Name: Key *Exam ID:*

PLEASE READ THIS BEFORE YOU DO ANYTHING ELSE!

1. Make sure that your exam contains 8 pages, including this one.
2. **NO** calculators, books, notes, other written material, or help from other students allowed.
3. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.
4. You will be graded on proper use of limit, sequence, and set notation.
5. You are free to use any theorem in the book, any theorem presented in class, or any result of an assigned homework problem without proof. The only exception is if it trivializes the problem.
6. If you use a named theorem in the book, you MUST cite the name when invoking the theorem. Common abbreviations are fine (i.e. PMI for the Principle of Mathematical Induction).
7. You have until 9:50am to finish this exam.
8. Read the statement below and sign your name.

I affirm that I neither will give nor receive unauthorized assistance on this examination. All the work that appears on the following pages is entirely my own.

Signature: ______________________

"You can profit from your mistakes,
but that does not mean the more mistakes, the more profit." – Anonymous

GOOD LUCK!!!

1. (*8 pts*) State the IN CLASS version of the Archimedean Property and The Denseness of $\mathbb{Q}$.

Archimedean Property

If $a>0$ and $b>0$, then $\exists n \in \mathbb{N}$ such that $na>b$

Denseness of $\mathbb{Q}$

If $a,b \in \mathbb{R}$ and $a<b$, then $\exists r \in \mathbb{Q}$ such that $a<r<b$.

2. (*12 pts*) Use the definition (i.e. ϵN-property) to prove that

$$\lim_{n\to\infty} \frac{5n}{n+2} = 5$$

Let $\varepsilon>0$ be given.

$|s_n - 5| = \left|\frac{5n}{n+2} - 5\right| = \left|\frac{5n-5n-10}{n+2}\right| = \frac{10}{n+2}$

Then, we get $|s_n - 5| < \varepsilon \iff \frac{10}{n+2} < \varepsilon \iff \frac{10}{\varepsilon} < n+2 \iff n > \frac{10}{\varepsilon} - 2$.

Choose $N = \frac{10}{\varepsilon} - 2$. Thus, $\forall n > N \Rightarrow \left|\frac{5n}{n+2} - 5\right| < \varepsilon$.

Therefore, we conclude $\lim_{n\to\infty} \frac{5n}{n+2} = 5$

3. (*6 pts*) For the following sequences, determine whether it converges and, if so, give its limit. Otherwise, state it diverges. You may use any method you wish to determine this, without justification.

(a) $a_n = \frac{(-1)^n}{n}$

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = \frac{\text{`}\pm 1\text{'}}{\infty} = \boxed{0}$$

(b) $b_n = \left(1 + \frac{7}{5n}\right)^n$ $= \left(1 + \frac{1}{\frac{5n}{7}}\right)^{n\frac{7}{5}\cdot\frac{7}{5}} = \left[\left(1+\frac{1}{\frac{5n}{7}}\right)^{\frac{5n}{7}}\right]^{\frac{7}{5}}$

$$\Rightarrow \lim_{n\to\infty} b_n = \lim_{n\to\infty}\left[\left(1+\frac{1}{\frac{5n}{7}}\right)^{\frac{5n}{7}}\right]^{\frac{7}{5}} = \boxed{e^{7/5}}$$

4. (*15 pts*) For the following sets, determine its maximum, minimum, supremum, and infimum. You do not need to provide any justification.

(a) $A = \{x \in \mathbb{R} : 3 \leq x < 7\}$

min $A = 3$, max A DNE

inf $A = 3$, sup $A = 7$

(b) $B = \cap_{n=1}^{\infty}\left(1 + \frac{1}{n}, 7 + \frac{1}{n}\right]$ $= (1+1, 7+1] \cap (1+\frac{1}{2}, 7+\frac{1}{2}] \cap (1+\frac{1}{3}, 7+\frac{1}{3}] \cap \ldots$

$\Rightarrow B = \lim_{n\to\infty} (1+\frac{1}{n}, 7+\frac{1}{n}] = (1, 7]$

min B DNE, max $B = 7$

inf $B = 1$, sup $B = 7$

(c) $C = \{n \in \mathbb{N} : 0 \leq n \leq 11\}$ $= \{1, 2, 3, \ldots, 11\}$

min $C = 1$, max $C = 11$

inf $C = 1$, sup $C = 11$

5. (*12 pts*) For this problem, you must use the Ordered Field Axioms or Theorems 3.1 and 3.2, which are provided on the last page. Prove if $a + c = b + c$, then $a = b$.

Ꝥ $a+c=b+c$. By A4, $\exists -c \in \mathbb{R}$ such that $c+(-c)=0$. Adding $-c$ to both sides gives us $(a+c)+(-c)=(b+c)+(-c) \Rightarrow a+(c+(-c))=b+(c+(-c))$ by A1.

$\Rightarrow a+0=b+0$ by A4 $\Rightarrow a=b$ by A3.

Therefore, we conclude if $a+c=b+c$, then $a=b$.

6. (*12 pts*) Prove that $\sqrt[3]{6}$ is irrational using the Rational Zero's Theorem.

Let $x=\sqrt[3]{6}$, then we have

$x=\sqrt[3]{6} \Leftrightarrow x^3-6=0$ (1).

So x solves equation (1). By Rational Zero's Theorem, since rational solutions must be of the form $\frac{p}{q}$ with $p|6$ and $q|1$, the only rational solutions to (1) are $\pm1, \pm2, \pm3, \pm6$. Clearly, none of these solve (1). Since $x=\sqrt[3]{6}$ solves (1), it cannot be rational

Therefore, $\sqrt[3]{6}$ must be irrational.

7. (*12 pts*) Using induction, prove $(n+3)! > 5^n$ for positive integers.

i) This is true for $n=1$, since $24 = (1+3)! > 5^1 = 5$.

ii) Let $n \in \mathbb{N}$. Assume $(n+3)! > 5^n$ is true.

Then $((n+1)+3)! = (n+4)! = (n+4)(n+3)! > (n+4)\,5^n \geq 5^{n+1}$

since $n+4 \geq 5 \ \forall n \in \mathbb{N}$. Thus we have $((n+1)+3)! > 5^{n+1}$,

so the inequality holds for $n+1$.

Therefore, by PMI, we conclude

$(n+3)! > 5^n$ for all positive integers.

8. (*10 pts*) Prove if x is irrational and r is rational with $r \neq 0$, then xr is irrational.

$\wp$ $x \in \mathbb{I}$ and $r \in \mathbb{Q}$ with $r \neq 0$, but $xr \notin \mathbb{I}$.

So $xr \in \mathbb{Q}$. Then, $\exists\, p, q, m, n \in \mathbb{Z}$ with $q \neq 0$ and $n \neq 0$

where $r = \frac{p}{q}$ and $xr = \frac{m}{n}$. Moreover, $p \neq 0$ since $r \neq 0$.

Combining these, we get

$\frac{m}{n} = xr = x\frac{p}{q} \implies x = \frac{mq}{np}$, so $x \in \mathbb{Q}$ since

$p, q, m, n \in \mathbb{Z}$ with $n \neq 0$ and $p \neq 0$. But $x \in \mathbb{I}$.

Contradiction!

Therefore, we conclude if $x \in \mathbb{I}$ and $r \in \mathbb{Q}$ with $r \neq 0$

then $xr \in \mathbb{I}$.

9. (*13 pts*) Prove if $a < b$, then there exists $x \in \mathbb{I}$ such that $a < x < b$.

§ $a<b$. Subtracting $\sqrt{2}$ from both sides gives us $a-\sqrt{2} < b-\sqrt{2}$. By the Denseness of $\mathbb{Q}$, $\exists r \in \mathbb{Q}$ such that $a-\sqrt{2} < r < b-\sqrt{2}$. Adding $\sqrt{2}$ to all sides, we obtain

$$a < r+\sqrt{2} < b.$$

Let $x = r+\sqrt{2}$, and notice $x \in \mathbb{I}$ since $r \in \mathbb{Q}$ and $\sqrt{2} \in \mathbb{I}$. Thus, we found $x \in \mathbb{I}$ with $a < x < b$.

Therefore, if $a<b$, then there exists $x \in \mathbb{I}$ such that $a < x < b$.