Math 25
Vogler

Properties of Lim Sup and Lim Inf

Lemma

For a given sequence $\{s_n\}$, define $u_N := \sup\{s_n : n > N\}$
$l_N := \inf\{s_n : n > N\}$ for any $N \in \mathbb{R}$. Then, we have

1) $\liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n$
2) $\limsup_{n\to\infty} s_n \leq u_{N+1} \leq u_N \quad \forall N \in \mathbb{N}$
3) $l_N \leq l_{N+1} \leq \liminf_{n\to\infty} s_n \quad \forall N \in \mathbb{N}$

Note: We can consider two new sequences, the sups of tails $\{u_N\}$ and infs of tails $\{l_N\}$. $\{u_N\}$ is a nondecreasing sequence, and $\{l_N\}$ is a nondecreasing sequence.

Thm (12.1)

Let $\{t_n\}$ be a sequence. If $\{s_n\}$ converges to a positive real number, then $\limsup_{n\to\infty} s_n t_n = s \cdot \limsup_{n\to\infty} t_n$.

Thm (12.2)

Let $\{s_n\}$ be a sequence of nonzero numbers. Then, we have

$$\liminf_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right| \leq \liminf_{n\to\infty} |s_n|^{1/n} \leq \limsup_{n\to\infty} |s_n|^{1/n} \leq \limsup_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right|.$$

Cor (12.3)

If $\lim_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right| = L$, then $\lim_{n\to\infty} |s_n|^{1/n}$.

Note: These theorems in Section 12 are basically groundwork for our study of Series, Sections 14 and 15.