Math 21D
Voyler

Line Integrals

\( \mathbf{r}(t): \mathbb{R} \to \mathbb{R}^3 \)

Exchange rate: \( \Delta s = \frac{ds}{dt} \Delta t \)

\( t_i \) such that \( \mathbf{r}(t_i) = P_i \)

- Let \( C \) be curve defined by \( \mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k} \) for \( t = a \) to \( t = b \)
- Let \( f(P) \) be function defined on \( C \) (i.e. \( f(x,y,z) = f(g(t), h(t), k(t)) \))
- Let \( C_1, \ldots, C_n \) be subdivision of \( C \) w/ corresponding arc lengths \( \Delta s_1, \Delta s_2, \ldots, \Delta s_n \).
- Let \( P_i = (x_i, y_i, z_i) \) be arbitrary pt. in \( C_i \) \( \forall i = 1, \ldots, n \)
- Let mesh be \( \Delta s_i = \max_{i \in \text{set}} \{ \Delta s_i \} \)

Then the line integral of \( f \) over curve \( C \) from \( t = a \) to \( t = b \) is

\[
\int_C f(x, y, z)\, ds = \lim_{\Delta t \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i) \cdot \Delta s_i = \lim_{\Delta t \to 0} \sum_{i=1}^{n} f(g(t_i), h(t_i), k(t_i)) \frac{ds}{dt} \Delta t
\]

Method of Evaluation

\[
\int_C f(x, y, z)\, ds = \int_{a}^{b} f(g(t), h(t), k(t)) \cdot \frac{ds}{dt} \, dt
\]

w/ \( \frac{ds}{dt} = |\dot{\mathbf{r}}(t)| = \sqrt{(g'(t))^2 + (h'(t))^2 + (k'(t))^2} \)

Extra info.
Applications of Line Integrals

Let \( \delta(P) = \delta(x,y,z) \) be density \( \frac{\text{mass}}{\text{length}} \) units be defined on \( C \).

1) Length = \( s := \int_C 1 \, ds \)

2) Mass = \( m := \int_C \delta(P) \, ds \)

3) 1st Moments about Arpibary Planes
   - About plane \( x=x_0 \): \( M_x = \bar{x} = \int_C (x-x_0) \delta(P) \, ds \)
   - About plane \( y=y_0 \): \( M_y = \bar{y} = \int_C (y-y_0) \delta(P) \, ds \)
   - About plane \( z=z_0 \): \( M_z = \bar{z} = \int_C (z-z_0) \delta(P) \, ds \)

4) 1st Moments about Coordinate Planes
   \( M_{yz} = \int_C x \delta(P) \, ds, \quad M_{xz} = \int_C y \delta(P) \, ds, \quad M_{xy} = \int_C z \delta(P) \, ds \)
   Note: \( M_{yz} = M_{x=0}, \quad M_{xz} = M_{y=0}, \quad M_{xy} = M_{z=0} \)

5) Center of Mass \(( \bar{x}, \bar{y}, \bar{z} )\)
   \( \bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m} \)

6) Centroid \(( \bar{x}, \bar{y}, \bar{z} )\)
   \( \bar{x} = \frac{\int_C x \, ds}{\text{Length}}, \quad \bar{y} = \frac{\int_C y \, ds}{\text{Length}}, \quad \bar{z} = \frac{\int_C z \, ds}{\text{Length}} \)

7) Moments of Inertia (2nd Moments)
   - About line \( L \) or pt. \( P_0 \): \( I = \int_C r^2 \delta(P) \, ds \)
     w/ \( r := r(x,y,z) \) is distance from line \( L \) or pt. \( P_0 \)
   - About \( x \)-axis: \( I_x = \int_C (y^2+z^2) \delta(P) \, ds \)
   - About \( y \)-axis: \( I_y = \int_C (x^2+z^2) \delta(P) \, ds \)
   - About \( z \)-axis: \( I_z = \int_C (x^2+y^2) \delta(P) \, ds \)