1. (*pts*) Prove that $2^n > (n+1)^2$ for all integers $n \geq 6$ using induction.

i) This is true for $n=6$, since $64 = 2^6 > 7^2 = 49$.

ii) Let $n \in \mathbb{N}$ with $n \geq 6$. & $2^n > (n+1)^2$ is true.

Then $2^{n+1} = 2(2^n) > 2(n+1)^2 = 2(n^2+2n+1) = 2n^2+4n+2$

$\geq n^2+4n+4 = (n+2)^2$

since $n^2 \geq 2$ for $n \geq 6$. Thus, we have $2^{n+1} > (n+2)^2$, so the inequality holds for $n+1$.

Therefore, by PMI, we conclude

$2^n > (n+1)^2 \quad \forall n \in \mathbb{Z}$ with $n \geq 6$.

2. (*pts*) Prove that $\sqrt[3]{6-\sqrt{3}}$ is irrational using the Rational Zero's Theorem.

Let $x = \sqrt[3]{6-\sqrt{3}}$, then we have

$x = \sqrt[3]{6-\sqrt{3}} \iff x^3 - 6 = -\sqrt{3} \iff (x^3-6)^2 = 3$

$\iff x^6 - 12x^3 + 36 = 3 \iff x^6 - 12x^3 + 33 = 0 \quad (1)$

So x solves equation (1). By Rational Zero Theorem, since solutions must be of the form $\frac{p}{q}$ with $p | 33$ and $q | 1$, the only rational solutions to (1) are $\pm 1, \pm 33$. Clearly, none of these solve (1)

Since $x = \sqrt[3]{6-\sqrt{3}}$ solves (1), it cannot be rational.

Thus, $\sqrt[3]{6-\sqrt{3}}$ must be irrational.

3. (*pts*) Use the definition (i.e. ϵN-property) to prove that

$$\lim_{n\to\infty} \frac{5n}{n+2} = 5$$

Let $\varepsilon > 0$ be given. Notice

$$|s_n - s| = \left|\frac{5n}{n+2} - 5\right| = \left|\frac{5n - 5n - 10}{(n+2)}\right| = \frac{10}{(n+2)} = \frac{10}{n+2}.$$

Then, we get $|s_n - s| < \varepsilon \Leftrightarrow \frac{10}{n+2} < \varepsilon \Leftrightarrow \frac{10}{\varepsilon} < n+2 \Leftrightarrow n > \frac{10}{\varepsilon} - 2.$

Choose $N = \frac{10}{\varepsilon} - 2$. Thus, $\forall n > N \Rightarrow \left|\frac{5n}{n+2} - 5\right| < \varepsilon.$

Therefore, we conclude $\lim\limits_{n\to\infty} \frac{5n}{n+2} = 5.$

4. (*pts*) For the following sets, determine its maximum, minimum, supremum, and infimum.

(a) $A = \cap_{n=1}^{\infty} \left[-\frac{1}{n}, 1 + \frac{1}{n}\right]$ $= [-1, 2] \cap [-\frac{1}{2}, \frac{3}{2}] \cap [-\frac{1}{3}, 1\frac{1}{3}] \cap [-\frac{1}{4}, 1\frac{1}{4}] \ldots$

$\max A = 1$, $\min A = 0$

$\sup A = 1$, $\inf A = 0$

(b) $B = \left\{\frac{1}{n} : n \in \mathbb{N} \text{ and } n \text{ is prime}\right\}$ $= \{\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots\}$

$\max B = \frac{1}{2}$, $\min B$ DNE

$\sup B = \frac{1}{2}$, $\inf B = 0$

(c) $C = \left\{\cos\left(\frac{n\pi}{3}\right) : n \in \mathbb{N}\right\}$ $= \{\frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, \ldots\}$

$\max C = 1$, $\min C = -1$

$\sup C = 1$, $\inf C = -1$

5. (*pts*) For the next two problems, use the Ordered Field Axioms and Theorems 3.1 and 3.2 (These will be given to you on the exam).

(a) Prove if $ab = 0$, then $a = 0$ or $b = 0$.

Pf $ab=0$. WLOG assume $a \neq 0$.

Then $\underset{(m3)}{b = 1 \cdot b} = \underset{(m4)}{(a^{-1}a)\, b} = \underset{(m1)}{a^{-1}(ab)} = a^{-1}0 = 0$ by Thm 3.1 2).

Thus $b=0$. Therefore, if $ab=0 \Rightarrow a=0$ or $b=0$

(b) Prove if $a \leq b$ and $c \leq d$, then $a + c \leq b + d$.

Pf $a \leq b$ and $c \leq d$. Since $a \leq b$, we have $a+c \leq b+c$ by (O4)

Also, since $c \leq d$, we have $b+c \leq b+d$ with (O4) and (A2).

Combining these with (O3), we have $a+c \leq b+d$.

Thus, if $a \leq b$ and $c \leq d \Rightarrow a+c \leq b+d$.

(c) (*pts*) Let $x \in \mathbb{R}$. Prove that if x^3 is irrational, then x is irrational.

Pf x is not irrational, then $x \in \mathbb{Q}$. So, $\exists\, m, n \in \mathbb{Z}$

such that $x = \frac{m}{n}$ where $n \neq 0$. Thus,

$x^3 = \left(\frac{m}{n}\right)^3 = \frac{m^3}{n^3} \in \mathbb{Q}$ since $m^3, n^3 \in \mathbb{Z}$.

So x^3 is not irrational.

Hence, if $x^3 \in \mathbb{I} \Rightarrow x \in \mathbb{I}$.

6. (*pts*) Consider the sequence $s_n = (-1)^n$. Use the definition (i.e. ϵN-property) to prove that

$$\lim_{n\to\infty} s_n \neq -1$$

Notice $S_n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even.} \end{cases}$

Choose $\varepsilon = \frac{1}{2}$, and let $N \in \mathbb{R}$ be given. By Archimedean Property, $\exists\, n^* \in \mathbb{N}$ such that $n^* > N$. Choose n to be next even number after n^*. Then, we have $|s_n - (-1)| = |1+1| = 2 \geq \frac{1}{2} = \varepsilon$. So $|s_n - (-1)| \geq \varepsilon$.

Thus, we conclude $\lim_{n\to\infty} S_n \neq -1$.

7. (*pts*) For the following sequences, determine whether it converges and, if so, give its limit. Otherwise, state it diverges. You may use any method you wish to determine this.

(a) $a_n = \dfrac{7n^4 + 8n}{2n^3 - 31}$

$\lim_{n\to\infty} \dfrac{7n^4+8n}{2n^3-31} = \infty$ or DNE. $\boxed{\text{So sequence diverges.}}$

(b) $b_n = 51 + \dfrac{\sin n}{n}$

$\lim_{n\to\infty} 51 + \dfrac{\sin n}{n} = \boxed{51}$

(c) $c_n = \left(1 + \dfrac{5}{n}\right)^n = \left(1+\frac{5}{n}\right)^{\frac{n}{5}\cdot 5} = \left[\left(1+\frac{5}{n}\right)^{\frac{n}{5}}\right]^5$

$\Rightarrow \lim_{n\to\infty} C_n = \lim_{n\to\infty} \left[\left(1+\frac{5}{n}\right)^{\frac{n}{5}}\right]^5 = \boxed{e^5}$

8. (*pts*) Prove that for all $x \in \mathbb{I}$ you can construct a sequence of rational numbers $\{x_n\}$ which converges to x. Note: Just show how to build the sequence. You DO NOT have to prove the convergence.

Let $x \in \mathbb{I}$. Construct a sequence of rationals $\{x_n\}$ as follows.

For x_1, consider $x - 1 < x$. By Denseness of $\mathbb{Q}$, $\exists x_1 \in \mathbb{Q}$ such that $x - 1 < x_1 < x$.

For x_2, consider $x - \frac{1}{2} < x$. By Denseness of $\mathbb{Q}$, $\exists x_2 \in \mathbb{Q}$ such that $x - \frac{1}{2} < x_2 < x$.

$\vdots$

For x_n with $n \in \mathbb{N}$, consider $x - \frac{1}{n} < x$. By Denseness of $\mathbb{Q}$, $\exists x_n \in \mathbb{Q}$ such that $x - \frac{1}{n} < x_n < x$.

Thus, we have a sequence $\{x_n\}$ with $x_n \in \mathbb{Q}$ $\forall n \in \mathbb{N}$ that converges to x.

Since $x \in \mathbb{I}$ was arbitrary, we have $\forall x \in \mathbb{I}$, $\exists \{x_n\}$ with $x_n \in \mathbb{Q}$ $\forall n \in \mathbb{N}$ such that $\lim_{n \to \infty} x_n = x$. ∎

Side note: Proving convergence of $\{x_n\}$ is straightforward with the Squeeze Theorem and Limit Theorems. since we have $x - \frac{1}{n} < x_n < x$ $\forall n \in \mathbb{N}$.