Discussion problems 5

*Note.* Please justify the answers to all yes-no questions (with a proof or with a counterexample). All proofs in this set are very short.

1. (a) Show that the series
\[ \sum_{n=1}^{\infty} \frac{1}{n(n+2)} \]
converges and find its sum.
(b) Is there a rearrangement of the series in (a) with a different sum?

2. Do the exercise 2.7.6(a) in the book first. Assume \((a_n)\) and \((b_n)\) are two sequences such that \(a_n > 0\) and \(b_n > 0\) for every \(n\), and such that \(\lim_{n \to \infty} b_n/a_n\) exists and is nonzero. Prove that \(\sum_{k=1}^{\infty} a_k\) converges if and only if \(\sum_{k=1}^{\infty} b_k\) converges. (This is known as the *Limit Comparison Test* in calculus).

3. For each of the following series, determine (with proof) whether it converges absolutely, converges conditionally, or diverges:
   (a) \(\sum_{k=1}^{\infty} \frac{1}{\sqrt{k(k+1)}}\)
   (b) \(\sum_{k=1}^{\infty} (-1)^k \frac{1}{\sqrt{k(k+1)}}\)
   (c) \(\sum_{k=1}^{\infty} (-1)^k \frac{1}{\sqrt{k(k+1)}}\)
   (d) \(1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \cdots\) (Every third term is negative.)

4. Assume \(a_k > 0\) for all \(k \in \mathbb{N}\). For each statement below, prove it or find a counterexample.
   (a) If \(\sum_{k=1}^{\infty} a_k\) converges, then \(\sum_{k=1}^{\infty} 1/a_k\) diverges.
   (b) If \(\sum_{k=1}^{\infty} a_k\) converges, then \((a_k)\) is a decreasing sequence.
   (c) If \(\lim sup_n n a_n > 1\), then \(\sum_{k=1}^{\infty} a_k\) diverges.
   (d) If \(\lim inf_n n a_n > 1\), then \(\sum_{k=1}^{\infty} a_k\) diverges.
   (e) If \(\lim a_n = 0\), then \(\sum_{k=1}^{\infty} (-1)^k a_k\) converges.