Discussion Problems 9

Note. These problems should be viewed as partial practice for the final exam. Problems from other discussions and homework problems are also good for preparation.

1. Assume that a sequence \((a_n)\) is unbounded. Show that there exists a subsequence \((a_{n_k})\) so that either \(\lim_{k} a_{n_k} = \infty\) or \(\lim_{k} a_{n_k} = -\infty\).

2. Show that a set \(A \subseteq \mathbb{R}\) is bounded if and only if its closure \(\overline{A}\) is compact.

3. (a) Assume that a sequence \((a_n)\) satisfies \(\lim_{n \to \infty} \frac{a_n}{a_n + 4} = 0\). Show that \(\lim a_n = 0\). (You may use Problem 1 to show first that \((a_n)\) is bounded.)
   (b) Does the conclusion from (a) hold if \(\lim_{n \to \infty} \frac{a_n}{a_n^2 + 4} = 0\)?
   (c) Does the conclusion from (a) hold if \(\lim_{n \to \infty} \frac{a_n}{\sqrt{|a_n|} + 4} = 0\)?

4. (a) State precisely what this means: \(A \subseteq \mathbb{R}\) is compact.
   For each of the following statements determine, with proof, whether it is true or false.
   (b) If \(A \subseteq \mathbb{R}\) is compact then \(A^c\) is open.
   (c) If \(A \subseteq \mathbb{R}\) is compact then \(A \cap (0,1)\) is compact.
   (d) If \(A \subseteq \mathbb{R}\) is compact, and \(x_n \in A\), then \((x_n)\) is a convergent sequence.
   (e) If \(A \subseteq \mathbb{R}\), and there exist an open cover of \(A\) with a finite subcover, then \(A\) is compact.
   (f) If \(A \subseteq \mathbb{R}\) is compact, and \((x_n)\) is a convergent sequence with limit in \(A\), then \(A \cup \{x_n : n \in \mathbb{N}\}\) is compact.

5. Assume the sequence \((x_n)\) is given recursively by \(x_1 = a > 0\) and
\[
x_{n+1} = \frac{1}{2} \sqrt{x_n + x_n^3} \quad \text{for } n \in \mathbb{N}.
\]
   (a) Assume \(a = 3\). Show that the sequence is decreasing and bounded and compute its limit.
   (b) Assume \(a = 4\). Show that the sequence is increasing and unbounded.

6. Assume \(a_n > 0\) for all \(n\). For each of the following statements determine, with proof, whether it is true or false.
   (a) If \(a_n \leq 4\) for all \(n\), then \((a_n)\) is convergent.
   (b) If \(a_n \geq 4\) for all \(n\), then \((a_n)\) is divergent.
   (c) If \(a_{n+1} \leq a_n\) for all \(n\), then \((a_n)\) is a Cauchy sequence.
   (d) If \((a_n)\) is convergent, then \(\lim \sup(-1)^n a_n = \lim a_n\).
   (e) If \(\sum_{n=1}^{\infty} a_n\) converges, then \(\sum_{n=1}^{\infty} (a_n + n^{-3/2})\) converges.
   (f) If \(\sum_{n=1}^{\infty} a_n\) converges, then \(\sum_{n=1}^{\infty} a_n^{2/3}\) converges.
   (g) If \(\sum_{n=1}^{\infty} (3 - a_n)\) converges, then \(\lim a_n = 3\).
   (h) If \(\sum_{n=1}^{\infty} a_n\) converges \(\sum_{n=1}^{\infty} (-1)^{17n^2+3n} a_n\) converges.
   (i) If \(\sum_{n=1}^{\infty} a_n\) diverges and \(s_n = a_1 + \ldots + a_n\), then \(\sum_{n=1}^{\infty} (-1)^n \frac{1}{s_n}\) converges.

7. (a) Assume that \(A \subseteq \mathbb{R}\) is nonempty and bounded. State the definition of \(\sup A\) and \(\max A\). Does
sup $A$ always exist? Does max $A$ always exist?

(b) Let $s_n = \sum_{i=1}^{n} 3^{-i}$. Let $A = \{s_n : n \in \mathbb{N}\}$. Determine sup $A$ and inf $A$. Does max $A$ exist? Does min $A$?

(c) Let $s_n = \sum_{i=1}^{n} \frac{(i)!^2}{(2i)!}$. Let $A = \{s_n : n \in \mathbb{N}\}$. Does sup $A$ exist?

(d) Let $A = \bigcup_{n=1}^{\infty} (1/(n+1), 1/n)$. Determine sup $A$ and inf $A$.

(e) Is the set $A$ from (d) connected?

(f) Compute $\overline{A}$ for the set in (d). Is $\overline{A}$ connected?

(g) Let $A = \left\{\frac{(-1)^n n}{(n+1)^{n+1}} : n \in \mathbb{N}\right\}$. Determine sup $A$ and inf $A$. Does max $A$ exist? Does min $A$?
1. Let $n_1 = 1$. Assume $n_1, \ldots, n_{k-1}$ are chosen for some $k \geq 2$. The sequence $(a_{n_k+1}, a_{n_k+2}, \ldots)$ is also unbounded, so there exists an $n_k > n_{k-1}$ so that $|a_{n_k}| > k$. Thus we have \( \lim_{k} |a_{n_k}| = \infty \). If only finitely many terms $a_{n_k}$ are positive, then \( \lim_{k} a_{n_k} = -\infty \). If, on the other hand, infinitely many terms $a_{n_k}$ are positive, modify the subsequence by eliminating all negative terms from it. The remaining terms form a subsequence whose terms are all positive so it diverges to $\infty$.

2. If $A$ is bounded, then $A \subset [-M, M]$ for some $M \geq 0$. As $[-M, M]$ is closed, $\bar{A} \subset [-M, M]$, and so $\bar{A}$ is closed and bounded, thus compact.

3. (a) Let 

$$b_n = \frac{a_n}{5|a_n| + 4}.$$ 

We first show that $(a_n)$ is bounded. If not, then by Problem 1, there is a subsequence $(a_{n_k})$ which either diverges to $\infty$ or $-\infty$. If the subsequence diverges to $\infty$, then 

$$\lim_{k \to \infty} b_{n_k} = \lim_{k \to \infty} \frac{a_{n_k}}{5|a_{n_k}| + 4} = \lim_{k \to \infty} \frac{a_{n_k}}{5a_{n_k} + 4} = \lim_{k \to \infty} \frac{1}{5 + 4/a_{n_k}} = \frac{1}{5},$$

which is a contradiction as this limit must be equal 0 (which is the limit of the full sequence $b_n$). A similar contradiction (with $-1/5$ instead of $1/5$) is obtained if the subsequence diverges to $-\infty$. Therefore $a_n$ is bounded, say $|a_n| \leq M$ for all $n$. As $a_n = b_n(5|a_n| + 4)$,

$$|a_n| \leq |b_n|(5M + 4)$$

and so $|a_n| \to 0$.

(b) No. Take $a_n = n$.

(c) Yes. As in (a), we let 

$$b_n = \frac{a_n}{5\sqrt{|a_n|} + 4}$$

and prove that $(a_n)$ must be bounded. If not, there is a subsequence $(a_{n_k})$ which either diverges to $\infty$ or $-\infty$. Assume the subsequence diverges to $\infty$ (the other case is similar), then 

$$\lim_{k \to \infty} b_{n_k} = \lim_{k \to \infty} \frac{a_{n_k}}{5\sqrt{a_{n_k}} + 4} = \lim_{k \to \infty} \frac{\sqrt{a_{n_k}}}{5 + 4/\sqrt{a_{n_k}}} = \infty,$$

which is a contradiction as this limit must be equal 0. Therefore $a_n$ is bounded, say $|a_n| \leq M$ for all $n$. As $a_n = b_n(5\sqrt{|a_n|} + 4)$,

$$|a_n| \leq |b_n|(5\sqrt{M} + 4)$$

and so $|a_n| \to 0$.

4. (a) Omitted.

(b) Yes. As $A$ is compact, it must be closed, and then $A^c$ is open.

(c) No. Take $A = [0, 1]$, then $A$ is compact, but $A \cap (0, 1) = (0, 1)$ is not closed.

(d) No. A subsequence is convergent, but the full sequence may not be. For example, take $A = [-1, 1]$ and $x_n = (-1)^n$. 

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(e) No. Every subset of \( \mathbb{R} \) has a finite open cover by a single set \( \mathbb{R} \). (The theorem says that \( A \) is compact if every (and not just one) open cover of \( A \) has finite subcover.

(f) Yes. Let \( x = \lim x_n \in A \). Then \( B = \{ x_n : n \in \mathbb{N} \} \cup \{ x \} \) is compact (as it contains its only limit point \( x \)), and

\[
A \cup \{ x_n : n \in \mathbb{N} \} = A \cup B,
\]

which is compact as the union of two compact sets.

5. (a) By obvious induction \( x_n > 0 \) for all \( n \). We prove by induction that \( x_{n+1} < x_n \). For \( n = 1 \), this is true as \( x_2 = \frac{1}{2} \sqrt{30} > 3 = x_1 \). For the \((n \to n + 1)\) step, observe

\[
x_{n+1} < x_n \implies x_{n+1}^3 < x_n^3 \implies x_{n+1} + x_{n+1}^3 < x_n + x_n^3 \implies \frac{1}{2} \sqrt{x_{n+1} + x_{n+1}^3} < \frac{1}{2} \sqrt{x_n + x_n^3},
\]

which states \( x_{n+2} < x_{n+1} \). Thus the sequence is decreasing, and bounded as \( 0 < x_n < 3 \) for all \( n \). Thus \( x = \lim x_n \) exits and satisfies the equation

\[
x = \frac{1}{2} \sqrt{x + x^3}.
\]

Then \( x(x^2 - 4x + 1) = 0 \), with three solutions \( x = 0 \) and \( x = 2 \pm \sqrt{3} \). We can immediately eliminate \( 2 + \sqrt{3} > 3 \).

To eliminate \( x = 0 \), we show \( x_n > 2 - \sqrt{3} \) for all \( n \) by induction. Clearly this is true for \( n = 1 \) and for \((n \to n + 1)\) step observe that under the hypothesis

\[
x_{n+1} \geq \frac{1}{2} \sqrt{(2 - \sqrt{3}) + (2 - \sqrt{3})^3} = 2 - \sqrt{3},
\]

as \( 2 - \sqrt{3} \) solves the above equation for \( x \). Thus \( \lim x_n = 2 - \sqrt{3} \).

(b) The proof that the sequence is increasing is almost the same as in (a) with reversed inequalities. If the sequence were bounded, it would have a limit \( x = \lim x_n > 4 \), but then \( x \) would have to satisfy the same equation as in (a), and that equation has no solution larger than 4, as the largest solution is \( 2 + \sqrt{3} < 4 \).

6. (a) No. For example, \( a_n = 2 + (-1)^n \) satisfies \( 1 \leq a_n \leq 3 \) for all \( n \), but diverges.

(b) No. For example, \( a_n = 4 + 1/n \to 4 \).

(c) Yes. It is decreasing an positive, therefore monotone and bounded, therefore convergent, and therefore Cauchy.

(d) Yes. The even terms converge to \( \lim a_n \) and the odd ones to \( -\lim a_n \).

(e) Yes. \( \sum_{n=1}^{\infty} n^{-3/2} \) converges and then \( \sum_{n=1}^{\infty} (a_n + n^{-3/2}) \) converges as the sum of two convergent series.

(f) No. Take \( a_n = n^{-5/4} \); then \( a_n^{2/3} = n^{-5/6} \) and \( \sum n^{-5/4} \) converges while \( \sum n^{-5/6} \) diverges.

(g) Yes. By the nth term test, \( \lim (3 - a_n) = 0 \), so \( \lim a_n = 3 \).

(h) Yes. The second sum converges absolutely (as \( |(-1)^{7n^2 + 3n} a_n| = a_n \).

(i) Yes. The sequence \( (s_n) \) is increasing, hence \( 1/s_n \) is decreasing. Also \( s_n \to \infty \), so \( 1/s_n \to 0 \). Thus \( \sum (-1)^n \frac{1}{s_n} \) converges by the alternating series test.

7. (a) A number \( b \in \mathbb{R} \) is the least upper bound of \( A \), also called sup \( A \), if \( b \) is (1) an upper bound, that is, \( a \leq b \) for all \( a \in A \); and (2) the smallest of all upper bounds, that is, if \( \beta \in \mathbb{R} \) is such that \( a \leq \beta \) for all \( a \in A \), then \( b \leq \beta \). If \( A \) is bounded and nonempty, sup \( A \) exists by the completeness axiom (in
fact, nonempty and bounded above is enough). We say that \( m = \max A \) if \( m \in A \) and \( a \leq m \) for all \( a \in A \). If \( \sup A \notin A \), then \( \max A \) does not exist, e.g., \( A = [0, 1) \) has \( \sup A = 1 \notin A \).

(b) The sequence \( s_n \) is increasing and converges to the sum of the geometric series \( 1/2 \). Thus \( \sup A = 1/2 \notin A \) so \( \max A \) does not exist, and \( \inf A = \min A = s_1 = \frac{1}{3} \).

(c) This is another way to ask whether \( \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \) converges. To determine this, denote the \( n \)th term by \( a_n \) and use the ratio test
\[
\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{2(n+1)(2n+1)} \to \frac{1}{4}
\]
to show that the series converges and that the answer is yes.

(d) We see that \( A = (0, 1) \setminus \{1/n : n = 2, 3, \ldots \} \). Thus \( \sup A = 1 \) and \( \inf A = 0 \).

(e) No. For example, \( 2/5 \in A \) and \( 3/5 \in A \), and \( 2/5 < 1/2 < 3/5 \), but \( 1/2 \notin A \).

(f) Every number in \( [0, 1] \) is a limit point of \( A \), so \( \overline{A} = [0, 1] \). As an interval, \( \overline{A} \) is connected.

(g) Let \( a_n = \frac{(-1)^n n}{(-1)^n + n+1} \). For even \( n \), \( a_n = \frac{n}{n+1} = -\frac{1}{1-1/n} \). So, for even \( n \), \( a_n \) is an increasing subsequence, converging to \(-1\). For odd \( n \), \( a_n = \frac{-n}{n+1} = -\frac{1}{1+1/n} \). So, for odd \( n \), \( a_n \) is an decreasing subsequence, converging to \(-1\). Thus \( \inf A = \min A = a_2 = -2 \), and \( \sup A = \max A = a_1 = -1/2 \).