Discussion problems 7

Note. These problems are optional.

1. Discrete L’Hôpital. Assume that \( b_n > 0 \) for every \( n \in \mathbb{N} \), and that \( \sum_{n=1}^{\infty} b_n \) diverges. Let \((a_n)\) be an arbitrary sequence, and \( s \in \mathbb{R} \). Show that \( \lim_{n \to \infty} a_n/b_n = s \) implies

\[
\lim_{n \to \infty} \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} = s.
\]

Show also that the reverse implication does not hold.

2. Let \( s_n = \sum_{i=1}^{n} 1/\sqrt{i} \).
   (a) Compute \( \lim_{n \to \infty} s_n/\sqrt{n} \).
   (b) Find a number \( \alpha \in \mathbb{Q} \) so that \( \lim_{n \to \infty} \frac{\sum_{k=1}^{n} s_k}{n^\alpha} \) is neither 0 nor \( \infty \).

3. A Cauchy square is the Cauchy product of a series by itself. The \( p \)th Cauchy power of a series is the \( p \)-fold Cauchy product of a series by itself. Show that for \( p \in \mathbb{N} \), the \( p \)th Cauchy power of a geometric series \( \sum_{k=0}^{\infty} a^k \) with \( a \in (0, 1) \) equals

\[
\sum_{k=0}^{\infty} \binom{k+p-1}{p-1} a^k,
\]

which equals \( 1/(1-a)^p \). (Hint. This is proved by induction using the formula \( \sum_{k=m}^{n} \binom{k}{m} = \binom{n+1}{m+1} \), valid for \( m \leq n \). This formula can be proved by dividing the number of ways of choosing \( m+1 \) numbers from \( \{0, \ldots, n\} \) according to the largest chosen number.)

4. Show that the Cauchy square of \( \sum_{k=1}^{\infty} (-1)^k/\sqrt{k} \) diverges, while the Cauchy square of \( \sum_{k=1}^{\infty} (-1)^k/k \) converges conditionally.

5. Show that there is no slowest divergent series in the following sense. Assume \( \sum_{k=1}^{\infty} a_k \) is divergent with all \( a_k > 0 \). Then there exists a divergent series \( \sum_{k=1}^{\infty} b_k \) so that all \( b_k > 0 \) and \( \lim_{n \to \infty} \frac{b_n}{a_n} = 0 \).