1. (*pts*) Using induction, prove $3^n > n^3$ $\forall n \in \mathbb{N}$ with $n > 3$.

(i) This is true for $n=4$, since $81 = 3^4 > 4^3 = 64$.
(ii) Let $n \in \mathbb{N}$ with $n > 3$. Assume $3^n > n^3$ is true.
Then, $3^{n+1} = 3 \cdot 3^n > 3n^3 = n^3 + n^3 + n^3 > n^3 + 3n^2 + 9n$,
since $n > 3$. Continuing we have
$$3^{n+1} > n^3 + 3n^2 + 9n > n^3 + 3n^2 + 3n + 1 = (n+1)^3,$$
where we used $6n > 1$ in last inequality. Thus, statement is true for $n+1$.

Therefore, by PMI, we conclude
$$3^n > n^3 \quad \forall n \in \mathbb{N} \text{ with } n > 3$$

2. (*pts*) Consider the sequence $s_n = \cos \frac{n\pi}{2}$. Use the definition (i.e. ϵN-property) to prove that
$$\lim_{n\to\infty} s_n \neq 1$$

Notice
$$s_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } 4 \mid n \\ -1 & \text{if } 4 \mid (n-2) \end{cases}$$

Choose $\varepsilon = \frac{1}{2}$, and let $N \in \mathbb{R}$ be given. By Archimedian Property, $\exists\, n^* \in \mathbb{N}$ such that $n^* > N$. Choose n to be next odd number after n^*. Then, we have

$|s_n - 1| = |0 - 1| = 1 \geq \frac{1}{2} = \varepsilon$. So $|s_n - 1| \geq \varepsilon$.

Thus, we conclude $\lim_{n\to\infty} s_n \neq 1$.

3. (*pts*) Prove $\lim_{n\to\infty} \frac{2n+5}{7n^2+3} = 0$ using Theorems 9.2-9.7. These Theorems (or any other necessary ones) will be provided on the exam.

$$\lim_{n\to\infty} \frac{2n+5}{7n^2+3} = \frac{\lim_{n\to\infty} 2n+5}{\lim_{n\to\infty} 7n^2+3} \quad \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \quad \text{by Theorem 9.6}$$

$$= \frac{\lim_{n\to\infty} \frac{2}{n} + \lim_{n\to\infty} \frac{5}{n^2}}{\lim_{n\to\infty} 7 + \lim_{n\to\infty} \frac{3}{n^2}} \quad \text{by Theorem 9.3 (twice)}$$

$$= \frac{2\lim_{n\to\infty} \frac{1}{n} + 5\lim_{n\to\infty} \frac{1}{n^2}}{7 + 3\lim_{n\to\infty} \frac{1}{n^2}} \quad \text{by Theorem 9.2 (three times)}$$

$$= \frac{0+0}{7+0} \quad \text{by Theorem 9.7 (three times)}$$

$$= 0$$

4. (*pts*) Let $t_1 = 1$ and $t_{n+1} = t_n\left(1 - \frac{1}{(n+1)^2}\right)$ for $n \geq 1$. Prove this sequence $\{t_n\}$ converges.

§ $t_1 = 1$ and $t_{n+1} = t_n\left(1 - \frac{1}{(n+1)^2}\right)$ for $n \geq 1$.

For $n \in \mathbb{N}$, $1 < (n+1)^2 \Rightarrow 0 < \frac{1}{(n+1)^2} < 1 \Rightarrow -1 < -\frac{1}{(1+n)^2} < 0$

$\Rightarrow 0 < 1 - \frac{1}{(n+1)^2} < 1$ (1). Then, we have $t_{n+1} = t_n\left(1 - \frac{1}{(1+n)^2}\right) < t_n \ \forall n \in \mathbb{N}$, so $\{t_n\}$ is non-increasing (or decreasing).

Also, this shows $\{t_n\}$ is bounded above by 1 since $t_1 = 1$.

Moreover, if $t_n > 0$ for $n \in \mathbb{N}$, (1) tells us

$$0 < \left(1 - \frac{1}{(n+1)^2}\right)t_n < t_n \Rightarrow 0 < t_{n+1} < t_n,$$

and $t_{n+1} > 0$. Since $n \in \mathbb{N}$ was arbitrary, $t_n > 0 \ \forall n \in \mathbb{N}$, and we have $\{t_n\}$ bounded below by 0. Hence, $\{t_n\}$ is bounded.

Therefore, $\{t_n\}$ is a bounded monotone sequence, and we conclude it converges.

5. (*pts*) Prove if $\{s_n\}$ is a convergent sequence of real numbers, then the limit is unique.

$\oint$ $\{s_n\}$ converges, but the limit is not unique. So $\exists x, y \in \mathbb{R}$ where $\lim_{n\to\infty} s_n = x$ and $\lim_{n\to\infty} s_n = y$ and $x \neq y$. Notice

$$0 = \lim_{n\to\infty} 0 = \lim_{n\to\infty} s_n - s_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_n = x - y$$

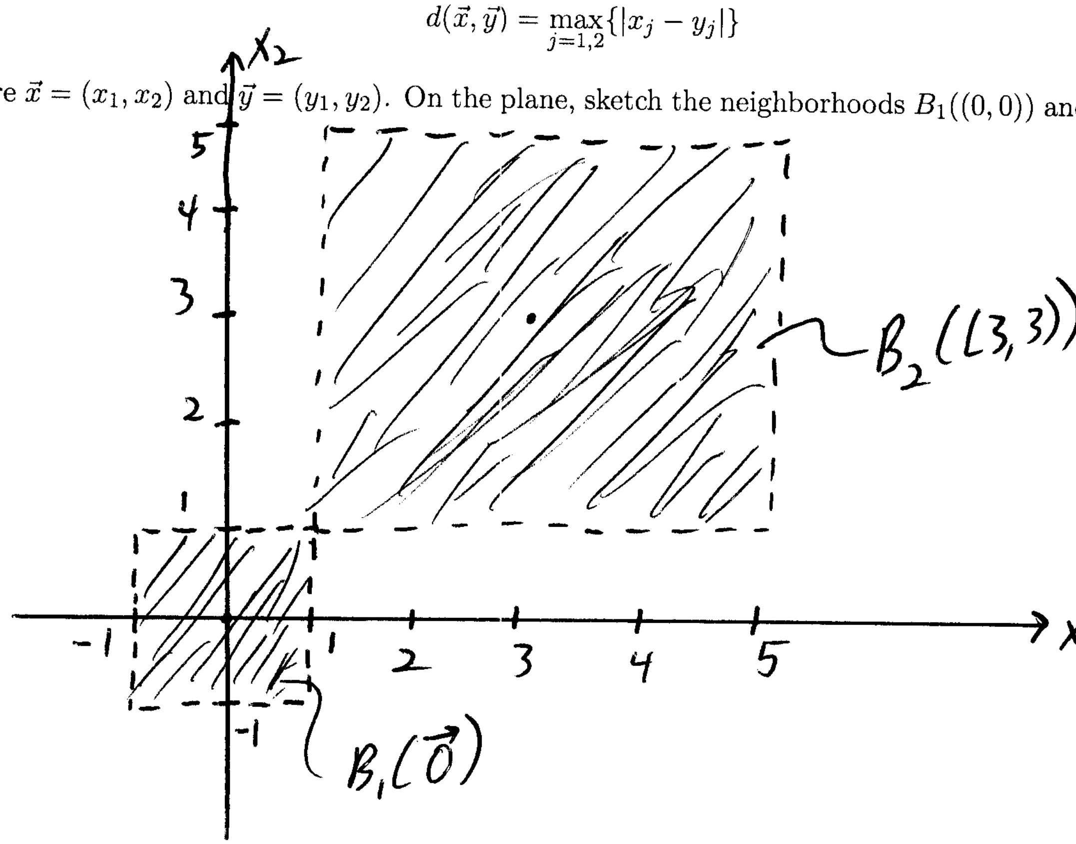
$\Rightarrow x - y = 0 \Rightarrow x = y$. Contradiction!

Thus, we conclude if $\{s_n\}$ converges, then the limit is unique.

6. (*pts*) Consider $\mathbb{R}^2$ with the following metric

$$d(\vec{x}, \vec{y}) = \max_{j=1,2}\{|x_j - y_j|\}$$

where $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$. On the plane, sketch the neighborhoods $B_1((0,0))$ and $B_2((3,3))$



7. (*pts*) Determine whether the following statements are TRUE or FALSE. If true, BRIEFLY justify the given statement; if false, give a counterexample to the statement.

(a) If $\{a_n\}$ is an unbounded sequence and $a_n \neq 0\ \forall n \in \mathbb{N}$, then $\{\frac{1}{a_n}\}$ is a bounded sequence.

False. Consider the sequence $a_n = \begin{cases} n & \text{if } n \text{ is odd} \\ \frac{1}{n} & \text{if } n \text{ is even.} \end{cases}$

Clearly, $\{a_n\}$ is unbounded and $a_n \neq 0\ \forall n \in \mathbb{N}$, but $\{\frac{1}{a_n}\}$ is unbounded.

(b) Every (infinite) sequence with only a finite number of distinct elements has a convergent subsequence.

True. If $\{t_n\}$ is a sequence with a finite number of terms, say $x_1, x_2, \ldots, x_n$ for some $n \in \mathbb{N}$, then at least one of them repeats an infinite number of times, say x_i with $i = 1, \ldots, n$. Construct subsequence $\{t_{n_k}\}$ which only accepts this term x_i. Clearly, $\lim_{k\to\infty} t_{n_k} = \lim_{k\to\infty} x_i = x_i$, and this subsequence converges.

(c) If E is compact in $\mathbb{R}^n$, then E^C is an open set.

True. Since E is compact in $\mathbb{R}^n$, it is closed and bounded by the Heine-Borel Theorem. If E is closed, then E^C is open by definition of a closed set.

8. (*pts*) For the following sets, determine whether it is open, closed, or neither. Also, find its interior, closure, and boundary. Plus, determine whether it is compact or not. You DO NOT need justification for your answers.

   (a) $A = [3, 7)$

$A^o = (3,7)$, $A^- = [3,7]$, $\partial A = \{3, 7\}$

A is neither open or closed and is not compact.

   (b) $B = \{\sin \frac{n\pi}{4} : n \in \mathbb{N}\}$ $= \{-1, -\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, 1\}$

$B^o = \emptyset$, $B^- = B$, $\partial B = B$

B is closed and compact

   (c) $C = \cap_{n=1}^{\infty} \left[-\frac{1}{n}, 1 + \frac{1}{n}\right]$ $= [0, 1]$

$C^o = (0,1)$, $C^- = [0,1]$, $\partial C = \{0, 1\}$

C is closed and compact.

9. (*pts*) Determine whether each of the following series converges absolutely, converges conditionally, or diverges. Write clear and complete solutions including the name of the series test that you use.

   (a) $\sum_{n=2}^{\infty} \frac{(\ln n)^n}{n^{2n}}$ $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{(\ln n)^n}{n^{2n}}} = \lim_{n\to\infty} \frac{\ln n}{n^2} = 0 < 1$

Hence, the series converges absolutely by the Root Test.

(b) $\sum_{n=1}^{\infty}\left(\frac{n-2}{n}\right)^n$ Notice $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n-2}{n}\right)^2 = \lim_{n\to\infty}\left(1+\frac{-2}{n}\right)^n = e^{-2} \neq 0.$

Thus, series diverges by nth Term Test.

(c) $1+\frac{1}{4}-\frac{1}{9}+\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\dots$ Let $\sum_1^{\infty} a_n = 1+\frac{1}{4}-\frac{1}{9}+\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\dots$

Then $\sum_1^{\infty}|a_n| = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots = \sum_1^{\infty}\frac{1}{n^2}$ which converges by the p-series Test ($p=2>1$).

Therefore, the original series $\sum a_n$ converges absolutely.

10. (*pts*) Prove if $\sum |a_n|$ converges and $\{b_n\}$ is a bounded sequence, then $\sum a_n b_n$ converges.

§ $\sum |a_n|$ converges and $\{b_n\}$ is bounded. Let $\varepsilon>0$ be given. Since $\{b_n\}$ is bounded, $\exists\, C>0$ such that $|b_n|\le C$ $\forall n\in\mathbb{N}$. Since $\sum|a_n|$ converges, it satisfies the Cauchy Criterion and $\exists N$ such that $\forall m>n>N \Rightarrow \left|\sum_{k=n}^{m}|a_k|\right| = \sum_{k=n}^{m}|a_k| < \frac{\varepsilon}{C}$ (1)

Choose this N. Thus, using (1) and the triangle inequality, we have

$$\forall m>n>N \Rightarrow \left|\sum_{k=n}^{m} a_k b_k\right| \le \sum_{k=n}^{m}|a_k||b_k| \le C\sum_{k=n}^{m}|a_k| < C\frac{\varepsilon}{C} = \varepsilon.$$

Hence, the series $\sum a_n b_n$ satisfies the Cauchy Criterion and converges.

Therefore, we conclude if $\sum|a_n|$ converges and $\{b_n\}$ is a bounded sequence, then $\sum a_n b_n$ converges. ∎

11. (*pts*) Prove $S = \{1 + \frac{1}{n} : n \in \mathbb{N}\}$ with metric $d(x, y) = |x - y|$ is NOT complete.

Consider sequence $\{s_n\}$ in S defined as $s_n = 1 + \frac{1}{n}$ $\forall n \in \mathbb{N}$.
We have $\lim_{n \to \infty} s_n = \lim_{n \to \infty} 1 + \frac{1}{n} = 1$, but $1 \notin S$. Since $\{s_n\}$ converges in $\mathbb{R}$, it is Cauchy in $\mathbb{R}$. Then, it is Cauchy in (S, d). So we have a Cauchy sequence in S that does not converge to a point in S.
Therefore, we conclude (S, d) is not complete.

12. (*pts*) Prove $d(x, y) = |x - 3y|$ is not a metric on $\mathbb{R}$.

Notice $d(x,x) = |x - 3x| = |2x| \neq 0$ when $x \neq 0$.
Thus, d is not a metric.

13. (*pts*) Let E be a subset of $\mathbb{R}^n$. Prove that E is compact if and only if every sequence in E has a subsequence that converges to a point in E.

Let $E \subseteq \mathbb{R}^n$.

($\Rightarrow$) $\oint$ E is compact. By Hiene-Boral Theorem, E is closed and bounded. Consider sequence $\{\vec{x}_k\}$ in E, which must be bounded. By the Bolzano-Weierstrass Thm, $\{\vec{x}_k\}$ has a convergent subsequence to a point $\vec{x} \in \mathbb{R}^n$. Since E is closed, $\vec{x} \in E$, proving the claim.

($\Leftarrow$) $\oint$ every sequence in E has a subsequence that converges in E, but E is not compact. By Hiene-Boral Thm, either E is unbounded or not closed.

If E was unbounded, $\exists$ sequence in E $\{\vec{x}_k\}$ where $\lim_{k \to \infty} d(\vec{x}_k, \vec{0}) = \infty$ and no subsequence would converge at all. Contradiction! Hence, E is bounded.

If E is not closed, then $\exists$ sequence $\{\vec{x}_k\}$ that converges $\vec{x}$, but $\vec{x} \notin E$ (Prop. 13.9b). Because $\{\vec{x}_k\}$ converges to $\vec{x}$, every subsequence of $\{\vec{x}_k\}$ converges to $\vec{x} \notin E$ (Thm 11.2). But we assumed every sequence in E has a subsequence that converges in E. Contradiction! Thus, E is closed.

Therefore, E is closed and bounded which contradicts E is not compact. We conclude E is compact.