Math 17A
Vogler
The Differential

Let $\Delta x$ be the change (error) in $x$, and assume $x$ changes from $a$ to $a+\Delta x$.

- Define the exact change in $f$ as
  \[ \Delta f = \Delta y = f(a+\Delta x) - f(a) \]

- Note: slope of line $L = \frac{\text{rise}}{\text{run}} \Rightarrow f'(a) = \frac{df}{\Delta x} \Rightarrow df = f'(a) \Delta x$

- Define the differential (approximate change) of $f$ as
  \[ df = f'(a) \Delta x \]

- Fact: If $\Delta x$ is ‘small’ then $df \approx \Delta f$

With this fact & the differential, we can approximate or simply functions by the following eqn for a line

\[ f(a+\Delta x) \approx f(a) + df = f(a) + f'(a) \Delta x \]

A more convenient form for this line is obtained by letting $x = a+\Delta x$ ($\Rightarrow \Delta x = x-a$) and rewriting as

\[ L(x) = f(a) + f'(a) (x-a) \]

This equation is called the linearization of $f @ x=a$.

Note: To use the linearization effectively, you must choose ‘$a$’ such that $f(a)$ & $f'(a)$ can be easily determined.
More examples using differentials

**Example 1:** For small \( h \), show that \( \sqrt{4+3h^2} \approx 2 + \frac{3}{4}h^2 \) using differentials.

**Soln** Let \( f(x) = \sqrt{x} \) & assume that \( x: 4 \rightarrow 4 + 3h^2 \)

\[ \Delta x = 3h^2 \quad \Rightarrow \quad f'(x) = \frac{1}{2\sqrt{x}} \].

Since \( \Delta x \) is small (b/c \( h \) is small)

\[ \Delta f \approx df \quad \Rightarrow \quad f(4 + 3h^2) - f(4) \approx f'(4) \cdot \Delta x \]

\[ = \sqrt{4 + 3h^2} - \sqrt{4} \approx \frac{1}{2\sqrt{4}} \cdot 3h^2 = \frac{3}{4}h^2 \]

\[ \Rightarrow \sqrt{4 + 3h^2} \approx 2 + \frac{3}{4}h^2 \]

**Example 2:** If the radius of a circle is measured w/ an absolute percentage error of @ most 3%, use differentials to estimate the maximum absolute percentage error in computing the circle's

a) circumference

\[ \text{Solution: Assume that} \quad \frac{|\Delta r|}{r} \leq 3\% \]

\[ \text{a)} \quad C = 2\pi r \quad \Rightarrow \quad C' = 2\pi, \text{ find} \quad \frac{|\Delta C|}{C} = \frac{1}{C} \cdot \frac{|\Delta r|}{r} \leq 3\% \]

\[ \left| \frac{\Delta A}{A} \right| \sim \frac{|\Delta A|}{A} = \frac{1}{A} \cdot \frac{|\Delta r|}{r} = \frac{|\Delta r|}{r} \leq 2(3\%) = 6\% \]

b) \( A = \pi r^2 \rightarrow A' = 2\pi r \), find \( \frac{1}{A} \)

b) \[ \frac{|\Delta A|}{A} \sim \frac{|\Delta A|}{A} = \frac{1}{2\pi r^2} \cdot \frac{|\Delta r|}{r} = \frac{|\Delta r|}{r} \leq 2(3\%) = 6\% \]