1. (*pts*) Using induction, prove $3^n > n^3$ $\forall n \in \mathbb{N}$ with $n > 3$.

2. (*pts*) Consider the sequence $s_n = \cos \frac{n\pi}{2}$. Use the definition (i.e. ϵN-property) to prove that

$$\lim_{n \to \infty} \quad s_n \neq 1$$

3. (*pts*) Prove $\lim_{n\to\infty} \dfrac{2n+5}{7n^2+3} = 0$ using Theorems 9.2-9.7. These Theorems (or any other necessary ones) will be provided on the exam.

4. (*pts*) Let $t_1 = 1$ and $t_{n+1} = t_n \left(1 - \dfrac{1}{(n+1)^2}\right)$ for $n \geq 1$. Prove this sequence $\{t_n\}$ converges.

5. (*pts*) Prove if $\{s_n\}$ is a convergent sequence of real numbers, then the limit is unique.

6. (*pts*) Consider $\mathbb{R}^2$ with the following metric

$$d(\vec{x}, \vec{y}) = \max_{j=1,2}\{|x_j - y_j|\}$$

where $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$. On the plane, sketch the neighborhoods $B_1((0,0))$ and $B_2((3,3))$

7. (*pts*) Determine whether the following statements are TRUE or FALSE. If true, BRIEFLY justify the given statement; if false, give a counterexample to the statement.

   (a) If $\{a_n\}$ is an unbounded sequence and $a_n \neq 0 \ \forall n \in \mathbb{N}$, then $\{\frac{1}{a_n}\}$ is a bounded sequence.

   (b) Every (infinite) sequence with only a finite number of distinct elements has a convergent subsequence.

   (c) If E is compact in $\mathbb{R}^n$, then E^C is an open set.

8. (*pts*) For the following sets, determine whether it is open, closed, or neither. Also, find its interior, closure, and boundary. Plus, determine whether it is compact or not. You DO NOT need justification for your answers.

   (a) $A = [3, 7)$

   (b) $B = \{\sin \frac{n\pi}{4} : n \in \mathbb{N}\}$

   (c) $C = \cap_{n=1}^{\infty} \left[-\frac{1}{n}, 1 + \frac{1}{n}\right]$

9. (*pts*) Determine whether each of the following series converges absolutely, converges conditionally, or diverges. Write clear and complete solutions including the name of the series test that you use.

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^n}{n^{2n}}$

(b) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n-2}{n}\right)^{n}$

(c) $1+\frac{1}{4}-\frac{1}{9}+\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\dots$

10. (*pts*) Prove if $\sum |a_n|$ converges and $\{b_n\}$ is a bounded sequence, then $\sum a_n b_n$ converges.

11. (*pts*) Prove $S = \{1 + \frac{1}{n} : n \in \mathbb{N}\}$ with metric $d(x, y) = |x - y|$ is NOT complete.

12. (*pts*) Prove $d(x, y) = |x - 3y|$ is not a metric on $\mathbb{R}$.

13. (*pts*) Let E be a subset of $\mathbb{R}^n$. Prove that E is compact if and only if every sequence in E has a subsequence that converges to a point in E.

The following extra credit problems are OPTIONAL and you are advised to finish the rest of the test before trying these problems.

1. (*pts*) Prove that if every subsequence $\{a_{n_k}\}$ of a sequence $\{a_n\}$ contains a subsequence $\{a_{n_{k_l}}\}$ converging to a, then the sequence $\{a_n\}$ also converges to a.

2. (*pts*) Let (S, d) be a metric space. We say $E \subseteq S$ is a *countable subset* if there exists a sequence $\{s_n\}$ in S where the set of elements in $\{s_n\}$ is E (i.e. $E = \{s_n : n \in \mathbb{N}\}$). The metric space is called *separable* if there exists a countable subset $E \subseteq S$ such that $E^- = S$. Prove $(\mathbb{R}, |\cdot|)$ is separable.