Some Basics for Proofs

Sets

Defn 1) A set $S$ is a collection of objects.
2) $x \in S$ means $x$ is an element of $S$.
3) $x \notin S$ means $x$ is not an element of $S$.
4) $\emptyset$ represents the empty set (i.e. one with no elements)

Note: For this class, we'll be working with sets of numbers.

Ex Here are a collection of sets we'll be dealing with.

a) $\mathbb{N} := \{1, 2, 3, \ldots \}$ are the natural numbers.
b) $\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots \}$ are the integers.
c) $\mathbb{Q} := \{\frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0\}$ are the rational numbers.
d) $\mathbb{R}$ is the set of all real numbers (including decimals)
e) $\mathbb{I}$ is the set of all irrational numbers.

Notes: 1) $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$  
   2) $\mathbb{I} = \mathbb{R} - \mathbb{Q}$  
   3) More info in appendix.

Quantifiers Let $P(x)$ be a proposition involving $x$.

Ex $P(x) = (|x| < 7)$

Defn 1) $(\forall x) P(x)$ means 'for all $x$, $P(x)$'
2) $(\exists x) P(x)$ means 'there exists an $x$ such that $P(x)$'
3) $(\exists! x) P(x)$ means 'there exists a unique $x$ such that $P(x)$'
Negation of Quantifiers \( \neg \) is the negation of a proposition.

Ex If \( P(x) = (3x = 9) \), then \( \neg P(x) = (3x \neq 9) \)

Lemma 1) \( \neg (\forall x) P(x) \) is equivalent to \( (\exists x) (\neg P(x)) \)
2) \( \neg (\exists x) P(x) \) is equivalent to \( (\forall x) (\neg P(x)) \)

Notes a) By (1), we can disprove \( (\forall x) P(x) \) by finding an \( x \) (at least one) for which \( P(x) \) is false, such an \( x \) is called a counterexample.
b) We cannot prove \( (\forall x) P(x) \) by giving an example(s).

Subsets
Defn Let \( A \) and \( B \) be sets. We say \( A \) is a subset of \( B \), written \( A \subseteq B \), if and only if (iff) every element of \( A \) is an element of \( B \). Using symbols, we have \( (\forall x)(x \in A \Rightarrow x \in B) \) or \( \forall x \in A \Rightarrow x \in B \)

Ex 1) \( N \subseteq Z \subseteq Q \subseteq R \)

Defn Sets \( A \) and \( B \) are equal, written \( A = B \), if \( A \subseteq B \) and \( B \subseteq A \)

Ex 1) \( \{ n: n \text{ is even}\} = \{ m+1: m \text{ is odd}\} \)

Conditionals
1) The conditional \( P \Rightarrow Q \) means 'if \( P \) then \( Q \)'
Note: A conditional is true only when \( P \) forces \( Q \) to be true.
2) The biconditional \( P \iff Q \) means \( P \Rightarrow Q \) and \( Q \Rightarrow P \).