Discussion problems 2

1. (a) Show that for every real numbers \(a, b > 0\), their geometric mean is no larger than their arithmetic mean, that is, \(\sqrt{ab} \leq \frac{a+b}{2}\).

(b) (Optional\textsuperscript{1}.) One can get a lot of very sophisticated inequalities out of the order field axioms; here is one from a math competition. Compute the minimum of the set

\[
\{(1 + x_1)(1 + x_2)(1 + x_3) : x_1, x_2, x_3 \geq 0, x_1x_2x_3 \geq 1\}.
\]

(Hints. Argue that you may assume \(x_1x_2x_3 = 1\). Then replace both \(x_1\) and \(x_2\) by \(x_1' = x_2' = \sqrt{x_1x_2}\). Show that \((1 + x_1')(1 + x_2') \leq (1 + x_1)(1 + x_2)\). Argue that this means that the minimum is achieved when all \(x_i\) are equal.)

2. Find the supremum of \(A\), if it exists, and the infimum of \(A\), if it exists, in each case below. You do not need to prove your assertions.

(a) \(A = \mathbb{N}\).

(b) \(A = \left\{ \frac{7}{n} : n \in \mathbb{N} \right\}\).

(c) \(A = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\}\).

(d) \(A = (1, 3) \cup [5, 7]\).

(e) \(A = \bigcup_{n=1}^{\infty} (n, n + 1/n)\).

(f) \(A = \left\{ n + \frac{(-1)^n}{n} : n \in \mathbb{Z} \right\}\).

(g) \(A = \{1 - r^2 : r \in \mathbb{Q}\}\).

3. Suppose \(A \subseteq \mathbb{R}\) and \(B \subseteq \mathbb{R}\) are both nonempty. Suppose also that \(a < b\) for all \(a \in A\) and \(b \in B\).

(a) Prove that \(\sup A \leq \inf B\).

(b) Must it be true that \(\sup A < \inf B\)?

4. Does the set \([0, \sqrt{2}] \cap \mathbb{Q}\) have a minimum? A maximum?

5. Prove that, for \(A \subseteq \mathbb{R}\), \(\inf A = \sup A\) if and only if \(A\) is a singleton (that is, if and only if \(A = \{a\}\) for some \(a \in \mathbb{R}\)).

6. Prove that for every \(a, b \in \mathbb{R}\), \(|a^2 - b^2| \leq (|a| + |b|)|a - b|\).

7. Find the minimum of the set

\[
\{|x| + |x - 1| + |x - 3| : x \in \mathbb{R}\}.
\]

\textsuperscript{1}Optional problems will not be covered in discussions and you will not be required to know how to do them.