*Name:* Key *Exam ID:*__________

**PLEASE READ THIS BEFORE YOU DO ANYTHING ELSE!**

1. Make sure that your exam contains 8 pages, including this one.
2. **NO** calculators, books, notes, other written material, or help from other students allowed.
3. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.
4. You will be graded on proper use of limit, sequence, and set notation.
5. You are free to use any theorem in the book, any theorem presented in class, or any result of an assigned homework problem without proof. The only exception is if it trivializes the problem.
6. If you use a named theorem in the book, you MUST cite the name when invoking the theorem. Common abbreviations are fine (i.e. PMI for the Principle of Mathematical Induction).
7. You have until 9:50am to finish this exam.
8. Read the statement below and sign your name.

*I affirm that I neither will give nor receive unauthorized assistance on this examination. All the work that appears on the following pages is entirely my own.*

Signature: ____________________________

*"You can profit from your mistakes,*
*but that does not mean the more mistakes, the more profit."* – *Anonymous*

*GOOD LUCK!!!*

1. (*12 pts*) State the IN CLASS version of the proof for Lemma 10.9:

*Convergent sequences are Cauchy sequences.*

$\oint \lim_{n\to\infty} s_n = s$. Notice

$|s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s_m - s|$ (1) by Triangle Inequality.

Let $\varepsilon > 0$ be given. Since $\{s_n\}$ converges to $s$,

$\exists\, N \in \mathbb{R}$ such that

$$\forall n > N \Rightarrow |s_n - s| < \frac{\varepsilon}{2}. \quad (2)$$

Moreover, we also have

$$\forall m > N \Rightarrow |s_m - s| < \frac{\varepsilon}{2}. \quad (3)$$

We choose this $N$. By (1), (2), & (3), we obtain

$$\forall m, n > N \Rightarrow |s_n - s_m| \leq |s_n - s| + |s_m - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, we conclude $\{s_n\}$ is Cauchy ∎

2. (*8 pts*) Use the definition (i.e. MN property) to prove that

$$\lim_{n\to\infty} n^3 = \infty.$$

Let $M > 0$ be given. Then, we have

$$s_n > M \Leftrightarrow n^3 > M \Leftrightarrow n > \sqrt[3]{M}.$$

Choose $N = \sqrt[3]{M}$. Thus, $\forall n > N$, then $n^3 > M$.

Therefore, we conclude $\lim_{n\to\infty} n^3 = \infty$.

3. (*12 pts*) For the following sequence, determine its set of subsequential limits, its $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$, whether it converges or diverges, whether it is monotonically non-increasing or non-decreasing, and whether it is bounded. You DO NOT need justification for your answers.

(a) $a_n = \begin{cases} \frac{n-1}{2} & \text{if n is odd} \\ -\frac{n}{2} & \text{if n is even} \end{cases}$ $\quad a_n: 0, -1, 1, -2, 2, \dots$

$S = \{-\infty, \infty\}$, $\limsup_{n\to\infty} a_n = \infty$, $\liminf_{n\to\infty} a_n = -\infty$

$\{a_n\}$ diverges, and it is not monotonic or bounded

(b) $b_n = \dfrac{3}{4n+1}$ $\quad b_n: \frac{3}{5}, \frac{3}{9}, \frac{3}{13}, \dots$

$S = \{0\}$, $\limsup_{n\to\infty} b_n = \liminf_{n\to\infty} b_n = 0$.

$\{b_n\}$ converges and it is non-increasing (or decreasing) and bounded.

4. (*10 pts*) Prove if $\lim_{n\to\infty} s_n = s$ and $k \in \mathbb{R}$, then $\lim_{n\to\infty} ks_n = ks$ using the definition (i.e. $\epsilon N$ property).

$\int$ $\lim_{n\to\infty} s_n = s$ and $k \in \mathbb{R}$. If $k=0$, then $\{ks_n\}$ clearly converges to zero. Assume $k \neq 0$ and let $\varepsilon > 0$ be given.

Since $\lim_{n\to\infty} s_n = s$, $\exists N$ such that

$$\forall n > N \Rightarrow |s_n - s| < \frac{\varepsilon}{|k|}$$

Choose this $N$. Thus, we have

$\forall n > N \Rightarrow \quad |ks_n - ks| = |k||s_n - s| < |k| \cdot \frac{\varepsilon}{|k|} = \varepsilon.$

Thus, we conclude $\lim_{n\to\infty} ks_n = ks$.

5. (*14 pts*) Consider the following sequence $\{s_n\}$ by letting $s_1 = \sqrt{5}$ and $s_{n+1} = \sqrt{5 + s_n}$ for $n \geq 1$. Notice that $0 \leq s_n \leq 3$ $\forall n \in \mathbb{N}$ (You DO NOT need to prove this bound and may assume it).

(a) Prove this sequence $\{s_n\}$ converges.

We show $s_n \leq s_{n+1}$ $\forall n \in \mathbb{N}$ (i.e. increasing) by induction:

(i) This is true for $n = 1$, since $s_1 = \sqrt{5} < \sqrt{5 + \sqrt{5}} = s_2$.

(ii) Let $n \in \mathbb{N}$. Assume $s_n \leq s_{n+1}$ is true. Then, we get

$5 + s_n \leq 5 + s_{n+1} \Rightarrow s_{n+1} = \sqrt{5 + s_n} \leq \sqrt{5 + s_{n+1}} = s_{n+2}$.

So the statement is true for $n+1$.

Hence, by PMI, we have $s_n \leq s_{n+1}$ $\forall n \in \mathbb{N}$, and $\{s_n\}$ is monotonically increasing.

Therefore, $\{s_n\}$ is a bounded monotone sequence, so we conclude that it must converge.

(b) Find $\lim\limits_{n \to \infty} s_n$.

Let $\lim\limits_{n \to \infty} s_n = s$. Then, we have

$\lim\limits_{n \to \infty} s_{n+1} = \lim\limits_{n \to \infty} \sqrt{5 + s_n} \Rightarrow s = \sqrt{5 + s} \Rightarrow s^2 - s - 5 = 0$

$\Rightarrow s = \dfrac{1 \pm \sqrt{21}}{2}$.

Since $s_1 = \sqrt{5} > 0$ & $\{s_n\}$ is increasing, we must have $s > 0$.

Therefore, we conclude $\boxed{s = \dfrac{1 + \sqrt{21}}{2}}$

6. (*12 pts*) Consider the sequence $s_n = (-1)^n + \cos\left(\frac{n\pi}{2}\right)$. Find the set of subsequential limits $S$ by providing a subsequence that converges to each subsequential limit point.

$n: 1, 2, 3, 4, 5, 6, 7, 8, \ldots$

$s_n: -1, 0, -1, 2, -1, 0, -1, 2, \ldots$

So $s_n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 0 & \text{if } 4 \mid (n-2) \\ 2 & \text{if } 4 \mid n \end{cases}$

- Consider subsequence $\{s_{2k+1}\}$. Then $s_{2k+1} = -1 \ \forall k \in \mathbb{N}$.
  So $\lim_{k\to\infty} s_{2k+1} = -1$.
- Consider subsequence $\{s_{4k+2}\}$. Then $s_{4k+2} = 0 \ \forall k \in \mathbb{N}$.
  So $\lim_{k\to\infty} s_{4k+2} = 0$.
- Consider subsequence $\{s_{4k}\}$. Then $s_{4k} = 2 \ \forall k \in \mathbb{N}$,
  and $\lim_{k\to\infty} s_{4k} = 2$.

Thus, the set of subsequential limits is

$S = \{-1, 0, 2\}$.

7. (*18 pts*) Determine whether the following statements are TRUE or FALSE. If true, BRIEFLY justify the given statement; if false, give a counterexample to the statement.

(a) For $a, b \in \mathbb{R}$, there does not exist a sequence where the set of subsequential limits is $S = (a, b]$.

True. If $S=(a,b]$, then I could easily construct a sequence $\{a_n\}$ with $a_n \in S$ $\forall n \in \mathbb{N}$ which $\lim_{n\to\infty} a_n = a$. Thus, $a \in S$ (by Thm 11.8). So no sequence can exist with $S=(a,b]$.

(b) In $\mathbb{R}$, if a sequence $\{b_n\}$ has no convergent subsequences, then $\lim_{n\to\infty} |b_n| = \infty$.

True. Consider the contrapositive. If $\lim_{n\to\infty} |b_n| \neq \infty$, then the sequence is bounded. By the Bolzano-Weierstrass Thm, there's a convergent subsequence. Thus, the contrapositive and the statement is true.

(c) If the sequences of real numbers $\{s_n\}$ and $\{t_n\}$ both converge and $s_n < t_n$ $\forall n \in \mathbb{N}$, then $\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n$.

False. Let $s_n = 0$ and $t_n = \frac{1}{n}$. Clearly, $s_n < t_n$ $\forall n \in \mathbb{N}$, but $\lim_{n\to\infty} s_n = 0 = \lim_{n\to\infty} t_n$, disproving the claim.

8. (*14 pts*) Prove that if $\{s_n\}$ and $\{t_n\}$ are both bounded sequences, then

$$\limsup_{n\to\infty} (s_n + t_n) \leq \limsup_{n\to\infty} s_n + \limsup_{n\to\infty} t_n.$$

Pf $\{s_n\}$ and $\{t_n\}$ are bounded sequences. For each $N \in \mathbb{N}$, define $A_N = \{s_n + t_n : n > N\}$, $B_N = \{s_n : n > N\}$, & $C_N = \{t_n : n > N\}$.

Since all sets are bounded, their sup's must exist, by Completeness Axiom.

Let $a_N = \sup A_N$, $b_N = \sup B_N$, and $c_N = \sup C_N$ $\forall N \in \mathbb{N}$.

Fix $N \in \mathbb{N}$. If $n \geq N \Rightarrow s_n + t_n \leq b_N + c_N$, so $b_N + c_N$ is an upper bound for $A_N$, and $a_N \leq b_N + c_N$.

Since $N \in \mathbb{N}$ was arbitrary, we have

$$a_N \leq b_N + c_N \quad \forall N \in \mathbb{N}.$$

Taking limits of both sides gives us

$$\lim_{N\to\infty} a_N \leq \lim_{N\to\infty} b_N + \lim_{N\to\infty} c_N.$$

Therefore, we conclude if $\{s_n\}$ and $\{t_n\}$ are bounded, then $\limsup_{n\to\infty} (s_n + t_n) \leq \limsup_{n\to\infty} s_n + \limsup_{n\to\infty} t_n$ ☐