Arc Length, Unit Tangent Vector, Unit Normal Vector, & Curvature

Position Vector: \( \vec{r}(t) = f(t) \hat{i} + g(t) \hat{j} + h(t) \hat{k} \)

Velocity Vector: \( \vec{v}(t) = f'(t) \hat{i} + g'(t) \hat{j} + h'(t) \hat{k} \)

Acceleration Vector: \( \vec{a}(t) = f''(t) \hat{i} + g''(t) \hat{j} + h''(t) \hat{k} \)

\[ |\vec{v}(t)| = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} \]

Let curve \( C \) be determined by vector function \( \vec{r}(t) : \mathbb{R} \to \mathbb{R}^3 \).

**Defn** The **arc length** \( s \) for \( t = a \) to \( t \) is

\[ s(t) = \int_a^t \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} \, dt = \int_a^t |\vec{v}(t)| \, dt \]

**Defn** The **unit tangent vector** for \( \vec{r}(t) \) is

\[ \hat{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} \]

**Notes:**
1) \( \hat{T}(t) \) points in direction of motion along \( C \).
2) \( \hat{T}(t) \) is a unit vector.

**Defn** The **principal unit normal vector** for \( \vec{r}(t) \) is

\[ \hat{N}(t) = \frac{\hat{T}'(t)}{|\hat{T}'(t)|} \]

**Thm**
1) \( \hat{N}(t) \) is a unit vector.
2) \( \hat{N}(t) \) is normal to the path \( C \) determined by \( \vec{r}(t) \), i.e., \( \hat{N}(t) \) is orthogonal to \( \hat{T}(t) \).
3) \( \hat{N}(t) \) points in the direction that curve \( C \) is turning.
Defn The curvature of the path $C$ is
$$\kappa = \left| \frac{dT}{ds} \right|$$

Formula for Computing Curvature
$$\kappa = \frac{1}{|\vec{N}(t)| \cdot |\vec{T}'(t)|}$$

Fact: The curvature of a circle of radius $a$ is $\kappa = \frac{1}{a}$

Defn The circle of curvature at a point $P$ on path $C$ is the circle in the plane of the curve that
1) is tangent to the curve at $P$ (has same tangent line)
2) has same curvature that curve $C$ has at $P$
3) lies toward the concave (inner) side of curve $C$
4) has radius $\rho = \frac{1}{\kappa}$, which is referred to as radius of curvature

Circle of Curvature

Small $\kappa \Rightarrow$ Big radius $\rho$
(Since small change in $\vec{T}$)

Big $\kappa \Rightarrow$ Small radius $\rho$
(Since Big change in $\vec{T}$)