( \frac{2k+3}{3k+2} \right)^k < \infty \).

So, \( \sum \left( \frac{2k+3}{3k+2} \right)^k \) converges absolutely.

[Alternative Way: Root Test.]

5.

(a). Note \( \frac{1}{1+a_k} < 1 \) given \( a_k > 0 \), \( \forall k \in \mathbb{N} \).

So, \( \frac{a_k}{1+a_k} < a_k \).

So, \( \sum \frac{a_k}{1+a_k} < \sum a_k < \infty \).

So, the statement is TRUE.
(b). The statement is FALSE

Counter example:

\[ a_k = 1, \forall k \in \mathbb{N}. \] Then \( \sum a_k \) diverges.

But every rearrangement of \( \sum a_k \) is \( \sum a_k \) itself, which diverges.

Remark: Every divergent series (with positive terms)

is a counter example.

(c). Let

\[ a_n = \begin{cases} \frac{1}{n^2} & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even.} \end{cases} \]

Then \( \lim \inf n^2 a_n = 0 \).

But \( \sum a_n \) diverges.