

Mathematical Induction and $\mathbb{N}$

The natural numbers $\mathbb{N}$ have the following properties:

- 1) $1 \in \mathbb{N}$
- 2) If $n \in \mathbb{N}$, then its successor $n+1 \in \mathbb{N}$
- 3) 1 is not a successor of any $n \in \mathbb{N}$
- 4) If $m, n \in \mathbb{N}$ have the same successor, then $m = n$.
- 5) Principle of Mathematical Induction (PMI)

Let $S \subseteq \mathbb{N}$ such that

- a) $1 \in S$.
 - b) If $n \in S$, then $n+1 \in S$.
- Then $S = \mathbb{N}$.

Note: 1) These properties are sometimes referred to as Peano Axioms or Peano Postulates.

2) Property 5) is the basis for Proofs by Induction, which was shown on the last handout.

Thm Well-Ordering Principle (WOP)

Every nonempty subset of $\mathbb{N}$ has a least element

Thm Principle of Complete Induction (PCI)

Let $S \subseteq \mathbb{N}$ such that

- a) $1 \in S$.
- b) If $1, 2, 3, \dots, n \in S$, then $n+1 \in S$.

Then $S = \mathbb{N}$.

Note: Analogous to PMI leads to Proofs by Induction, PCI leads to Proofs by Strong Induction, which is covered in Math 108.

Thm

PMI, PCI, and WOP are all equivalent statements.

- Notes
- 1) This means that any of these three statements can be used as the fifth property in describing IN.
 - 2) Either the WOP or a Proof by Strong Induction can be used instead of a Proof by Induction because of this theorem. Proof by Induction tends to be the easiest to use in most scenarios, so people stress this more than the other two.
 - 3) This theorem along with the WOP and PCI will be discussed further in Math 108.
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Defn A number x is called an algebraic number if it satisfies an algebraic equation

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

with $a_0, a_1, \dots, a_n \in \mathbb{Z}$, $a_n \neq 0$, and $n \in \mathbb{N}$.

- Notes:
- 1) Any number described with the six basic algebraic operations $(+, -, \cdot, \div, ()^n, \sqrt{\quad})$ is an algebraic number, which includes all rationals.
 - 2) The transcendental number π is not an algebraic number

Rational Zeros Thm (2.2)

Suppose $a_0, a_1, \dots, a_n \in \mathbb{Z}$ and $r \in \mathbb{Q}$ satisfying

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0 \quad (1)$$

where $n \geq 1$, $a_n \neq 0$, and $a_0 \neq 0$. Write $r = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ having no common factors and $q \neq 0$. Then q divides a_n and p divides a_0 .

- Notes:
- 1) This theorem tells us the only rational candidates for solutions of (1) must have the form $\frac{p}{q}$ with $p|a_0$ and $q|a_n$.
 - 2) Recall, a divides b , written $a|b$, if $\exists k \in \mathbb{Z}$ such that $b = ka$.