

Fields and Ordered FieldsDefn

Any set of numbers is a field if it satisfies the following:

- A1) $a + (b + c) = (a + b) + c \quad \forall a, b, c \in S.$
- A2) $a + b = b + a \quad \forall a, b \in S.$
- A3) $a + 0 = a \quad \forall a \in S.$
- A4) $\forall a \in S, \exists -a \in S$ such that $a + (-a) = 0.$
- M1) $a(bc) = (ab)c \quad \forall a, b, c \in S.$
- M2) $ab = ba \quad \forall a, b \in S.$
- M3) $a \cdot 1 = a \quad \forall a \in S.$
- M4) $\forall a \in S$ with $a \neq 0, \exists a^{-1} \in S$ such that $aa^{-1} = 1.$
- DL) $a(b+c) = ab+bc \quad \forall a, b, c \in S.$

Notes: 1) (A1) and (M1) are called the associative laws of multiplication and addition, respectively.

2) (A2) and (M2) are the commutative laws.

3) (A3) and (M3) state the existence of an identity element, 0 for addition and 1 for multiplication.

4) (A4) and (M4) state the existence of inverses.

5) (DL) is called the distributive law.

6) Any set of numbers and a binary operation (i.e. $+$, $\cdot$) with the associative law, an identity element, and inverses is called a group, which is studied in Math 150.

Defn

Any field S is called an ordered field if it satisfies the following for $a, b, c \in S$:

- 01) Either $a \leq b$ or $b \leq a$.
- 02) If $a \leq b$ and $b \leq a$, then $a = b$.
- 03) If $a \leq b$ and $b \leq c$, then $a \leq c$.
- 04) If $a \leq b$, then $a + c \leq b + c$.
- 05) If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

Notes: 1) (03) is called the transitive law.

2) $\mathbb{Q}$ and $\mathbb{R}$ are the classic ordered fields, but there are many more which are studied in Math 150.

Thm (3.1)

For any field S , it has the following properties for $a, b, c \in S$:

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|--|---|
| 1) $a + c = b + c \Rightarrow a = b$. | 4) $(-a)(-b) = ab$. |
| 2) $a \cdot 0 = 0$. | 5) $ac = bc$ and $c \neq 0 \Rightarrow a = b$. |
| 3) $(-a)b = -ab$. | 6) $ab = 0 \Rightarrow a = 0$ or $b = 0$. |

Thm (3.2)

For any ordered field S , it has the following properties for $a, b, c \in S$:

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|--|---|
| 1) If $a \leq b \Rightarrow -b \leq -a$. | 5) $0 < 1$. |
| 2) If $a \leq b$ and $c \leq 0 \Rightarrow bc \geq ac$. | 6) If $0 < a \Rightarrow 0 < a^{-1}$. |
| 3) If $0 \leq a$ and $0 \leq b \Rightarrow 0 \leq ab$. | 7) If $0 < a < b \Rightarrow 0 < b^{-1} < a^{-1}$. |
| 4) $0 \leq a^2$. | |