Math 25

Vogler

Lemma used to Prove Alternating Series Test

Lemma

Consider a series of the form $\sum (-1)^k a_k$. If the sequence $\{a_k\}$ is positive and decreasing (or nonincreasing), then
$$\left|\sum_{k=m}^{n} (-1)^k a_k\right| \le a_m \quad \forall n, m \in \mathbb{N} \text{ with } n \ge m.$$

Pf

$\S$ we have a series $\sum (-1)^k a_k$ where the sequence $\{a_k\}$ is positive and decreasing. Fix $n, m \in \mathbb{N}$ with $n \ge m$. Define
$$A = a_m - a_{m+1} + a_{m+2} - a_{m+3} + \dots \pm a_n$$
Then, we have
$$\sum_{k=m}^{n} (-1)^k a_k = (-1)^m A. \quad (*)$$
We consider 2 cases: 1) $n-m$ is odd or 2) $n-m$ is even.

For case 1), if $n-m$ is odd, we have
$$A = [a_m - a_{m+1}] + [a_{m+2} - a_{m+3}] + \dots + [a_{n-1} - a_n] \ge 0$$
and
$$A = a_m - [a_{m+1} - a_{m+2}] - [a_{m+3} - a_{m+4}] - \dots - [a_{n-2} - a_{n-1}] - a_n \le a_m,$$
since each bracket $[\sim]$ is positive since $\{a_k\}$ is positive and decreasing.

For case 2), if $n-m$ is even, we have same inequalities. (Optional HW).

Either way, we have $0 \le A \le a_m$. By (1), we get
$$\left|\sum_{k=m}^{n} (-1)^k a_k\right| = |(-1)^m A| = A \le a_m.$$
Since $n, m \in \mathbb{N}$ with $n \ge m$ was arbitrary, we conclude
$$\left|\sum_{k=m}^{n} (-1)^k a_k\right| \le a_m \quad \forall n, m \in \mathbb{N} \text{ with } n \ge m \;\blacksquare$$