

Section 2.2

$$2.) a_n = 3n^2 : \begin{array}{cccccc} 0 & 3 & 12 & 27 & 48 & 75 \\ n & : & 0 & 1 & 2 & 3 & 4 & 5 \end{array}$$

$$9.) a_n = (-1)^n n : \begin{array}{cccccc} 0 & -1 & 2 & -3 & 4 & -5 \\ n & : & 0 & 1 & 2 & 3 & 4 & 5 \end{array}$$

$$10.) a_n = \frac{(-1)^n}{(n+1)^2} : \begin{array}{cccccc} 1 & -\frac{1}{4} & \frac{1}{9} & -\frac{1}{16} & \frac{1}{25} & -\frac{1}{36} \\ n & : & 0 & 1 & 2 & 3 & 4 & 5 \end{array}$$

$$19.) 1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}, \frac{1}{49}, \frac{1}{64}, \frac{1}{81}, \dots$$

$$20.) -1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}, -\frac{1}{25}, \frac{1}{36}, -\frac{1}{49}, \frac{1}{64}, -\frac{1}{81}, \dots$$

$$21.) \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \frac{9}{10}, \dots$$

$$22.) \frac{1}{5}, \frac{4}{10}, \frac{9}{17}, \frac{16}{26}, \frac{25}{37}, \frac{36}{50}, \frac{49}{65}, \frac{64}{82}, \frac{81}{101}, \dots$$

$$26.) \begin{array}{cccccc} a_n : & 0 & 2 & 4 & 6 & 8 & \dots \\ a_n : & 0 & 2 \cdot 1 & 2 \cdot 2 & 2 \cdot 3 & 2 \cdot 4 & \dots \\ n : & 0 & 1 & 2 & 3 & 4 & \dots \end{array} \quad \boxed{2n}$$

$$27.) \begin{array}{cccccc} a_n : & 1 & 2 & 4 & 8 & 16 & \dots \\ a_n : & 2^0 & 2^1 & 2^2 & 2^3 & 2^4 & \dots \\ n : & 0 & 1 & 2 & 3 & 4 & \dots \end{array} \quad \boxed{2^n}$$

$$28.) a_n: 1, 3, 5, 7, 9, \dots$$

$$a_n: 0+1, 2+1, 2 \cdot 2+1, 2 \cdot 3+1, 2 \cdot 4+1, \dots \quad \boxed{2n+1}$$

$$n: 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \dots \quad n$$

$$29.) a_n: 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots$$

$$a_n: \frac{1}{3^0}, \frac{1}{3^1}, \frac{1}{3^2}, \frac{1}{3^3}, \frac{1}{3^4}, \dots \quad \boxed{\frac{1}{3^n}}$$

$$n: 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \dots \quad n$$

$$30.) a_n: \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \dots$$

$$a_n: \frac{1}{2+1}, \frac{2}{2 \cdot 2+1}, \frac{3}{2 \cdot 3+1}, \frac{4}{2 \cdot 4+1}, \frac{5}{2 \cdot 5+1}, \dots \quad \boxed{\frac{n+1}{2(n+1)+1}}$$

$$n: 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \dots \quad n$$

$$31.) a_n: -1, 2, -3, 4, -5, \dots$$

$$a_n: (-1)^1 \cdot 1, (-1)^2 \cdot 2, (-1)^3 \cdot 3, (-1)^4 \cdot 4, (-1)^5 \cdot 5, \dots, \quad \boxed{(-1)^{n+1} (n+1)}$$

$$n: 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \dots \quad n$$

$$34.) a_n: \frac{1}{2}, -\frac{1}{8}, \frac{1}{18}, -\frac{1}{32}, \frac{1}{50}, \dots$$

$$a_n: \frac{(-1)^0}{2 \cdot 1^2}, \frac{(-1)^1}{2 \cdot 2^2}, \frac{(-1)^2}{2 \cdot 3^2}, \frac{(-1)^3}{2 \cdot 4^2}, \frac{(-1)^4}{2 \cdot 5^2}, \dots \quad \boxed{\frac{(-1)^n}{2 \cdot (n+1)^2}}$$

$$n: 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \dots \quad n$$

$$38.) a_n = \frac{2}{n+1}: 2, 1, \frac{2}{3}, \frac{2}{4}, \frac{2}{5}, \dots$$

$$\lim_{n \rightarrow \infty} \frac{2}{n+1} = \frac{2}{\infty} = 0$$

$$39.) a_n = \frac{n}{n+1} : 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n}{n+1} &\stackrel{\text{"}\infty\text{"}}{=} \lim_{n \rightarrow \infty} \frac{n}{n+1} \cdot \frac{1/n}{1/n} \\ &= \lim_{n \rightarrow \infty} \frac{1}{1 + 1/n} = \frac{1}{1+0} = 1 \end{aligned}$$

$$43.) a_n = \frac{(-1)^n}{n+1} : 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \dots$$

$$\frac{-1}{n+1} \leq \frac{(-1)^n}{n+1} \leq \frac{+1}{n+1} \quad \text{and}$$

$$\lim_{n \rightarrow \infty} \frac{-1}{n+1} = 0 = \lim_{n \rightarrow \infty} \frac{1}{n+1}, \quad \text{so}$$

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n+1} = 0 \quad \text{by Squeeze Principle}$$

$$45.) a_n = \frac{n^2}{n+1} : 0, 1, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}, \dots$$

$$\lim_{n \rightarrow \infty} \frac{n^2}{n+1} \stackrel{\text{"}\infty\text{"}}{=} \lim_{n \rightarrow \infty} \frac{n^2}{n+1} \cdot \frac{1/n}{1/n}$$

$$= \lim_{n \rightarrow \infty} \frac{n}{1 + 1/n} = \frac{\infty}{1+0} = \infty \quad (\text{DNE})$$

$$49.) a_n = 2^n : 1, 2, 4, 8, 16, \dots$$

$$\lim_{n \rightarrow \infty} 2^n = \infty \quad (\text{DNE})$$

$$50.) a_n = \left(\frac{1}{2}\right)^n : 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$$

$$\lim_{n \rightarrow \infty} \left(\frac{1}{2}\right)^n = 0$$

$$72.) \lim_{n \rightarrow \infty} \left(\frac{2}{n} - \frac{1}{n^2+1} \right) = \lim_{n \rightarrow \infty} \frac{2}{n} - \lim_{n \rightarrow \infty} \frac{1}{n^2+1}$$

$$= \frac{2}{\infty} - \frac{1}{\infty} = 0 - 0 = 0$$

$$75.) \lim_{n \rightarrow \infty} \left(\frac{n^2+1}{n^2} \right) = \lim_{n \rightarrow \infty} \left(\frac{n^2}{n^2} + \frac{1}{n^2} \right)$$

$$= \lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{1}{n^2} = 1 + \frac{1}{\infty} = 1 + 0 = 1$$

$$78.) \lim_{n \rightarrow \infty} \left(\frac{n+2}{n^2-4} \right) = \lim_{n \rightarrow \infty} \frac{n+2}{(n+2)(n-2)}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n-2} = \frac{1}{\infty} = 0$$

$$81.) \lim_{n \rightarrow \infty} \frac{n+2^{-n}}{n} = \lim_{n \rightarrow \infty} \left(\frac{n}{n} + \frac{2^{-n}}{n} \right)$$

$$= \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n \cdot 2^n} \right) = \lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{1}{n \cdot 2^n}$$

$$= 1 + \frac{1}{\infty} = 1 + 0 = 1$$

$$83.) \quad a_{n+1} = 2a_n, \quad a_0 = 1 :$$

$$a_1 = 2a_0 = 2(1) = 2$$

$$a_2 = 2a_1 = 2(2) = 4$$

$$a_3 = 2a_2 = 2(4) = 8$$

$$a_4 = 2a_3 = 2(8) = 16$$

$$a_5 = 2a_4 = 2(16) = 32$$

$$88.) \quad a_{n+1} = 4 - 2a_n, \quad a_0 = \frac{4}{3} :$$

$$a_1 = 4 - 2a_0 = 4 - 2\left(\frac{4}{3}\right) = 4 - \frac{8}{3} = \frac{4}{3}$$

$$a_2 = 4 - 2a_1 = 4 - 2\left(\frac{4}{3}\right) = \frac{4}{3}$$

$$a_3 = \frac{4}{3}, \quad a_4 = \frac{4}{3}, \quad a_5 = \frac{4}{3}$$

$$91.) \quad a_{n+1} = a_n + \frac{1}{a_n}, \quad a_0 = 1 :$$

$$a_1 = a_0 + \frac{1}{a_0} = 1 + \frac{1}{1} = 2$$

$$a_2 = a_1 + \frac{1}{a_1} = 2 + \frac{1}{2} = \frac{5}{2}$$

$$a_3 = a_2 + \frac{1}{a_2} = \frac{5}{2} + \frac{1}{5/2} = \frac{5}{2} + \frac{2}{5} = \frac{29}{10}$$

$$a_4 = a_3 + \frac{1}{a_3} = \frac{29}{10} + \frac{1}{29/10} = \frac{29}{10} + \frac{10}{29} = \frac{941}{290}$$

$$a_5 = a_4 + \frac{1}{a_4} = \frac{941}{290} + \frac{1}{941/290} = \frac{941}{290} + \frac{290}{941} = \frac{969,581}{272,890}$$

93.) assume $\lim_{n \rightarrow \infty} a_n = L$; then

$$a_{n+1} = \frac{1}{2} a_n + 2 \rightarrow (n \rightarrow \infty) \quad L = \frac{1}{2} L + 2 \rightarrow \frac{1}{2} L = 2 \rightarrow L = 4$$

96.) assume $\lim_{n \rightarrow \infty} a_n = L$; then

$$a_{n+1} = -\frac{1}{3} a_n + \frac{1}{4} \rightarrow (n \rightarrow \infty) \quad L = -\frac{1}{3} L + \frac{1}{4} \rightarrow \frac{4}{3} L = \frac{1}{4} \rightarrow L = \frac{1}{4} \div \frac{4}{3} = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}$$

100.) assume $\lim_{n \rightarrow \infty} a_n = L$; then

$$a_{n+1} = \frac{3}{a_n - 2} \rightarrow (n \rightarrow \infty) \quad L = \frac{3}{L - 2} \rightarrow$$

$$L^2 - 2L = 3 \rightarrow L^2 - 2L - 3 = 0 \rightarrow$$

$$(L - 3)(L + 1) = 0 \rightarrow L = 3 \text{ or } L = -1$$

105.) assume $\lim_{n \rightarrow \infty} a_n = L$; then

$$a_{n+1} = \sqrt{2a_n} \rightarrow (n \rightarrow \infty) \quad L = \sqrt{2L} \rightarrow$$

$$L^2 = 2L \rightarrow L^2 - 2L = 0 \rightarrow$$

$$L(L - 2) = 0 \rightarrow$$

$$L = 0 \text{ or } L = 2$$

the table suggests the limit is 2 for $a_0 = 1$.

n	a{n}
0	1.00
1	1.41
2	1.68
3	1.83
4	1.92
5	1.96
6	1.98
7	1.99
8	1.99
9	2.00
10	2.00

106.) If $a_{n+1} = \sqrt{2a_n}$ and $a_0 = 0$, then
 $a_1 = \sqrt{2a_0} = \sqrt{2(0)} = \sqrt{0} = 0$,
 $a_2 = \sqrt{2a_1} = \sqrt{2(0)} = \sqrt{0} = 0$,
 $a_3 = 0, a_4 = 0, a_5 = 0, \dots$ so
 $\lim_{n \rightarrow \infty} a_n = 0$ if $a_0 = 0$

109.) Assume $\lim_{n \rightarrow \infty} a_n = L$; then

$$a_{n+1} = \frac{1}{2} \left(a_n + \frac{4}{a_n} \right) \rightarrow (n \rightarrow \infty) \quad L = \frac{1}{2} \left(L + \frac{4}{L} \right)$$

$$\rightarrow 2L = L + \frac{4}{L} \rightarrow 2L^2 = L^2 + 4 \rightarrow L^2 = 4$$

$$\rightarrow L = 2 \text{ or } L = -2;$$

the table suggests
the limit is 2
if $a_0 = 1$.

n	a{n}
0	1.00000
1	2.50000
2	2.05000
3	2.00061
4	2.00000
5	2.00000

E, N Proof worksheet

Reminder: 1.) Start the proof with 3 sentences.

2.) The middle of the proof consists of your "iff" statements.

3.) Finish the proof with 2 sentences.

$$1.) \lim_{n \rightarrow \infty} \frac{500}{n^3 + 2} = \frac{500}{\infty} = 0.$$

Proof: Let $\epsilon > 0$ be given. Find an integer N so that if $n > N$, then $|a_n - L| < \epsilon$. Begin with $|a_n - L| < \epsilon$ and solve for n .

$$\text{Then } |a_n - L| < \epsilon$$

$$\text{iff } \left| \frac{500}{n^3 + 2} - 0 \right| < \epsilon$$

$$\text{iff } \frac{500}{n^3 + 2} < \epsilon \quad (n^3 + 2 > 0)$$

$$\text{iff } \frac{500}{\epsilon} < n^3 + 2$$

$$\text{iff } \frac{500}{\epsilon} - 2 < n^3$$

$$\text{iff } n^3 > \frac{500}{\epsilon} - 2$$

$$\text{iff } n > \left(\frac{500}{\epsilon} - 2 \right)^{1/3}.$$

Now choose integer $N \geq \left(\frac{500}{\epsilon} - 2\right)^{1/3}$.
 Thus, if $n > N$ it follows that
 $|a_n - L| < \epsilon$.

QED

$$2.) \lim_{n \rightarrow \infty} \frac{2n+3}{n-10} \stackrel{\text{"}\infty\text{"}}{=} \lim_{n \rightarrow \infty} \frac{2n+3}{n-10} \cdot \frac{1/n}{1/n}$$

$$= \lim_{n \rightarrow \infty} \frac{2 + 3/n}{1 - 10/n} = \frac{2+0}{1-0} = 2.$$

Proof: Let $\epsilon > 0$ be given. Find
 integer N so that if $n > N$, then
 $|a_n - L| < \epsilon$. Begin with $|a_n - L| < \epsilon$
 and solve for n . Then

$$|a_n - L| < \epsilon \text{ iff } \left| \frac{2n+3}{n-10} - 2 \right| < \epsilon$$

$$\text{iff } \left| \frac{2n+3 - 2(n-10)}{n-10} \right| < \epsilon$$

$$\text{iff } \left| \frac{2n+3 - 2n+20}{n-10} \right| < \epsilon$$

$$\text{iff } \frac{23}{|n-10|} < \epsilon$$

$$\text{iff } \frac{23}{n-10} < \epsilon \quad (\text{assume } n > 10 \text{ since } n \rightarrow \infty.)$$

$$\text{iff } \frac{23}{\epsilon} < n-10$$

$$\text{iff } \frac{23}{\varepsilon} + 10 < n$$

$$\text{iff } n > \frac{23}{\varepsilon} + 10.$$

Now choose integer $N \geq \frac{23}{\varepsilon} + 10$.
Thus, if $n > N$, it follows that $|a_n - L| < \varepsilon$.

$$\begin{aligned} 3) \quad \lim_{n \rightarrow \infty} \frac{3n+2}{5-4n} &\stackrel{\text{"}\infty\text{"}}{=} \lim_{n \rightarrow \infty} \frac{3n+2}{5-4n} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} \frac{3 + \frac{2}{n}}{\frac{5}{n} - 4} = \frac{3+0}{0-4} = -\frac{3}{4}. \end{aligned}$$

Proof: Let $\varepsilon > 0$ be given. Find integer N so that if $n > N$, then $|a_n - L| < \varepsilon$. Begin with $|a_n - L| < \varepsilon$ and solve for n . Then

$$|a_n - L| < \varepsilon \text{ iff } \left| \frac{3n+2}{5-4n} - \left(-\frac{3}{4}\right) \right| < \varepsilon$$

$$\text{iff } \left| \frac{4(3n+2) + 3(5-4n)}{4(5-4n)} \right| < \varepsilon$$

$$\text{iff } \left| \frac{12n + 8 + 15 - 12n}{4(5-4n)} \right| < \varepsilon$$

$$\text{iff } \frac{23}{|4(5-4n)|} < \varepsilon$$

$$\text{iff } \frac{23}{-4(5-4n)} < \varepsilon \quad (\text{since } n \rightarrow \infty \text{ we have } 5-4n < 0.)$$

$$\text{iff } \frac{23}{-4\varepsilon} > 5-4n$$

$$\text{iff } 4n > 5 + \frac{23}{4\varepsilon}$$

$$\text{iff } n > \frac{1}{4} \left(5 + \frac{23}{4\varepsilon} \right).$$

Now choose integer $N \geq \frac{1}{4} \left(5 + \frac{23}{4\varepsilon} \right)$.

Thus, if $n > N$, it follows that $|a_n - L| < \varepsilon$.

QED

$$\begin{aligned} 4.) \quad \lim_{n \rightarrow \infty} \frac{\sqrt{n} + 2}{4 - \sqrt{n}} &\stackrel{\text{"}\infty\text{"}}{=} \lim_{n \rightarrow \infty} \frac{\sqrt{n} + 2}{4 - \sqrt{n}} \cdot \frac{1/\sqrt{n}}{1/\sqrt{n}} \\ &= \lim_{n \rightarrow \infty} \frac{1 + 2/\sqrt{n}}{\frac{4}{\sqrt{n}} - 1} = \frac{1+0}{0-1} = -1. \end{aligned}$$

Proof: Let $\varepsilon > 0$ be given. Find integer N so that if $n > N$, then $|a_n - L| < \varepsilon$. Begin with $|a_n - L| < \varepsilon$ and solve for n . Then

$$|a_n - L| < \varepsilon \quad \text{iff} \quad \left| \frac{\sqrt{n} + 2}{4 - \sqrt{n}} - (-1) \right| < \varepsilon$$

$$\text{iff } \left| \frac{(\sqrt{n}+2) + (4-\sqrt{n})}{4-\sqrt{n}} \right| < \varepsilon$$

$$\text{iff } \frac{6}{|4-\sqrt{n}|} < \varepsilon$$

$$\text{iff } \frac{6}{-(4-\sqrt{n})} < \varepsilon \quad (\text{Since } n \rightarrow \infty \text{ we know that } 4-\sqrt{n} < 0.)$$

$$\text{iff } \frac{6}{-\varepsilon} > 4-\sqrt{n}$$

$$\text{iff } \sqrt{n} > 4 + \frac{6}{\varepsilon}$$

$$\text{iff } n > \left(4 + \frac{6}{\varepsilon}\right)^2.$$

Now choose integer $N \geq \left(4 + \frac{6}{\varepsilon}\right)^2$.
Thus, if $n > N$, it follows that $|a_n - L| < \varepsilon$.

QED

$$5.) \lim_{n \rightarrow \infty} (0.999)^n = 0 \text{ since } -1 < 0.999 < 1.$$

Proof: Let $\varepsilon > 0$ be given. Find integer N so that if $n > N$ then $|a_n - L| < \varepsilon$. Begin with $|a_n - L| < \varepsilon$ and solve for n . Then

$$|a_n - L| < \varepsilon \text{ iff } |(0.999)^n - 0| < \varepsilon$$

$$\text{iff } (0.999)^n < \varepsilon \quad (\text{Since } (0.999)^n > 0.)$$

$$\text{iff } \ln (0.999)^n < \ln \varepsilon$$

$$\text{iff } n \ln (0.999) < \ln \varepsilon$$

$$\text{iff } n > \frac{\ln \varepsilon}{\ln (0.999)} \quad (\text{Since } \ln (0.999) < 0).$$

$$\text{Now choose integer } N \geq \frac{\ln \varepsilon}{\ln (0.999)}.$$

Thus, if $n > N$,
it follows that $|a_n - L| < \varepsilon$.

QED