1. (*pts*) Use the definition (i.e. MN property) to prove that

$$\lim_{n\to\infty} \frac{n^3+4n+6}{3n^2+5n+7} = \infty.$$

Let $M>0$ be given. Notice

$$\frac{n^3+4n+6}{3n^2+5n+7} > \frac{n^3}{3n^2+5n^2+7n^2} = \frac{n^3}{15n^2} = \frac{n}{15}.$$

So we have

$$s_n > M \iff \frac{n^3+4n+6}{3n^2+5n+7} > \frac{n}{15} > M \iff \frac{n}{15} > M \iff n > 15M.$$

Choose $N = 15M$. Thus, $\forall n > N$, then $\frac{n^3+4n+6}{3n^2+5n+7} > M$.

Therefore, we conclude $\lim_{n\to\infty} \frac{n^3+4n+6}{3n^2+5n+7} = \infty$.

2. (*pts*) Prove $\lim_{n\to\infty} \frac{3n+7}{6n-5} = \frac{1}{2}$ using Theorems 9.2-9.7. These Theorems (or any other necessary ones) will be provided on the exam.

$$\lim_{n\to\infty} \frac{3n+7}{6n-5} = \frac{\lim_{n\to\infty} 3n+7 \cdot \frac{1}{n}}{\lim_{n\to\infty} 6n-5 \cdot \frac{1}{n}} \quad \text{by Theorem 9.6.}$$

$$= \frac{\lim_{n\to\infty} 3 + \lim_{n\to\infty} \frac{7}{n}}{\lim_{n\to\infty} 6 + \lim_{n\to\infty} \frac{-5}{n}} \quad \text{by Theorem 9.3 (twice)}$$

$$= \frac{3 + 7\left(\lim_{n\to\infty} \frac{1}{n}\right)}{6 + (-5)\left(\lim_{n\to\infty} \frac{1}{n}\right)} \quad \text{by Theorem 9.2 (twice)}$$

$$= \frac{3+0}{6+0} \quad \text{by Theorem 9.7 (twice)}$$

$$= \frac{1}{2}$$

3. (*pts*) For the following sequence, determine its set of subsequential limits, its $\limsup_{n\to\infty}$ and $\liminf_{n\to\infty}$, whether it converges or diverges, whether it is monotonically non-increasing or non-decreasing, and whether it is bounded.

(a) $a_n = (-2)^n$ $\quad a_n: -2, 4, -8, 16, -32, \ldots$

$S = \{-\infty, \infty\}$, $\limsup_{n\to\infty} a_n = \infty$, $\liminf_{n\to\infty} a_n = -\infty$,

Sequence diverges, and it is not monotonic or bounded.

(b) $b_n = 5^{(-1)^n}$ $\quad b_n: \frac{1}{5}, 5, \frac{1}{5}, 5, \frac{1}{5}, \ldots$

$S = \{\frac{1}{5}, 5\}$, $\limsup_{n\to\infty} b_n = 5$, $\liminf_{n\to\infty} b_n = \frac{1}{5}$

Sequence diverges and is bounded, but it is not monotonic.

(c) $c_n = -\frac{n}{4} + \left\lceil\frac{n}{4}\right\rceil + (-1)^n$ $\quad c_n: -\frac{1}{4}, \frac{3}{2}, -\frac{3}{4}, 1, -\frac{1}{4}, \frac{3}{2}, -\frac{3}{4}, 1, \ldots$

$S = \{-\frac{3}{4}, -\frac{1}{4}, 1, \frac{3}{2}\}$, $\limsup_{n\to\infty} c_n = \frac{3}{2}$, $\liminf_{n\to\infty} c_n = -\frac{3}{4}$,

Sequence diverges and is bounded, but it is not monotonic.

4. (*pts*) Let $t_1 = 1$ and $t_{n+1} = \left(1 - \frac{1}{4n^2}\right) t_n$ for $n \geq 1$. Prove this sequence $\{t_n\}$ converges.

$\mathcal{P}$ $t_1 = 1$ and $t_{n+1} = (1 - \frac{1}{4n^2}) t_n$ for $n \geq 1$.

Notice $t_{n+1} = (1 - \frac{1}{4n^2}) t_n < t_n$ $\forall n \in \mathbb{N}$, so $\{t_n\}$ is non-increasing (or decreasing).

Also, this shows $\{t_n\}$ is bounded above by 1 since $t_1 = 1$.

Now we show $\{t_n\}$ is bounded below by 0 thru induction:

(i) It's true for $n=1$ since $t_1 = 1 > 0$.

(ii) Let $n \in \mathbb{N}$. Assume $0 < t_n$ is true. Then,

$4n^2 > 1 \Rightarrow \frac{1}{4n^2} < 1 \Rightarrow 1 - \frac{1}{4n^2} > 0 \Rightarrow t_{n+1} = (1 - \frac{1}{4n^2}) t_n > 0.$

So statement is true for $n+1$.

Thus, by PMI, we have $0 < t_n$ $\forall n \in \mathbb{N}$, and $\{t_n\}$ is bounded.

Therefore, $\{t_n\}$ is a bounded monotone sequence, and we conclude it converges.

5. (*pts*) Prove if $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$, then $\lim_{n\to\infty} s_n + t_n = s + t$ using the definition (i.e. ϵN property).

$\mathcal{P}$ $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$. Let $\varepsilon > 0$ be given.

Notice $|s_n + t_n - (s+t)| = |s_n - s + t_n - t| \leq |s_n - s| + |t_n - t|$ (1) by $\Delta \leq$.

Since $\lim_{n\to\infty} s_n = s$, $\exists N_1 \in \mathbb{R}$ such that

$\forall n > N_1 \Rightarrow |s_n - s| < \varepsilon/2$ (2)

Since $\lim_{n\to\infty} t_n = t$, $\exists N_2 \in \mathbb{R}$ such that

$\forall n > N_2 \Rightarrow |t_n - t| < \varepsilon/2$ (3)

Choose $N := \max\{N_1, N_2\}$. Thus, by (1), (2), (3), we have

$\forall n > N \Rightarrow |s_n + t_n - (s+t)| \leq |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

Therefore, we conclude $\lim_{n\to\infty} s_n + t_n = s + t$.

6. (*pts*) Consider the sequence $s_n = (-1)^n + \cos\left(\frac{n\pi}{2}\right)$. Find the set of subsequential limits S by providing a subsequence that converges to each subsequential limit point.

Pf $s_n = (-1)^n + \cos\left(\frac{n\pi}{2}\right)$. $s_n: -1, 0, -1, 2, -1, 0, -1, 2, \ldots$

So $s_n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 0 & \text{if } 4 \mid (n-2) \\ 2 & \text{if } 4 \mid n \end{cases}$

- Consider subsequence $\{s_{2k+1}\}$. Then $s_{2k+1} = -1$ $\forall k \in \mathbb{N}$, so $\lim_{k\to\infty} s_{2k+1} = -1$, and -1 is subsequential limit.
- Consider subsequence $\{s_{4k+2}\}$. Then $s_{4k+2} = 0$ $\forall k \in \mathbb{N}$, so $\lim_{k\to\infty} s_{4k+2} = 0$, and 0 is subsequential limit.
- Consider subsequence $\{s_{4k}\}$. Then $s_{4k} = 2$ $\forall k \in \mathbb{N}$, so $\lim_{k\to\infty} s_{4k} = 2$, and 2 is subsequential limit.

Thus, the set of subsequential limits is $S = \{-1, 0, 2\}$

7. (*pts*) Let $\{s_n\}$ and $\{t_n\}$ be bounded sequences with $s_n \le t_n$ $\forall n \in \mathbb{N}$. Prove that $\limsup_{n\to\infty} s_n \le \limsup_{n\to\infty} t_n$. (Note: Since the sequences are bounded, their lim sup's must be finite).

Pf $\{s_n\}$ and $\{t_n\}$ are bounded sequences with $s_n \le t_n$ $\forall n \in \mathbb{N}$.
For any $N \in \mathbb{N}$, let $u_N := \sup\{s_n : n > N\}$ and $w_N := \sup\{t_n : n > N\}$.
Then $\{u_N\}$ and $\{w_N\}$ are monotone bounded sequences with $u_N \le w_N$ $\forall N \in \mathbb{N}$, and therefore they must converge.
Let $u = \lim_{N\to\infty} u_N$ and $w = \lim_{N\to\infty} w_N$. Since $u_N \le w_N$ $\forall N \in \mathbb{N}$, then $u_N - w_N \le 0$ $\forall N \in \mathbb{N}$ $\Rightarrow \lim_{N\to\infty} u_N - w_N = u - w \le 0$
$\Rightarrow u \le w$. By defn, $u = \limsup_{n\to\infty} s_n$ and $w = \limsup_{n\to\infty} t_n$

Therefore, we conclude $\limsup_{n\to\infty} s_n \le \limsup_{n\to\infty} t_n$.

8. (*pts*) Determine whether the following statements are TRUE or FALSE. If true, BRIEFLY justify the given statement; if false, give a counterexample to the statement.

(a) For $a, b \in \mathbb{R}$, there exists a sequence where the set of subsequential limits is $[a, b]$.

True. By E.C. problem, we know there exists $\{a_n\}$ with set of subsequential limits as $[0,1]$. Construct new sequence $\{s_n\}$ where $s_n = (b-a)a_n + a \ \forall n \in \mathbb{N}$. This sequence's set of subsequential limits is $[a,b]$.

(b) If $\{s_n\}$ is a Cauchy sequence, then $\{s_n\}$ is a monotone sequence.

False. Consider sequence $s_n = \frac{(-1)^n}{n}$. It is Cauchy since it converges, but it is clearly not monotone.

(c) For a given sequence $\{t_n\}$, if the corresponding sequence $\{\sin(t_n)\}$ converges, then $\{t_n\}$ converges.

False. Consider sequence $\{t_n\}$ with $t_n = 2\pi n$.
Then, sequence $\sin(t_n) = \sin(2\pi n) = 0 \ \forall n \in \mathbb{N}$.
Clearly, $\{\sin(t_n)\}$ converges to 0, but $\{t_n\}$ diverges to $+\infty$.

9. (*pts*)

(a) Let $\{s_n\}$ be sequence such that $|s_{n+1} - s_n| < \frac{1}{7^n}$ $\forall n \in \mathbb{N}$. Prove that

$$|s_m - s_n| < \frac{1}{7^{n-1}} \quad \forall n, m \in \mathbb{N} \text{ with } m > n. \tag{1}$$

§ $\{s_n\}$ is sequence w/ $|s_{n+1} - s_n| < \frac{1}{7^n}$ $\forall n \in \mathbb{N}$. By addition by zero, we get for $m, n \in \mathbb{N}$ where $m > n$

$$|s_m - s_n| = |s_m - s_{m-1} + s_{m-1} - s_{m-2} + \dots + s_{n+1} - s_n|$$

$$\leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \dots + |s_{n+1} - s_n| \quad \text{by } \Delta \leq.$$

Using supposition $\leq$, we have

$$|s_m - s_n| < \frac{1}{7^{m-1}} + \frac{1}{7^{m-2}} + \dots + \frac{1}{7^{m+1}} + \frac{1}{7^m}$$

$$< \frac{1}{7^n}\left(1 + \frac{1}{7} + \left(\frac{1}{7}\right)^2 + \dots + \left(\frac{1}{7}\right)^{m-n-2} + \left(\frac{1}{7}\right)^{m-n-1}\right)$$

$$= \frac{1}{7^n}\left(\frac{1 - \left(\frac{1}{7}\right)^{m-n}}{1 - \frac{1}{7}}\right) = \frac{1}{6}\left(\frac{1}{7^{n-1}} - \frac{1}{7^{m-1}}\right) \quad \text{by Proposition 1 in HW 11.}$$

Finally, because $m > n$, we get

$$|s_m - s_n| < \frac{1}{6}\left(\frac{1}{7^{n-1}} - \frac{1}{7^{m-1}}\right) < \frac{1}{6} \cdot \frac{1}{7^{n-1}} < \frac{1}{7^{n-1}} \text{ since } \frac{1}{7^{n-1}} > 0 \ \forall n \in \mathbb{N}.$$

Since $m, n \in \mathbb{N}$ with $m > n$ was arbitrarty, we conclude

$$|s_m - s_n| < \frac{1}{7^{n-1}} \quad \forall m, n \in \mathbb{N} \text{ with } m > n$$

(b) Use inequality (1) found in part (a) to prove the sequence $\{s_n\}$ converges.

Let $\varepsilon > 0$ be given. Since $\lim_{n \to \infty} \frac{1}{7^{n-1}} = 0$, $\exists N \in \mathbb{R}$ such that $\forall n > N \Rightarrow \left|\frac{1}{7^{n-1}} - 0\right| < \varepsilon$.

Choose this N. WLOG assume $m > n$.

Thus, $\forall m, n > N \Rightarrow |s_m - s_n| < \frac{1}{7^{n-1}} < \varepsilon$ by equation (1) since $m > n$.

Therefore, we conclude the sequence is Cauchy and must converge.